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Basic Control Theory for Linear Fractional Differential Equations With Constant Coefficients

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In this paper we present an analogous result of the famous Kalman controllability criterion for first order linear ordinary differential equations with constant coefficients that applies to the case of linear differential equations of fractional order with constant coefficients. We use the fractional Gramian matrix, the range space and the Kalman matrix as main tools to derive a sufficient and necessary condition for the controllability of the fractional system. Moreover, we provide some simple examples, including a linear fractional harmonic oscillator, to illustrate our results. Finally, several open problems arising from this topic are suggested, including another simple linear system of incommensurate fractional orders.

Keywords: linear differential equations, controllability, fractional Gramian, fractional differential equations, Kalman matrix

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1. INTRODUCTION

Controllability is a mathematical problem consisting in determining the targets to which one can drive the state of a dynamical system by means of a control input appearing in the equation. We have a dynamical system on which we can exert a certain influence. Is it possible to use this to make the system reach a desirable state? In other words, given a future time, an initial state and a target state, is it possible to find a control function such that the solution of the system starting from the initial state reaches the desirable state at the prescribed future time? For some classical and modern references on control theory we refer to references [1–3].

On the other hand, fractional calculus and fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences. We refer the reader to the monographs [4–7] and the articles [8, 9]. In particular, there are a growing number of research areas in physics which employ fractional calculus [10] and it has many applications among its different branches, ranging from imaging processing to fractional quantum harmonic oscillator [11]. Recently, in Yıldız [12] the dynamics of a waterborne pathogen fractional model under the influence of environmental pollution has been studied and the solutions of a generalized fractional kinetic equations are obtained [13] using the generalized fractional integrations of the generalized Mittag-Leffler type function. Finally, we highlight that different fractional systems have also been considered in the framework of control theory [14–18].

In the context of the latter application of fractional calculus, we present the current work, which deals with the controllability of a linear fractional differential equation with constant coefficients. The paper is organized as follows: In section 2, we recall the Kalman criterion for controllability of a linear system of first order. In section 3 we consider a linear system of fractional order, whose general solution is presented in terms of the Mittag-Leffler function. By using that representation we finally give in section 4 a new criterion for controllability.

Although this criterion is known since 1996 [19] we give another approach and use some elements of fractional calculus and a different proof to obtain the results. Also we reveal some interesting connections between linear differential equations of fractional order, control problems, linear algebra, Mittag-Leffler functions, geometry and physics.

For the relation between controllability of standard and fractional systems, see Klamka [20]. The calculation of the Gramian is useful to find a control to steer a given initial state to another prescribed final state.

2. CLASSICAL LINEAR CONTROL

Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $f:[0,\infty) \to \mathbb{R}$ be continuous. Consider the linear system

$$x'(t) = Ax(t) + f(t), \tag{1}$$

with the initial condition

$$x(0) = x_0 \in \mathbb{R}^n. \tag{2}$$

The solution of problem (1) and (2) is given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}f(s)ds.$$

Now consider the same system with a control function, so (1) is written like

$$x'(t) = Ax(t) + Bu(t), \tag{3}$$

where $B \in \mathcal{M}_{n\times m}(\mathbb{R})$ and $u:[0,\infty) \to \mathbb{R}^m$ is a possible control. For a given continuous control input u, the solution of (3) with initial condition $x(0) = x_0$ is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds.$$

For a given time $t > 0$ and an initial state $x(0) = x_0$, the reachable set of (1) at time $t > 0$ related to x_0 is the set $\mathcal{R}_t(x_0)$ of all states $x(t)$ that can be reached from x_0 by any control input. The linear system (3) is controllable if for any $x_0, x_1 \in \mathbb{R}^n$, there exists a control u such that the corresponding solution satisfies $x(0) = x_0$ and $x(t) = x_1$.

There is a simple criterion, the celebrated Kalman criterion for controllability.

Theorem 1. *The linear system (3) is controllable if and only if the Kalman matrix*

$$K = (B|AB|A^2B|\dots|A^{n-1}B)$$

has full rank.

