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The Maximum Principle for Variable-Order Fractional Diffusion Equations and the Estimates of Higher Variable-Order Fractional Derivatives

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In this paper, the maximum principle of variable-order fractional diffusion equations and the estimates of fractional derivatives with higher variable order are investigated. Firstly, we deduce the fractional derivative of a function of higher variable order at an arbitrary point. We also give an estimate of the error. Some important inequalities for fractional derivatives of variable order at arbitrary points and extreme points are presented. Then, the maximum principles of Riesz-Caputo fractional differential equations in terms of the multi-term space-time variable order are proved. Finally, under the initial-boundary value conditions, it is verified via the proposed principle that the solutions are unique, and their continuous dependance holds.

Keywords: maximum principle, fractional diffusion equation, fractional derivative with variable order, extreme point, boundary value problem

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1. INTRODUCTION

Fractional calculus Podlubny [1]; as a natural extension of traditional integer calculus, has become a classical and essential branch of mathematics through a long historical development. Recently Al-Refai and Baleanu [2], obtained the estimates of fractional derivatives with higher order for extreme points, providing an approach to the establishment of the maximum principles, as well as the results of the existence and uniqueness of solutions for the fractional differential equations (FDEs). As a kind of well-known technique for handling FDEs, the maximum principle may facilitate to acquire the key access to the solutions in the absence of any prior detailed knowledge about the solutions Protter and Weinberger [3]. Liu et al. [4] derived a maximum principle for fractional differential equations (VOFDEs, for short) with multi-term time variable order $0 < \alpha(\zeta,\tau) \leq 1$ and space variable orders $0 < \gamma(\zeta,\tau) \leq 1$ and $1 < \beta(\zeta,\tau) \leq 2$ in the sense of Riesz-Caputo, and showed the uniqueness of solutions as well as continuous of VOFDEs via the dependance. Ye et al. [5] investigated the solutions maximum principle. More researches in this area can be consulted in Luchko [6–8]; Li et al. [9]; Al-Refai and Luchko [10]; Yang et al. [11]; Coronelescamilla et al. [12]; Hajipour et al. [13].

However, the restriction for most of the aforesaid fractional diffusion equations is that their orders are constant. Such a restriction was relaxed by Samko and Ross [14] via a proposed variable-order (VO) operator to describe the diffusion process. In fact, VOFDEs are widerly used as powerful tools in many research topics, such as visco-elasticity Coimbra [15]; oscillation Ingman and Suzdalnitsky [16]; anomalous diffusion Sun et al. [17]; etc. For more applications of fractional differential equations, please refer to Cooper and Cowan [18]; Liu [19]; Sun et al. [20]; Liu and Li [21]; Yang [22], etc.

The contributions of this paper can be summarized as follows:

(1) The higher derivative of fractional function with variable order is given. On the basis of it, three useful theorems are given, which provide theoretical guarantee for the applications.
(2) The maximum principle for one-dimensional multi-term space-time higher VOFDEs is given.
(3) Based on the proposed method, a concrete example is given for the practical applications.

The paper is structured as the following. In **Section 2**, we recall some fundamental definitions that will be used in this paper. In **Section 3**, we derive some equalities and inequalities of the higher VOFDEs at arbitrary points and extreme points. We also give an estimate of the error. In **Section 4**, by virtue of these important inequalities, we establish the maximum principle for Riesz-Caputo FDEs with multi-term time variable order and space variable orders. In **Section 5**, based on the given principle, the uniqueness of solutions with their continuous dependance in the present of initial-boundary value conditions are strictly proved.

Notations: Throughout this paper, ζ denotes the space variable and τ denotes the time variable. $\Omega_T := (0,L)\times(0,T]$, $\overline{\Omega}_T$ and $\partial\Omega_T$ are the closure and the boundary of Ω_T, respectively. $\alpha(\cdot,\cdot)$, $\gamma(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ represent binary VO functions. It is supposed that the VO functions $\alpha, \alpha_1, \ldots, \alpha_n$, β and γ satisfy that

$$1<\alpha_n(\zeta,\tau)<\cdots<\alpha_1(\zeta,\tau)<\alpha(\zeta,\tau)\le 2,\quad (\zeta,\tau)\in\overline{\Omega}_T,$$

where $(\zeta,\tau)\in\overline{\Omega}_T$, $\beta(\zeta,\tau)\in(1,2]$ and $\gamma(\zeta,\tau)\in(0,1]$. Also, the functions $e(\zeta,\tau)$, $m(\zeta,\tau)$, $n(\zeta,\tau)$ and $a_i(\zeta,\tau), i=1,2,\ldots,n$ are supposed to be all continuous on $\overline{\Omega}_T$ with $m(\zeta,\tau)>0$, $n(\zeta,\tau)\ge 0$ and $e(\zeta,\tau)\le 0$.

2. PRELIMINARIES

Throughout this paper, $\mathbb{R}_+$ denotes the set of all positive real numbers. Let $C^n[0,T]=\{f: f^{(n)}\in C[0,T]\}$ be a Banach space with the norm $f_{C^n}=\max_{t\in[0,T]}[|f(t)|,|f'(t)|,\ldots,|f^{(n)}(t)|]$. For more details about the relevant concepts and results, please see Podlubny [1]; Liu et al. [4]; Kilbas et al. [23].

Definition 1. Let $f\in C[0,T]$ and $\alpha:(0,L)\times(0,T)\to\mathbb{R}_+$ be a VO function. The Riemann-Liouville fractional integrals of left-side VO and right-side VO are defined as

$$I_{0,\tau}^{\alpha(\zeta,\tau)}f(\tau)=\begin{cases}\dfrac{1}{\Gamma[\alpha(\zeta,\tau)]}\displaystyle\int_0^\tau(\tau-\vartheta)^{\alpha(\zeta,\tau)-1}f(\vartheta)\,d\vartheta, & \alpha(\zeta,\tau)>0,\\ f(\tau), & \alpha(\zeta,\tau)=0,\end{cases}$$

$$I_{\tau,T}^{\alpha(\zeta,\tau)}f(\tau)=\begin{cases}\dfrac{(-1)^{[\alpha(\zeta,\tau)]}}{\Gamma[\alpha(\zeta,\tau)]}\displaystyle\int_0^\tau(\tau-\vartheta)^{\alpha(\zeta,\tau)-1}f(\vartheta)\,d\vartheta, & \alpha(\zeta,\tau)>0,\\ f(\tau), & \alpha(\zeta,\tau)=0,\end{cases}$$

respectively, where $\Gamma[\alpha(\zeta,\tau)]=\int_0^\infty\theta^{\alpha(\zeta,\tau)-1}e^{-\theta}d\theta$ and $[\alpha(\zeta,\tau)]$ is the smallest integer not less than $\alpha(\zeta,\tau)$.

