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Exact Values for Some Size Ramsey Numbers of Paths and Cycles

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For the graphs G_1, G_2, and G, if every 2-coloring (*red* and *blue*) of the edges of G results in either a copy of *blue* G_1 or a copy of *red* G_2, we write $G \to (G_1, G_2)$. The size Ramsey number $\hat{R}(G_1, G_2)$ is the smallest number e such that there is a graph G with size e satisfying $G \to (G_1, G_2)$, i.e., $\hat{R}(G_1, G_2) = \min\{|E(G)| : G \to (G_1, G_2)\}$. In this paper, by developing the procedure and algorithm, we determine exact values of the size Ramsey numbers of some paths and cycles. More precisely, we obtain that $\hat{R}(C_4, C_5) = 19$, $\hat{R}(C_6, C_6) = 26$, $\hat{R}(P_4, C_5) = 14$, $\hat{R}(P_4, P_5) = 10$, $\hat{R}(P_4, P_6) = 14$, $\hat{R}(P_5, P_5) = 11$, $\hat{R}(P_3, P_5) = 7$ and $\hat{R}(P_3, P_6) = 8$.

Keywords: size Ramsey number, 2-coloring, connected graphs, connectivity, paths, cycles

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1. INTRODUCTION

We use standard notions and symbols from the field of graph theory, see [1]. By $G = G(V, E)$, we denote a simple graph with vertex and edge sets V and E having cardinalities $|V(G)|$ and $|E(G)|$, respectively. For $S_1, S_2 \subseteq V(G)$, we denote $E(S_1) = \{uv \in E(G)|v, u \in S_1\}$ and $E(S_1, S_2) = \{uv \in E(G)|u \in S_1, v \in S_2\}$. Moreover, we denote: the degree of a vertex v in G by $d(v|G)$ (or $d(v)$), the minimum degree among the vertices of G by $\delta(G)$, a path and a cycle having i vertices by P_i and C_i, respectively. For the graphs G_1, G_2, and G, if every 2-coloring (*red* and *blue*) of the edges of G results in either a copy of *blue* G_1 or a copy of *red* G_2, we call it Ramsey property of G and write $G \to (G_1, G_2)$. The size Ramsey number $\hat{R}(G_1, G_2)$ is the smallest number e such that there is a graph G with size e satisfying $G \to (G_1, G_2)$, i.e., $\hat{R}(G_1, G_2) = \min\{|E(G)| : G \to (G_1, G_2)\}$. For $k \in \mathbb{N}$, a non-complete graph G is called k-connected if $|V(G)| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. The greatest integer k such that G is k-connected is the connectivity $\kappa(G)$ of G. For the complete graph K_n, we define $\kappa(K_n) = n - 1$.

In 1978, Erdös et al. initiated the study of the size Ramsey number, and later it was continued by Faudree [2, 3], Lortz and Mengersen [4], and Pikhurko [5]. From these studies, we can see that the size Ramsey number $\hat{R}(G_1, G_2)$ exists for the graphs G_1 and G_2. Su and Shao applied a backtracking algorithm to find some upper bounds for the size Ramsey numbers. The study of the size Ramsey numbers based on the graph coloring is implicitly connected to several branches of science, such as: the energies of the status level "fully functional nodes," "partially functional nodes," and "non-functional nodes" can be interpreted by the way of graph coloring [6], frequency channel assignment [7, 8], time tabling [9], and CAD problems [10, 11]. For more literature regarding the Ramsey numbers, we refer [12–16] to the readers. This paper is devoted to study the properties of the graphs G with the smallest size for which $G \to (G_1, G_2)$ for given graphs G_1 and G_2. Moreover, by developing the procedure and algorithm, we determined size Ramsey numbers of some paths and cycles.

2. THE APPROACH

Lemma 1. *Let G be a graph with the smallest size for which $G \to (G_1, G_2)$. Then any G', obtained by removing all the isolated vertices of G, is connected.*

Proof: By the definition of G', we have $G' \to (G_1, G_2)$. Suppose to the contrary that there are at least two components H_1, H_2 in G'. Let $G' = H_1 \cup H_2 \cup \ldots \cup H_n$ with $n \geq 2$. Since H_i is not an isolated vertex for any i, we have $|E(H_i)| < |E(G')|$ for any i. Then there is a 2-coloring (red and blue) f_i of the edges of H_i such that H_i contains neither red G_1 nor blue G_2. Now, consider a 2-edge coloring f of the edges of G' with $f(e) = f_i(e)$ for any $e \in H_i$ for $i = 1, 2, \cdots, n$. Then G contains neither red G_1 nor blue G_2 under f, and so $G' \nrightarrow (G_1, G_2)$, a contradiction.