To prove this (see [21]), one of the main ingredients is the controllability Gramian of matrices A and B:

$$W(t) := \int_0^t e^{As}BB^*e^{A^*s}ds \in \mathcal{M}_{n\times n}(\mathbb{R}), \tag{4}$$

where A^* and B^* are, respectively, the adjoint matrices of A and B. The matrix $W(t)$ is positive semi-definite and its range coincides with the range of the Kalman matrix.

Also a relevant ingredient is the following property for a matrix A. Let $\varphi(z) = \sum_{k=0}^{\infty} a_k z^k$ be an analytic complex function. An application of the Cayley-Hamilton Theorem implies that there exists a polynomial p of degree less than n such that $\varphi(A) = p(A)$, i.e.,

$$\varphi(A) = p(A) = \sum_{k=0}^{n-1} c_k A^k,$$

for certain $c_0, \dots, c_{n-1} \in \mathbb{C}$.

We recall a relevant geometric interpretation. The subspace

$$\mathcal{R} := \mathcal{R}(B|AB|\dots|A^{n-1}B)$$

is the smallest A-invariant subspace containing $\mathcal{R}(B)$. The linear system (3) is controllable if and only if $W(t)$ is non-singular for $t > 0$. A physical interpretation of the controllability Gramian is that the input of the system is white Gaussian noise. Then, $W(t)$ is the covariance of the state (see p. 854 in [22]).

3. LINEAR CONTROL OF FRACTIONAL ORDER

Consider now the linear differential equation of fractional order $\alpha \in (0, 1]$

$$D^\alpha x(t) = Ax(t) + f(t), \tag{5}$$

with initial condition

$$x(0) = x_0 \in \mathbb{R}^n, \tag{6}$$

where, as before, $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, $f \in \mathcal{C}([0,\infty), \mathbb{R}^n)$ and $D^\alpha x$ is the fractional derivative of x. We use here the Caputo fractional derivative, which can be defined for any $x:[0,\infty) \to \mathbb{R}^n$ absolutely continuous and has the following form:

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} x'(s)ds,$$

where Γ is the classical gamma function. For some applications of fractional differential equations we refer, for example, to references [5, 23, 24].

Note that $D^\alpha x(t) = I^{1-\alpha}x'(t)$, where $I^{1-\alpha}$ is the fractional integral of Riemann-Liouville. In fact, for $\beta > 0$ and $x \in L^1_{loc}(0,\infty)$,

$$I^\beta x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s)ds.$$

Now, let $B \in \mathcal{M}_{n\times m}(\mathbb{R})$ and $u \in \mathcal{C}([0,\infty), \mathbb{R}^m)$. Letting $f(t) = Bu(t)$, we rewrite the Equation (5) as

$$D^\alpha x(t) = Ax(t) + Bu(t). \tag{7}$$

Analogously to the ordinary case, for any $t > 0$, $\mathcal{R}_t^\alpha$ will be defined as the reachable set of (7) related to the origin, which is the set of all states $x(t)$ that can be reached from the initial state zero for some continuous control input. We say that the system (7) is controllable if for any $x_0, x_1 \in \mathbb{R}^n$ there exists a control u such that the solution of (7) with $x(0) = x_0$ satisfies $x(t) = x_1$.

The solution of the first order equation (1) is given in terms of f and the exponential of A:

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

For the fractional order equation (5) the role of this exponential is played by the Mittag-Leffler functions: for $\alpha > 0$ and for $z \in \mathbb{C}$,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Note that for $\alpha = 1$, $E_\alpha(z) = e^z$.

In general, for $\alpha, \beta > 0$, the function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{8}$$

is well-defined in $\mathbb{C}$, since the series in (8) is convergent for every $z \in \mathbb{C}$ [25]. For instance, if $\beta = 1$, we recover the previous case: $E_{\alpha,1}(z) = E_\alpha(z)$.