Definition 2. Let $f\in C^n[0,T]$ and $\alpha:[0,L]\times[0,T]\to\mathbb{R}_+$ be a VO function. The Caputo fractional derivatives of left-side VO and right-side VO are defined respectively as

$$\begin{aligned}{}^{C}D_{0,\tau}^{\alpha(\zeta,\tau)}f(\tau)&=I_{0,\tau}^{n-\alpha(\zeta,\tau)}\frac{d^n}{d\tau^n}f(\tau)\\&=\begin{cases}\dfrac{1}{\Gamma[n-\alpha(\zeta,\tau)]}\displaystyle\int_0^\tau(\tau-\vartheta)^{n-\alpha(\zeta,\tau)-1}f^{(n)}(\vartheta)d\vartheta, & n-1<\alpha(\zeta,\tau)<n,\\ f^{(n)}(\tau), & \alpha(\zeta,\tau)=n,\end{cases}\end{aligned}$$

$$\begin{aligned}{}^{C}D_{\tau,T}^{\alpha(\zeta,\tau)}f(\tau)&=I_{\tau,T}^{n-\alpha(\zeta,\tau)}\frac{d^n}{d\tau^n}f(\tau)\\&=\begin{cases}\dfrac{(-1)^n}{\Gamma[n-\alpha(\zeta,\tau)]}\displaystyle\int_0^\tau(\tau-\vartheta)^{n-\alpha(\zeta,\tau)-1}f^{(n)}(\vartheta)d\vartheta, & n-1<\alpha(\zeta,\tau)<n,\\ f^{(n)}(\tau), & \alpha(\zeta,\tau)=n.\end{cases}\end{aligned}$$

Definition 3. The VO Riesz-Caputo fractional operator ${}^{C}R_\zeta^{\beta(\zeta,\tau)}$ of VO $\beta(\zeta,\tau)$ with $n-1<\beta(\zeta,\tau)\le n$ and $0\le\zeta\le L$ is defined as

$${}^{C}R_\zeta^{\beta(\zeta,\tau)}w(\zeta,\tau):=-\rho_{\beta(\zeta,\tau)}\left({}^{C}D_{0,\zeta}^{\beta(\zeta,\tau)}+{}^{C}D_{\zeta,L}^{\beta(\zeta,\tau)}\right)w(\zeta,\tau),$$

where $\Gamma[\alpha(\zeta,\tau)]=\int_0^\infty\theta^{\alpha(\zeta,\tau)-1}e^{-\theta}d\theta$, $\rho_{\beta(\zeta,\tau)}=2^{-1}\cos^{-1}[\beta(\zeta,\tau)\pi/2]$ is the coefficient with $\beta(\zeta,\tau)\ne 1,2,3,\ldots,$ and

$${}^{C}D_{0,\zeta}^{\beta(\zeta,\tau)}w(\zeta,\tau)=\frac{1}{\Gamma(n-\beta(\zeta,\tau))}\int_0^\zeta(\zeta-\vartheta)^{n-\beta(\zeta,\tau)-1}\frac{\partial^n w(\vartheta,\tau)}{\partial\vartheta^n}d\vartheta,$$

$${}^{C}D_{\zeta,L}^{\beta(\zeta,\tau)}w(\zeta,\tau)=\frac{(-1)^n}{\Gamma(n-\beta(\zeta,\tau))}\int_\zeta^L(\vartheta-\zeta)^{n-\beta(\zeta,\tau)-1}\frac{\partial^n w(\vartheta,\tau)}{\partial\vartheta^n}d\vartheta.$$

Moreover, if $\beta(\zeta,\tau)=n$, ${}^{C}R_\zeta^{\beta(\zeta,\tau)}w(\zeta,\tau)=[\partial^n w(\zeta,\tau)/\partial\zeta^n]$.

In this paper, we are interested in the following VOFDEs:

$$\begin{aligned}P_{\alpha,\alpha_1,\ldots,\alpha_n}({}_0^CD_\tau)w(\zeta,\tau)=&-\Big[m(\zeta,\tau)\,{}^{C}R_\zeta^{\beta(\zeta,\tau)}w(\zeta,\tau)\\&+n(\zeta,\tau)\,{}^{C}R_\zeta^{\gamma(\zeta,\tau)}w(\zeta,\tau)\\&+e(\zeta,\tau)w(\zeta,\tau)\Big]\\&+F(\zeta,\tau,w),\ (\zeta,\tau)\in\Omega_T,\end{aligned}\qquad(1)$$

where $P_{\alpha,\alpha_1,\ldots,\alpha_n}({}_0^CD_\tau)$ denotes the multi-term time VO Caputo fractional derivative operator, i.e.,

$$\begin{aligned}P_{\alpha,\alpha_1,\ldots,\alpha_n}({}_0^CD_t)w(\zeta,\tau)=&\,{}_0^CD_t^{\alpha(\zeta,\tau)}w(\zeta,\tau)\\&+\sum_{i=1}^n a_i(\zeta,\tau)\,{}_0^CD_t^{\alpha_i(\zeta,\tau)}w(\zeta,\tau).\end{aligned}\qquad(2)$$

3. THE VARABLE-ORDER FRACTIONAL DERIVTIVES AT ARBITRARY POINTS AND EXTREME POINTS

In this section, we are in position to give some basic results.