We can substitute z by a matrix A in (8) and the corresponding series converges. Hence, we can define the Mittag-Leffler function of a matrix A as

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}.$$

The solution of (5) is given by the variation of constants formula for fractional differential equations (see Theorem 5.15, p. 323 in [5] or Theorem 7.2, p. 135 in [23]):

$$x(t) = E_\alpha(t^\alpha A)c + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A) f(s) ds,$$

where c is any constant. Imposing the initial condition $x(0) = x_0$, then $c = x_0$.

In the case of (7) with the initial condition (6), the solution is [5, 17]

$$x(t) = E_\alpha(t^\alpha A)x_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A) Bu(s) ds.$$

At this point, we raise the following questions: In the fractional case, is there an analogous rule to the Kalman criterion? What is the Gramian matrix in such case?

4. PROOF OF THE FRACTIONAL CONTROL

We are ready to provide the reasoning that will lead us toward a controllability criterion for system (5).

Let $\alpha \in (0, 1)$. By applying the definition of the Mittag-Leffler function, the expression

$$\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A) Bu(s) ds$$

is equal to

$$\int_0^t (t-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{[(t-s)^\alpha A]^k}{\Gamma(\alpha k + \alpha)} Bu(s) ds.$$

Then, by using the uniform convergence, we arrive to the following expression

$$\sum_{k=0}^{\infty} \int_0^t (t-s)^{\alpha-1} \frac{(t-s)^{\alpha k} A^k}{\Gamma(\alpha k + \alpha)} Bu(s) ds,$$

which is obviously equal to

$$\lim_{N\to\infty} \sum_{k=0}^{N} A^k B \int_0^t (t-s)^{\alpha-1} \frac{(t-s)^{\alpha k}}{\Gamma(\alpha k + \alpha)} u(s) ds.$$

In the previous series, each term is a linear combination of the columns of $B, AB, A^2B, \ldots, A^N B$. Any of these matrices is a linear combination of $B, AB, A^2B, \ldots, A^{n-1}B$. Hence, the vector

$$\sum_{k=0}^{N} A^k B \int_0^t (t-s)^{\alpha-1} \frac{(t-s)^{\alpha k}}{\Gamma(\alpha k + \alpha)} u(s) ds \tag{9}$$

is a linear combination of the columns of $B, AB, A^2B, \ldots, A^{n-1}B$, i.e., it belongs to the range space of the Kalman matrix K. Therefore, as in the ordinary case, we get $\mathcal{R}_t^\alpha \subset \mathcal{R}(K)$. This is a necessary condition for controllability of the linear fractional system (7): The Kalman matrix has full rank. We cannot reach any state outside the range of the Kalman matrix.

The question is how can we get a control u so that

$$\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A) Bu(s) ds = x_1.$$

In order to do that, we define the α-Gramian as

$$\begin{aligned} W_t^\alpha &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A) BB^* E_{\alpha,\alpha}((t-s)^\alpha A^*)(t-s)^{\alpha-1} ds \\ &= \int_0^t (t-s)^{2\alpha-2} E_{\alpha,\alpha}((t-s)^\alpha A) BB^* E_{\alpha,\alpha}((t-s)^\alpha A^*) ds. \end{aligned}$$

Note that for $\alpha = 1$ we recover the Gramian in (4).

If we prove that $\mathcal{R}(K) \subset \mathcal{R}(W_t^\alpha)$, then, for $x_1 \in \mathcal{R}(K)$, we get $x_1 \in \mathcal{R}(W_t^\alpha)$ and there exists y such that $W_t^\alpha y = x_1$. By taking the control

$$u(s) = B^* E_{\alpha,\alpha}((t-s)^\alpha A^*)(t-s)^{\alpha-1} y,$$

we see that

$$\begin{aligned} x_1 &= W_t^\alpha y \\ &= \int_0^t (t-s)^{2\alpha-2} E_{\alpha,\alpha}((t-s)^\alpha A)BB^* E_{\alpha,\alpha}((t-s)^\alpha A^*) y ds \\ &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A) B u(s) ds. \end{aligned}$$

Thus, we steers the initial condition 0 to the state x_1 at time $t > 0$. This proves that $\mathcal{R}(K) \subset \mathcal{R}_t^\alpha$. It only remains proving that $\mathcal{R}(K) \subset \mathcal{R}(W_t^\alpha)$.