Theorem 1. Let $f\in C^n[0,T]$. and $\eta_n(\cdot,\cdot)$ be a VO function. If η_n satisfies

$$n-1<\eta_n(\zeta,\tau)<n,\quad\forall(\zeta,\tau)\in\overline{\Omega}_T,$$

then for any arbitrary point $\tau_0 \in (0, T)$, the following equation holds

$$ {}^{C}D_{0,\tau_0}^{\eta_n(\zeta,\tau_0)} f(\tau_0) = -\sum_{k=0}^{n-1} \frac{1}{\Gamma[k+1-\eta_n(\zeta,\tau_0)]} \tau_0^{k-\eta_n(\zeta,\tau_0)} h_{n-1}^{(k)}(0) $$

$$ + \frac{1}{\Gamma[-\eta_n(\zeta,\tau_0)]} \int_0^{\tau_0} (\tau_0 - s)^{-\eta_n(\zeta,\tau_0)-1} h_{n-1}(s) ds, $$

where $h_{n-1}(\tau) = f(\tau) - \sum_{k=0}^{n-1} [f^{(k)}(\tau_0)(\tau-\tau_0)^k / k!]$.

PROOF. We shall prove this by induction argument. If $0 < \eta_1(\zeta,\tau_0) < 1$, the result has been obtained in Liu et al. [4]. Assume that this is true for $n-1 < \eta_n(\zeta,\tau_0) < n$. Now we check that it still holds whenever $n < \eta_{n+1}(\zeta,\tau_0) < n+1$.

Let $\eta_{n+1}(\zeta,\tau_0) = \delta(\zeta,\tau_0) + n$, where $0 < \delta(\zeta,\tau_0) < 1$. Then $n-1 < n-1+\delta(\zeta,\tau_0) < n$. Define $\eta_n(\zeta,\tau_0) = n-1+\delta(\zeta,\tau_0)$. Then. $n-1 < \eta_n(\zeta,\tau_0) < n$.

By the induction hypothesis, one obtains

$$ {}^{C}D_{0,\tau_0}^{n-1+\delta(\zeta,\tau_0)} f(\tau_0) = -\sum_{k=0}^{n-1} \frac{1}{\Gamma[k+2-n-\delta(\zeta,\tau_0)]} \tau_0^{k+1-n-\delta(\zeta,\tau_0)} h_{n-1}^{(k)}(0) $$

$$ + \frac{1}{\Gamma[1-n-\delta(\zeta,\tau_0)]} \int_0^{\tau_0} (\tau_0 - s)^{-n-\delta(\zeta,\tau_0)} h_{n-1}(s) ds. $$

Substituting $f'(\tau)$ for $f(\tau)$ in the preceding equation, one has

$$ {}^{C}D_{0,\tau_0}^{n-1+\delta(\zeta,\tau_0)} f'(\tau_0) = -\sum_{k=0}^{n-1} \frac{1}{\Gamma[k+2-n-\delta(\zeta,\tau_0)]} \tau_0^{k+1-n-\delta(\zeta,\tau_0)} z_{n-1}^{(k)}(0) $$

$$ + \frac{1}{\Gamma[1-n-\delta(\zeta,\tau_0)]} \int_0^{\tau_0} (\tau_0 - s)^{-n-\delta(\zeta,\tau_0)} z_{n-1}(s) ds, $$

where $z_{n-1}(\tau) = f'(\tau) - \sum_{k=0}^{n-1} [f^{(k+1)}(\tau_0)(\tau-\tau_0)^k / k!]$.

Obviously, we have:

(1) $h_n'(\tau) = z_{n-1}(\tau)$,
(2) $h_n(\tau_0) = h_n'(\tau_0) = h_n''(\tau_0) = \cdots = h_n^{(n)}(\tau_0) = 0$.

Hence,

$$ h_n(\tau) = (\tau_0 - \tau)^{n+1} \mu_n(\tau), $$

where $\mu_n(\tau) \in C[0,T]$ and $h_n^{(k+1)}(0) = z_{n-1}^{(k)}(0)$.

Integrating by parts, we have

$$ \int_0^{\tau_0} (\tau_0 - s)^{-n-\delta(\zeta,\tau_0)} z_{n-1}(s) ds = (\tau_0 - s)^{-n-\delta(\zeta,\tau_0)} h_n(s)\big|_0^{\tau_0} $$

$$ - (n+\delta(\zeta,\tau_0)) \times \int_0^{\tau_0} (\tau_0 - s)^{-1-n-\delta(\zeta,\tau_0)} h_n(s) ds. $$

So

$$ \lim_{s\to\tau_0} \frac{h_n(s)}{(\tau_0 - s)^{n+\delta(\zeta,\tau_0)}} = \lim_{s\to\tau_0} (\tau_0 - s)^{1-\delta(\zeta,\tau_0)} \mu_n(\tau) = 0, $$

$$ \forall \zeta \in [0, L], $$

and

$$ \frac{1}{\Gamma(1-n-\delta(\zeta,\tau_0))} \int_0^{\tau_0} (\tau_0 - s)^{-n-\delta(\zeta,\tau_0)} z_{n-1}(s) ds $$

$$ = -\frac{\tau_0^{-n-\delta(\zeta,\tau_0)} h_n(0)}{\Gamma(1-n-\delta(\zeta,\tau_0))} - \frac{n+\delta(\zeta,\tau_0)}{\Gamma(1-n-\delta(\zeta,\tau_0))} $$

$$ \int_0^{\tau_0} (\tau_0 - s)^{-n-\delta(\zeta,\tau_0)-1} h_n(s) ds $$

$$ = -\frac{\tau_0^{-n-\delta(\zeta,\tau_0)} h_n(0)}{\Gamma(1-n-\delta(\zeta,\tau_0))} + \frac{1}{\Gamma(-n-\delta(\zeta,\tau_0))} $$

$$ \int_0^{\tau_0} (\tau_0 - s)^{-n-\delta(\zeta,\tau_0)-1} h_n(s) ds. $$

Thus,

$$ {}^{C}D_{0,\tau_0}^{n-1+\delta(\zeta,\tau_0)} f'(\tau_0) = -\sum_{k=0}^{n-1} \frac{1}{\Gamma(k+2-n-\delta(\zeta,\tau_0))} \tau_0^{k+1-n-\delta(\zeta,\tau_0)} $$

$$ h_n^{(k+1)}(0) - \frac{\tau_0^{-n-\delta(\zeta,\tau_0)} h_n(0)}{\Gamma(1-n-\delta(\zeta,\tau_0))} $$

$$ + \frac{1}{\Gamma(-n-\delta(\zeta,\tau_0))} \int_0^{\tau_0} (\tau_0 - s)^{-n-\delta(\zeta,\tau_0)-1} h_n(s) ds $$

$$ = -\sum_{k=1}^{n} \frac{1}{\Gamma(k+1-n-\delta(\zeta,\tau_0))} \tau_0^{k-n-\delta(\zeta,\tau_0)} h_n^{(k)}(0) $$

$$ - \frac{\tau_0^{-n-\delta(\zeta,\tau_0)} h_n(0)}{\Gamma(1-n-\delta(\zeta,\tau_0))} $$

$$ + \frac{1}{\Gamma(-n-\delta(\zeta,\tau_0))} \int_0^{\tau_0} (\tau_0 - s)^{-n-\delta(\zeta,\tau_0)-1} h_n(s) ds $$

$$ = -\sum_{k=0}^{n} \frac{1}{\Gamma(k+1-n-\delta(\zeta,\tau_0))} \tau_0^{k-n-\delta(\zeta,\tau_0)} h_n^{(k)}(0) $$

$$ + \frac{1}{\Gamma(-n-\delta(\zeta,\tau_0))} \int_0^{\tau_0} (\tau_0 - s)^{-n-\delta(\zeta,\tau_0)-1} h_n(s) ds $$

$$ = -\sum_{k=0}^{n-1} \frac{1}{\Gamma(k+1-\eta_{n+1}(\zeta,\tau_0))} \tau_0^{k-\eta_{n+1}(\zeta,\tau_0)} h_n^{(k)}(0) $$

$$ + \frac{1}{\Gamma(-\eta_{n+1}(\zeta,\tau_0))} \int_0^{\tau_0} (\tau_0 - s)^{-\eta_{n+1}(\zeta,\tau_0)-1} h_n(s) ds. $$

Hence ${}^{C}D_{0,\tau_0}^{n-1+\delta(\zeta,\tau_0)} f'(\tau_0) = {}^{C}D_{0,\tau_0}^{n+\delta(\zeta,\tau_0)} f(\tau_0) = {}^{C}D_{0,\tau_0}^{\eta_{n+1}(\zeta,\tau_0)} f(\tau_0)$. This complete the proof.

Remark 1. If $\eta_n(\zeta,\tau) \equiv \bar{\alpha}$ in $\bar{\Omega}_T$ $(n-1 < \bar{\alpha} \le n)$ and τ_0 is an extreme point, then Theorem 1 coincides with Al-Refai and Baleanu [2]'s result. Thus, our result generalizes AL-Refai and Baleanu's original idea.

Theorem 2.

Let $f \in C^n[0,T]$. Suppose that the VO function $\eta_n(\zeta,\tau)$ satisfies

$$n-1<\eta_n(\zeta,\tau)<n,\quad \forall(\zeta,\tau)\in\overline{\Omega}_T.$$

For any arbitrary point $\tau_0\in(0,T)$, one gets

(1) For any nonnegative $f^{(n)}(\tau)$ with $\tau\in[0,\tau_0]$, then

$$^{C}D_{0,\tau_0}^{\eta_n(\zeta,\tau_0)}f(\tau_0)\geq-\sum_{k=0}^{n-1}\frac{1}{\Gamma(k+1-\eta_n(\zeta,\tau_0))}\tau_0^{k-\eta_n(\zeta,\tau_0)}h_{n-1}^{(k)}(0)$$

(2) For any non-positive $f^{(n)}(\tau)$ with $\tau\in[0,\tau_0]$, then

$$^{C}D_{0,\tau_0}^{\eta_n(\zeta,\tau_0)}f(\tau_0)\leq-\sum_{k=0}^{n-1}\frac{1}{\Gamma(k+1-\eta_n(\zeta,\tau_0))}\tau_0^{k-\eta_n(\zeta,\tau_0)}h_{n-1}^{(k)}(0)$$

where $h_{n-1}(\tau)=f(\tau)-\sum_{k=0}^{n-1}(f^{(k)}(\tau_0)(\tau-\tau_0)^k/k!)$.

Proof. Employing the Taylor series expansion, we know that there is some τ_0 with $\tau<\vartheta_n(\tau)<\tau_0$ such that

$$h_{n-1}(\tau)=f(\tau)-\sum_{k=0}^{n-1}\frac{f^{(k)}(\tau_0)(\tau-\tau_0)^k}{k!}=\frac{f^{(n)}(\vartheta_n(\tau))(\tau-\tau_0)^n}{n!}$$

So, we have

$$\begin{aligned}m_n&=\frac{1}{\Gamma(-\eta_n(\zeta,\tau_0))}\int_0^{\tau_0}(\tau_0-s)^{-\eta_n(\zeta,\tau_0)-1}h_{n-1}(s)ds.\\&=\frac{1}{\Gamma(-\eta_n(\zeta,\tau_0))}\int_0^{\tau_0}(\tau_0-s)^{-\eta_n(\zeta,\tau_0)-1}\frac{f^{(n)}(\vartheta_n(\tau))(\tau-\tau_0)^n}{n!}ds.\\&=\frac{(-1)^n}{n!\Gamma(-\eta_n(\zeta,\tau_0))}\int_0^{\tau_0}(\tau_0-s)^{n-\eta_n(\zeta,\tau_0)-1}f^{(n)}(\vartheta_n(s))ds.\end{aligned}\tag{3}$$

Note that $n-1<\eta_n(\zeta,\tau_0)<n$, and

$$\Gamma(-\eta_n(\zeta,\tau_0))\begin{cases}>0, & \text{if } n \text{ is even,}\\ <0, & \text{otherwise.}\end{cases}$$

Therefore, we get $((-1)^n/\Gamma(-\eta_n(\zeta,\tau_0)))>0$, and

$$m_n=\begin{cases}\geq 0, & \text{if } f^{(n)}(\tau)\geq 0,\\ <0, & \text{otherwise.}\end{cases}$$