Let $z \in \mathbb{R}^n$ and suppose that $(W_t^\alpha)^* z \equiv z^* W_t^\alpha = 0$. For every $z \in \mathbb{R}^n$, this leads to

$$\begin{aligned} 0 = z^* W_t^\alpha z &= \langle z, W_t^\alpha z \rangle \\ &= \int_0^t (t-s)^{2\alpha-2} z^* E_{\alpha,\alpha}((t-s)^\alpha A)BB^* E_{\alpha,\alpha}((t-s)^\alpha A^*) z ds \\ &= \int_0^t (t-s)^{2\alpha-2} || z^* E_{\alpha,\alpha}((t-s)^\alpha A) B ||^2 ds \geq 0. \end{aligned}$$

For $s \in [0, t]$, $(t-s)^\alpha \in [0, t^\alpha]$. Therefore,

$$z^* \sum_{k=0}^{\infty} \frac{x^k A^k}{\Gamma(\alpha k + \alpha)} B = 0, \quad x \in [0, t^\alpha].$$

Differentiating k times ($k = 0, 1, 2, \ldots$) with respect to x and taking the limit when $x \to 0^+$ implies that $z^* A^k B = 0$, for $k = 0, \ldots, n-1$; i.e.,

$$z^* B = 0, \ldots, z^* A^{n-1} B = 0.$$

This gives us that

$$z^* \in \mathcal{N}(W_t^\alpha) = \mathcal{N}((W_t^\alpha)^*) \Rightarrow z^* \in \mathcal{N}(K^*).$$

We have that $\mathcal{N}((W_t^\alpha)^*) \subset \mathcal{N}(K^*)$ and we can write $\mathcal{N}(K^*)^\perp \subset \mathcal{N}((W_t^\alpha)^*)^\perp$. Therefore, we arrive to $\mathcal{R}(K) \subset \mathcal{R}(W_t^\alpha)$.

By gathering all the previous reasonings, we can finally state the following result.

Theorem 2. *The fractional system (7) is controllable if and only if the Kalman matrix K has full rank.*

As a direct implication, given $\alpha', \alpha'' \in (0, 1]$, there exists a link between the controllability of the system (7) for order α' and the one for order α":

Corollary 3. *If the fractional system (7) is controllable for a certain order $\hat{\alpha} \in (0, 1]$, then the system is controllable for every order $\alpha \in (0, 1]$.*

To conclude, we give several examples showing how Theorem 2 can be applied.

Example 1. Let $\alpha \in (0, 1)$. Consider the case $n = 2$, $m = 1$, with

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The system can be written as

$$\begin{cases} D^\alpha x_1(t) = x_1(t) + u(t), \\ D^\alpha x_2(t) = x_2(t). \end{cases} \tag{10}$$

The system is not controllable since the second equation is independent of the control as in the first order case ($\alpha = 1$). Nevertheless, it is possible to control x_1 and in that sense one may say that (10) is partially controllable. In this example,

$$K = \left(\begin{array}{c|c} 1 & 1 \\ 0 & 0 \end{array} \right)$$

and $rank(K) = 1 < 2$.

Example 2. Let $n = 2$, $m = 1$, $\alpha \in (0, 1]$ and consider the system

$$D^\alpha x(t) = Ax(t) + Bu(t),$$

where

$$A = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \in \mathcal{M}_{2\times1}(\mathbb{R}).$$

The Kalman matrix would be a 2×2 real matrix, whose columns are identified with B and AB. Moreover, if B is identified with an eigenvector of A, the system will not be controllable. For example, if

$$B = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

the Kalman matrix takes the form of

$$K = \left(\begin{array}{c|c} 1 & 2 \\ 2 & 4 \end{array} \right),$$

which has not full rank [$rank(K) = 1 < 2$]. The system would not be therefore controllable.