Theorem 3. Let $f\in C^n[0,T]$, and $|f^{(n)}(\tau)|\leq M$, for all $\tau\in[0,T]$. If the VO function $\eta_n(\zeta,\tau)$ satisfies

$$n-1<\eta_n(\zeta,\tau)<n,\quad \forall(\zeta,\tau)\in\overline{\Omega}_T,$$

then for any arbitrary point $\tau_0\in(0,T)$, the following equation holds:

$$^{C}D_{0,\tau_0}^{\eta_n(\zeta,\tau_0)}f(\tau_0)=-\sum_{k=0}^{n-1}\frac{1}{\Gamma(k+1-\eta_n(\zeta,\tau_0))}\tau_0^{k-\eta_n(\zeta,\tau_0)}h_{n-1}^{(k)}(0)+m_n,$$

where $h_{n-1}(\tau)=f(\tau)-\sum_{k=0}^{n-1}(f^{(k)}(\tau_0)(\tau-\tau_0)^k/k!)$, and

$$|m_n|\leq\frac{Mt_0^{n-\eta_n(\zeta,\tau_0)}}{n!(n-\eta_n(\zeta,\tau_0))|\Gamma(-\eta_n(\zeta,\tau_0))|}.$$

Proof. According to **Eq. 3**, one has

$$m_n=\frac{(-1)^n}{n!\Gamma(-\eta_n(\zeta,\tau_0))}\int_0^{\tau_0}(\tau_0-s)^{n-\eta_n(\zeta,\tau_0)-1}f^{(n)}(\vartheta_n(s))ds.$$

As a result,

$$\begin{aligned}|m_n|&\leq\frac{M}{n!|\Gamma(-\eta_n(\zeta,\tau_0))|}\int_0^{\tau_0}(\tau_0-s)^{n-\eta_n(\zeta,\tau_0)-1}ds\\&=\frac{Mt_0^{n-\eta_n(\zeta,\tau_0)}}{n!(n-\eta_n(\zeta,\tau_0))|\Gamma(-\eta_n(\zeta,\tau_0))|}.\end{aligned}$$

Theorem 4. Given a VO function $\alpha:[0,L]\times[0,T]\to\mathbb{R}_+$ with $1<\alpha(\zeta,\tau)<2$ for all $(\zeta,\tau)\in\overline{\Omega}_T$. If $f\in C^2[0,T]$ attains its maximum at $\tau_0\in(0,T)$, then it holds that

$$^{C}D_{0,\tau_0}^{\alpha(\zeta,\tau_0)}f(\tau_0)\leq\frac{\alpha(\zeta,\tau_0)-1}{\Gamma(2-\alpha(\zeta,\tau_0))}\tau_0^{-\alpha(\zeta,\tau_0)}[f(0)-f(\tau_0)]-\frac{\tau_0^{1-\alpha(\zeta,\tau_0)}}{\Gamma(2-\alpha(\zeta,\tau_0))}f'(0).$$

Moreover, if $f'(0)\geq 0$, then $^{C}_{0}D_{0,\tau_0}^{\alpha(\zeta,\tau_0)}f(\tau_0)\leq 0,\forall\zeta\in[0,L]$.

Proof. Let $\phi(\tau):=f(\tau)-f(\tau_0)\in C^2[0,T]$. Obviously, we have

(1) $\phi(\tau)\leq 0,\tau\in[0,T]$;
(2) $\phi(\tau_0)=\phi'(\tau_0)=0$ and $\phi''(\tau_0)\leq 0$;
(3) $\phi(\tau)=(\tau_0-\tau)^2\cdot\nu(\tau)$ where $\nu\in C[0,T]$ and $\nu(\tau)\leq 0$, $\forall\tau\in[0,T]$.

It can be easily verified that

$$^{C}D_{0,\tau}^{\alpha(\zeta,\tau)}\phi(\tau)={}^{C}D_{0,\tau}^{\alpha(\zeta,\tau)}f(\tau),\forall(\zeta,\tau)\in\overline{\Omega}$$

By Theorem 1, we obtain

$$\begin{aligned}^{C}D_{0,\tau_0}^{\alpha(\zeta,\tau_0)}\phi(\tau_0)&=-\frac{\tau_0^{1-\alpha(\zeta,\tau_0)}}{\Gamma(2-\alpha(\zeta,\tau_0))}\phi'(0)\\&+\frac{\alpha(\zeta,\tau_0)-1}{\Gamma(2-\alpha(\zeta,\tau_0))}\tau_0^{-\alpha(\zeta,\tau_0)}\phi(0)\\&+\frac{(\alpha(\zeta,\tau_0)-1)\cdot\alpha(\zeta,\tau_0)}{\Gamma(2-\alpha(\zeta,\tau_0))}\int_0^{\tau_0}(\tau_0-s)^{-\alpha(\zeta,\tau_0)-1}\phi(s)ds\end{aligned}$$

Since for all $\tau\in[0,\tau_0],\phi(\tau)\leq 0$ and $\phi(\tau)=(\tau_0-\tau)^2\nu(\tau)$, it follows that $M:=\max_{\tau\in[0,\tau_0]}\nu(\tau)\leq 0$.

Hence,

$$\begin{aligned}&\int_0^{\tau_0}(\tau_0-s)^{-\alpha(\zeta,\tau_0)-1}\phi(s)ds\\&=\int_0^{\tau_0}(\tau_0-s)^{1-\alpha(\zeta,\tau_0)}\nu(s)ds\\&\leq M\int_0^{\tau_0}(\tau_0-s)^{1-\alpha(\zeta,\tau_0)}ds\\&=M\frac{-1}{2-\alpha(\zeta,\tau_0)}(\tau_0-s)^{2-\alpha(\zeta,\tau_0)}\Big|_0^{\tau_0}\\&=M\frac{\tau_0^{2-\alpha(\zeta,\tau_0)}}{2-\alpha(\zeta,\tau_0)}\leq 0,\forall\zeta\in[0,L].\end{aligned}$$

Therefore

$$ {}^{C}D_{0,\tau_0}^{\alpha(\zeta,\tau_0)} f(\tau_0) = -\frac{\tau_0^{1-\alpha(\zeta,\tau_0)}}{\Gamma(2-\alpha(\zeta,\tau_0))}\phi'(0) $$

$$ + \frac{\alpha(\zeta,\tau_0)-1}{\Gamma(2-\alpha(\zeta,\tau_0))}\tau_0^{-\alpha(\zeta,\tau_0)}\phi(0) $$

$$ + \frac{(\alpha(\zeta,\tau_0)-1)\cdot\alpha(\zeta,\tau_0)}{\Gamma(2-\alpha(\zeta,\tau_0))}\int_0^{\tau_0}(\tau_0-s)^{-1-\alpha(\zeta,\tau_0)}v(s)ds $$

$$ \leq -\frac{\tau_0^{1-\alpha(\zeta,\tau_0)}}{\Gamma(2-\alpha(\zeta,\tau_0))}\phi'(0) + \frac{\alpha(\zeta,\tau_0)-1}{\Gamma(2-\alpha(\zeta,\tau_0))}\tau_0^{-\alpha(\zeta,\tau_0)}\phi(0) $$

$$ = \frac{\alpha(\zeta,\tau_0)-1}{\Gamma(2-\alpha(\zeta,\tau_0))}\tau_0^{-\alpha(\zeta,\tau_0)}[f(0)-f(\tau_0)] - \frac{\tau_0^{1-\alpha(\zeta,\tau_0)}}{\Gamma(2-\alpha(\zeta,\tau_0))}f'(0). $$

Consequently, ${}^{C}D_{0,\tau_0}^{\alpha(\zeta,\tau_0)} f(\tau_0) \leq 0$ for all $\zeta \in [0,L]$ whenever $f'(0) \geq 0$,

4. THE MAXIMUM PRINCIPLE

In this section, we will display and show the maximum principle for one-dimensional multi-term space-time higher VOFDEs.

For convenience, the symbol $Q_{\beta,\gamma}$ is used to denote the operator given by

$$ Q_{\beta,\gamma}w(\zeta,\tau) = m(\zeta,\tau)\,{}^{C}R_{\zeta}^{\beta(\zeta,\tau)}w(\zeta,\tau) + n(\zeta,\tau)\,{}^{C}R_{\zeta}^{\gamma(\zeta,\tau)}w(\zeta,\tau) + e(\zeta,\tau)w(\zeta,\tau). $$

It is easy to see that $Q_{\beta,\gamma}$ is a space VO operator on ζ.

Theorem 5. Suppose $w(\zeta,\tau) \in C^{2,2}(\overline{\Omega}_T)$ and

$$ P_{\alpha,\alpha_1,\ldots,\alpha_n}({}^{C}D_{0,t})w(\zeta,\tau) + Q_{\beta,\gamma}w(\zeta,\tau) \geq 0, \quad \forall(\zeta,\tau)\in\Omega_T. $$

If $(\partial w/\partial\zeta)|_{\zeta=0} \geq 0$ but $(\partial w/\partial\zeta)|_{\zeta=L} \leq 0$ whenever $0 \leq \tau \leq T$, then

$$ \max_{(\zeta,\tau)\in\overline{\Omega}_T} w(\zeta,\tau) \leq \max\left\{\max_{(\zeta,\tau)\in\partial\Omega_T} w(\zeta,\tau), 0\right\}, $$

Proof. We prove this by contradiction. Assume that there exits $(\zeta_0,\tau_0)\in\Omega_T$ such that

$$ w(\zeta_0,\tau_0) > \max\left\{\max_{(\zeta,\tau)\in\partial\Omega_T} w(\zeta,\tau), 0\right\} = M \geq 0. $$

Let $w^*(\zeta,\tau) = w(\zeta,\tau) + (\epsilon/2)((T-\tau)/T)^2$ for all $(\zeta,\tau)\in\overline{\Omega}_T$, where $\varepsilon = w(\zeta_0,\tau_0) - M > 0$.

Precisely, we have

$$ \begin{cases} {}^{C}D_{0,\tau}^{\alpha(\zeta,\tau)}w^*(\zeta,\tau) = {}^{C}D_{0,\tau}^{\alpha(\zeta,\tau)}w(\zeta,\tau) + \frac{\varepsilon}{T^2}\frac{\tau^{2-\alpha(\zeta,\tau)}}{\Gamma(3-\alpha(\zeta,\tau))}, \\ {}^{C}D_{0,\tau}^{\alpha_i(\zeta,\tau)}w^*(\zeta,\tau) = {}^{C}D_{0,\tau}^{\alpha_i(\zeta,\tau)}w(\zeta,\tau) + \frac{\varepsilon}{T^2}\frac{\tau^{2-\alpha_i(\zeta,\tau)}}{\Gamma(3-\alpha_i(\zeta,\tau))}, \quad i=1,2,\ldots,n, \end{cases} $$

and

$$ \begin{cases} {}^{C}R_{\zeta}^{\gamma(\zeta,\tau)}w^*(\zeta,\tau) = {}^{C}R_{\zeta}^{\gamma(\zeta,\tau)}w(\zeta,\tau), \\ {}^{C}R_{x}^{\beta(\zeta,\tau)}w^*(\zeta,\tau) = {}^{C}R_{\zeta}^{\beta(\zeta,\tau)}w(\zeta,\tau). \end{cases} $$

This implies that

$$ w^*(\zeta,\tau) = w(\zeta,\tau) + \frac{\varepsilon}{2}\left(\frac{T-\tau}{T}\right)^2 \leq w(\zeta,\tau) + \frac{\varepsilon}{2}, \quad (\zeta,\tau)\in\overline{\Omega}_T, $$

Thus,

$$ w^*(\zeta_0,\tau_0) > w(\zeta_0,\tau_0) = M + \varepsilon \geq \varepsilon + w(\zeta,\tau) \geq w^*(\zeta,\tau) + \frac{\varepsilon}{2}, \quad (\zeta,\tau)\in\partial\Omega_T. $$

This means w^* fails to reach the maximum value on the boundary $\partial\Omega_T$. Assume that w^* obtains the maximum value at $(\zeta_1,\tau_1)\in\Omega_T$. It follows that

$$ w^*(\zeta_1,\tau_1) \geq w^*(\zeta_0,\tau_0) > \varepsilon + M \geq \varepsilon > 0. $$