Something similar happens with the choice

$$B = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Nonetheless, any other choice of B which is not a multiple of one of the previous cases, leads to a controllable system regardless the value of $\alpha \in (0, 1]$.

Example 3. The classical linear harmonic oscillator ξ" $+\xi = u$ is equivalent to the system (3) taking the position $x_1 = \xi$ and the velocity $x_2 = \xi'$ with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A fractional control harmonic oscillator would be (7), which takes the form

$$\begin{cases} D^\alpha x_1 = x_2, \\ D^\alpha x_2 = u - x_1. \end{cases} \tag{11}$$

The first equation is independent of the control, but it appears in the second equation, involving both components and the fractional control system (11) is controllable. Indeed, the Kalman matrix

$$K = \left(\begin{array}{c|c} 0 & 1 \\ 1 & 0 \end{array}\right)$$

has full rank.

Example 4. Another possibility is to consider a coupled system of linear incommesurate fractional differential control system $\alpha_1, \alpha_2 \in (0, 1)$:

$$\begin{cases} D^{\alpha_1} x_1 = a_{11}x_1 + a_{12}x_2 + u_1, \\ D^{\alpha_2} x_2 = a_{21}x_1 + a_{22}x_2 + u_2, \end{cases} \tag{12}$$

but, to the best of our knowledge, no analytical solution is known.

5. CONCLUSIONS

In this work, we have studied the controllability of the linear fractional differential equation

$$D^\alpha x(t) = Ax(t) + Bu(t),$$

where the Caputo fractional derivative is considered, $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, $B \in \mathcal{M}_{n\times m}(\mathbb{R})$ and u is a m-dimensional control function. In particular, we have shown that such a system is controllable if and only if the Kalman matrix has full rank, which constitutes the main result, namely Theorem 2.

Although the criterion given in Theorem 2 does not depend on α and thus it becomes an analogous result to the classic one (ordinary case), the tools that we have used actually involve some adaptated reasonings. There are still several relations between the controllability of the system, the corresponding Gramian matrix W_t^α, the kernel of the associated operator, the range space $\mathcal{R}_t^\alpha$ and the Kalman matrix, but some arguments depend on the fractional order α. For instance, we recall that the Gramian matrix W_t^α has a singularity if $\alpha \in (0, 1)$ and the control steering the initial data x_0 to a final state x_1 depends on α, so as the coefficients of the linear combination of the matrices $B, \ldots, A^{n-1}B$ (which form the Kalman matrix) do in Equation (9).

In the future, some research deserves to be done with respect to further questions related to this work. For example, a couple of crucial problems are the cases where the matrices A and B are not constant, that is, the control system

$$D^\alpha x(t) = A(t)x(t) + B(t)u(t),$$

and the non-linear case

$$D^\alpha x(t) = f(x(t), u(t)),$$

which is also very relevant in applications and will be considered in detail. In general, in many situations, delay may also appear and functional fractional differential equations of the type

$$D^\alpha x(t) = f(x(t), x(t-\tau), u(t))$$

have to be considered.

In addition to the former comments, systems with impulses due to impacts are of interest too. Indeed, in Spong [26] and Nieto and Tisdell [27], the problem of controlling a physical object through impacts, called impulsive manipulation, is studied and it arises in a number of robotic applications [28, 29].

Another interesting line is to address the controllability of fractional order systems in the light of other fractional derivatives, such as Riemann-Liouville, Hadamard, Caputo-Fabrizio, etc.

Furthermore, some physical models will be considered under those fractional calculus approaches and the relations among them will be scrutinized.

Moreover, the incommesurate fractional system of Example 4 will also be a relevant problem to consider.

Finally, partial differential equations of fractional order could be treated both from the mathematical point of view and from the physical point of view too.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

SB-F and JN have contributed equally and significantly to the contents of this paper. All authors contributed to the article and approved the submitted version.

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