Trivially, one has

$$ P_{\alpha,\alpha_1,\ldots,\alpha_n}({}^{C}D_{0,\tau})w^*(\zeta,\tau) = P_{\alpha,\alpha_1,\ldots,\alpha_n}({}^{C}D_{0,\tau})w(\zeta,\tau) + \frac{\varepsilon}{T^2}\left[\frac{\tau^{2-\alpha(\zeta,\tau)}}{\Gamma(3-\alpha(\zeta,\tau))} + \sum_{i=1}^{n}\frac{a_i(\zeta,\tau)\cdot\tau^{2-\alpha_i(\zeta,\tau)}}{\Gamma(3-\alpha_i(\zeta,\tau))}\right]. \quad (4) $$

and

$$ \begin{aligned} Q_{\beta,\gamma}w^*(\zeta_1,\tau_1) &= p(\zeta_1,\tau_1)\,{}^{C}R_{x}^{\beta(\zeta_1,\tau_1)}w^*(\zeta_1,\tau_1) \\ &+ q(\zeta_1,\tau_1)\,{}^{C}R_{x}^{\gamma(\zeta_1,\tau_1)}w^*(\zeta_1,\tau_1) + e(\zeta_1,\tau_1)w^*(\zeta,\tau) \\ &= p(\zeta_1,\tau_1)\,{}^{C}R_{x}^{\beta(\zeta_1,\tau_1)}w(\zeta_1,\tau_1) + q(\zeta_1,\tau_1)\,{}^{C}R_{x}^{\gamma(\zeta_1,\tau_1)}w(\zeta_1,\tau_1) \\ &+ e(\zeta_1,\tau_1)w^*(\zeta_1,\tau_1) = Q_{\beta,\gamma}w(\zeta_1,\tau_1) - e(\zeta_1,\tau_1)w(\zeta_1,\tau_1) \\ &+ e(\zeta_1,\tau_1)w^*(\zeta_1,\tau_1) = Q_{\beta,\gamma}w(\zeta_1,\tau_1) + e(\zeta_1,\tau_1)\frac{\varepsilon}{2}\left(\frac{T-t_1}{T}\right)^2. \end{aligned} \quad (5) $$

Note that $q(\zeta_1,\tau_1) \geq 0$ and $p(\zeta_1,\tau_1) > 0$, which follow by applying Theorem four in this paper along with Theorems 3.2 and 3.3 in Liu et al. [4]. By virtue of **Eqs 4** and **5**, we have

$$ \begin{aligned} &P_{\alpha,\alpha_1,\ldots,\alpha_n}({}^{C}D_{0,\tau})w(\zeta_1,\tau_1) + Q_{\beta,\gamma}w(\zeta_1,\tau_1) \\ &= P_{\alpha,\alpha_1,\ldots,\alpha_n}({}^{C}D_{0,\tau})w^*(\zeta_1,\tau_1) - e(\zeta_1,\tau_1)\frac{\varepsilon}{2}\left(\frac{T-\tau_1}{T}\right)^2 \\ &- \frac{\varepsilon}{T^2}\left[\frac{\tau_1^{2-\alpha(\zeta_1,\tau_1)}}{\Gamma(3-\alpha(\zeta_1,\tau_1))} + \sum_{i=1}^{n}\frac{a_i(\zeta_1,\tau_1)\cdot\tau_1^{2-\alpha_i(\zeta_1,\tau_1)}}{\Gamma(3-\alpha_i(\zeta_1,\tau_1))}\right] \\ &+ Q_{\beta,\gamma}w^*(\zeta_1,\tau_1) \leq -\frac{\varepsilon}{T^2}\left[\frac{\tau_1^{2-\alpha(\zeta_1,\tau_1)}}{\Gamma(3-\alpha(\zeta_1,\tau_1))}\right. \\ &\left.+ \sum_{i=1}^{n}\frac{a_i(\zeta_1,\tau_1)\cdot\tau_1^{2-\alpha_i(\zeta_1,\tau_1)}}{\Gamma(3-\alpha_i(\zeta_1,\tau_1))}\right] + e(\zeta_1,\tau_1)\varepsilon\left[1 - \frac{1}{2}\left(\frac{T-\tau_1}{T}\right)^2\right] < 0. \end{aligned} $$

This is a contradiction to our assumption that

$$ P_{\alpha,\alpha_1,\ldots,\alpha_n}({}^{C}D_{0,\tau})w(\zeta,\tau) + Q_{\beta,\gamma}w(\zeta,\tau) \geq 0, \quad \forall(\zeta,\tau)\in\Omega_T. $$

This completes the proof.

If we substitute $-w$ for w in Theorem 5, the minimum principle is obtained as follows.

Theorem 6. Suppose $w(\zeta,\tau)\in C^{2,2}(\overline{\Omega}_T)$, and

$$P_{\alpha,\alpha_1,\ldots,\alpha_n}\left({}^{C}D_{0,\tau}\right)w(\zeta,\tau)+Q_{\beta,\gamma}w(\zeta,\tau)\le 0,\quad \forall(\zeta,\tau)\in\Omega_T. \tag{6}$$

If $(\partial w/\partial\zeta)|_{\zeta=0}\le 0$ and $(\partial w/\partial\zeta)|_{\zeta=L}\ge 0$, for all $\tau\in[0,T]$, then

$$\min_{(\zeta,\tau)\in\overline{\Omega}_T} w(\zeta,\tau)\ge\min\left\{\min_{(\zeta,\tau)\in\partial\Omega_T} w(\zeta,\tau),0\right\},$$

where $\partial\Omega_T$ is the boundary of Ω_T.

5. APPLICATIONS

In this section, we discuss multi-term space-time higer VOFDEs in the one-dimensional case:

$$P_{\alpha,\alpha_1,\ldots,\alpha_n}\left({}_{0}^{C}D_{\tau}\right)w(\zeta,\tau)+Q_{\beta,\gamma}w(\zeta,\tau)=f(\zeta,\tau),\quad(\zeta,\tau)\in\Omega_T, \tag{7}$$

with the initial conditions

$$w(\zeta,0)=\Theta(\zeta),\zeta\in[0,L]. \tag{8}$$

The boundary conditions are taken into consideration as below:

$$\begin{cases} w(0,\tau)=k_1(\tau), & \tau\in[0,T],\\ w(L,\tau)=k_2(\tau), & \tau\in[0,T]. \end{cases} \tag{9}$$

By Theorems 5 and 6, we can get the following theorems.

Theorem 7. Suppose $f(\zeta,\tau)\ge 0,(\zeta,\tau)\in\Omega_T;\Theta(\zeta)\le 0,\zeta\in[0,L];k_1(\tau)\le 0,k_2(\tau)\le 0,\tau\in[0,T]$. If $w(\zeta,\tau)\in C^{2,2}(\overline{\Omega}_T)$ is a solution of the problem **Eqs 7–9** with $(\partial w/\partial\zeta)|_{\zeta=0}\ge 0$ and $(\partial w/\partial\zeta)|_{\zeta=L}\le 0$ for all $\tau\in[0,T]$, then $w(\zeta,\tau)\le 0,(\zeta,\tau)\in\overline{\Omega}_T$.

Theorem 8. Suppose $f(\zeta,\tau)\le 0,(\zeta,\tau)\in\Omega_T;\Theta(\zeta)\ge 0,\zeta\in[0,L];k_1(\tau)\ge 0,k_2(\tau)\ge 0,\tau\in[0,T]$. If $w(\zeta,\tau)\in C^{2,2}(\overline{\Omega}_T)$ is a solution of the problem **Eqs 7–9** with $(\partial w/\partial\zeta)|_{\zeta=0}\le 0$ and $(\partial w/\partial\zeta)|_{\zeta=L}\ge 0$ for all $\tau\in[0,T]$, then $w(\zeta,\tau)\ge 0,(\zeta,\tau)\in\overline{\Omega}_T$,.

Remark 2. If $f(\zeta,\tau)=0$, then, according to Theorem 7 and 8, we know that the diffusion problem **Eqs 7–9** with zero initial and boundary conditions permits only zero solution in $C^{2,2}(\overline{\Omega}_T)$.

Consider the next nonlinear diffusion equation

$$\begin{aligned} P_{\alpha,\alpha_1,\ldots,\alpha_n}\left({}_{0}^{C}D_{0,\tau}\right)w(\zeta,\tau) &= -\Big[m(\zeta,\tau)^{C}R_{\zeta}^{\beta(\zeta,\tau)}w(\zeta,\tau)\\ &+n(\zeta,\tau)^{C}R_{\zeta}^{\gamma(\zeta,\tau)}w(\zeta,\tau)+e(\zeta,\tau)w(\zeta,\tau)]\\ &+F(\zeta,\tau,w),\,(\zeta,\tau)\in\Omega_T. \end{aligned} \tag{10}$$

Theorem 9. Assume that the partial derivative $\partial_w F=\partial_w F(\zeta,\tau,w)$ exists and satisfies $\partial_w F(\zeta,\tau,w)-e(\zeta,\tau)\le 0$ for all $(\zeta,\tau,w)\in\Omega_T\times\mathbb{R}$. If $(\partial w/\partial\zeta)|_{\zeta=0}=0$ and $(\partial w/\partial\zeta)|_{\zeta=L}=0$ for all $\tau\in[0,T]$, then the problem **Eqs 8–10** has at most one solution $w=w(\zeta,\tau),(\zeta,\tau)\in\overline{\Omega}_T$ in $C^{2,2}(\overline{\Omega}_T)$.

PROOF. Suppose that $w_1,w_2\in C^{2,2}(\overline{\Omega}_T)$ are two solutions of the problem **Eqs 8–10**. Let $w=w_1-w_2$. Then

$$\begin{aligned} P_{\alpha,\alpha_1,\ldots,\alpha_n}\left({}^{C}D_{0,\tau}\right)w(\zeta,\tau) &= -\Big[m(\zeta,\tau)^{C}R_{x}^{\beta(\zeta,\tau)}w(\zeta,\tau)\\ &+n(\zeta,\tau)^{C}R_{x}^{\gamma(\zeta,\tau)}w(\zeta,\tau)+e(\zeta,\tau)w(\zeta,\tau)]+F(\zeta,\tau,w_1)\\ &-F(\zeta,\tau,w_2). \end{aligned}$$

Since the homogeneous initial and boundary conditions are fulfilled by w, one has

$$w(\zeta,\tau)=0,\,(\zeta,\tau)\in\partial\Omega_T.$$

Owing to the existence of $\partial_w F=\partial_w F(\zeta,\tau,w)$, it holds that

$$F(\zeta,\tau,w_1)-F(\zeta,\tau,w_2)=\frac{\partial F}{\partial w}(w^*)(w_1(\zeta,\tau)-w_2(\zeta,\tau))$$

for all $(\zeta,\tau)\in\Omega_T$, where $w^*=(1-\varrho)w_1+\varrho w_2$ for some $0\le\varrho\le 1$.

Consequently,

$$\begin{cases} P_{\alpha,\alpha_1,\ldots,\alpha_n}\left({}^{C}D_{0,\tau}\right)w(\zeta,\tau)=-\Big[m(\zeta,\tau)^{C}R_{\zeta}^{\beta(\zeta,\tau)}w(\zeta,\tau)+\\ +n(\zeta,\tau)^{C}R_{\zeta}^{\gamma(\zeta,\tau)}w(\zeta,\tau)\Big]+h(\zeta,\tau)w(\zeta,\tau),\\ w(\zeta,\tau)=0,\,(\zeta,\tau)\in\partial\Omega_T, \end{cases} \tag{11}$$

where $h(\zeta,\tau)=(\partial F/\partial w)(w^*)-e(\zeta,\tau)\le 0$ for all $(\zeta,\tau)\in\Omega_T$.

By Theorem 7, $w(\zeta,\tau)\le 0$ holds for all $(\zeta,\tau)\in\overline{\Omega}_T$. Conversely, $w(\zeta,\tau)\ge 0$ is also true by using Theorem 8. So, $w(\zeta,\tau)=0$, i.e.,

$$w_1(\zeta,\tau)=w_2(\zeta,\tau),\quad\forall(\zeta,\tau)\in\overline{\Omega}_T.$$

This completes the proof.

6. CONCLUSIONS

This paper serves as a survey on the maximum principle and the estimates of time higher VOFDEs. The proposed maximum principle contributes to verify some important properties of solutions, including the uniqueness and the continuous dependance with initial-boundary value conditions being taken account. In the future, we will put attention to the solutions for problem **Eq. 1** in more general forms, and investigate the numerical solutions with their applications.

DATA AVAILABILITY STATEMENT

All datasets presented in this study are included in the article.

AUTHOR CONTRIBUTIONS

GX, FL, and GS contributed conception and layout of the research; GX organized the literature; FL completed the initial draft of the paper; GS carried out the proof; The main idea of this paper was proposed by GX; All authors approved the submitted paper.

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