Remark 1: Given the graphs G, G_1, G_2 with $G \to (G_1, G_2)$, by the Lemma 1, we only need to consider the connected graphs for G.

Lemma 2. *If G is a graph with the smallest size for which $G \to (G_1, G_2)$, and G is a connected graph, then $\kappa(G) \geq \min\{\kappa(G_1), \kappa(G_2)\}$.*

Proof: Assume on contrary that we have $\kappa(G) < \min\{\kappa(G_1), \kappa(G_2)\}$. Let $S \subseteq V(G)$ such that $|S| = \kappa(G)$ and $G - S$ is disconnected and assume $G - S = H_1 \cup H_2 \cup \ldots \cup H_n$ with $n \geq 2$. Let $V(T_i) = V(H_i) \cup S$ and $E(T_i) = E(H_i) \cup E(H_i, S)$. Since G is a graph with the smallest size for which $G \to (G_1, G_2)$, there is a red-blue coloring f_i of the edges of T_i such that T_i contains neither red G_1 nor blue G_2 for any i. Let $E(S) = \{e_1, e_2, \cdots, e_k\}$ for some k. Now consider a 2-edge coloring f of the edges of G with $f(e) = f_i(e)$ for any $e \in H_i$ for $i = 1, 2, \cdots, n$, $f(e_1) = red$, $f(e_i) = blue$ for any $i = 2, 3, \cdots, k$. Then G contains neither red G_1 nor blue G_2 under f, and so $G' \nrightarrow (G_1, G_2)$. Now, we consider the following two cases:

Case 1: If there is a *red* copy of G_1 as a subgraph of G.

Subcase 1.1: $E(G_1) \subseteq E(T_i) \cup E(S)$ with $i \in \{1, \ldots, n\}$.

Since f_i is a *red-blue* coloring of the edges of T_i such that T_i contains no red G_1. Then $E(G_1) \cap E(S) \neq \emptyset$. Since $G_1[E(G_1) \cap E(S)]$ is not a clique with $|S|$ vertices, there is a cut-set S_1 of G_1 with $S_1 \subseteq S$. Then $|S_1| \leq |S| < \kappa(G_1)$ by the assumption, a contraction.

Subcase 1.2: $E(G_1) \cap E(H_i) \neq \emptyset, E(G_1) \cap E(H_j) \neq \emptyset$ with $i \neq j$.

Then S is a cut-set of G_1 with $|S| < \kappa(G_1)$ by the assumption, a contraction.

Case 2: If there is a *blue* copy of G_2 as a subgraph of G.

Subcase 2.1: $E(G_2) \subseteq E(T_i) \cup E(S)$ with $i \in \{1, \ldots, n\}$.

Since f_i is a red-blue coloring of the edges of T_i such that T_i contains no blue G_2. Then $E(G_2) \cap E(S) \neq \emptyset$. Since $G_2[E(G_2) \cap E(S)]$ is not a clique with $|S|$ vertices, there is a cut-set S_2 of G_2 with $S_2 \subseteq S$. Then $|S_2| \leq |S| < \kappa(G_2)$ by the assumption, a contraction.

Subcase 2.2: $E(G_2) \cap E(H_i) \neq \emptyset, E(G_2) \cap E(H_j) \neq \emptyset$ with $i \neq j$.

Then S is a cut-set of G_2 with $|S| < \kappa(G_2)$ by the assumption, a contraction.

Lemma 3. *For the graphs G, G_1 and G_2, if there exist vertices $v_1, \ldots, v_t$ for some $1 \leq t \leq |V(G)|$ satisfying that $d(v_i|G^{i-1}) < \delta(G_1) + \delta(G_2) - 1$ for any $i = 1, 2, \cdots, t$ and $G_t \nrightarrow (G_1, G_2)$, where $G^i = G - \{v_1, \ldots, v_i\}$ and $G^0 = G$. Then $G \nrightarrow (G_1, G_2)$.*

Proof: We apply induction on t to prove it. Firstly, it is clear that the lemma holds if $t = 1$. Now, we suppose the stated result holds for $t = i$, we need to prove it for $t = i + 1$. Since the lemma holds if $t = i$, we have $G^1 \nrightarrow (G_1, G_2)$. Then there is a *red-blue* coloring g of the edges of G^1 such that there is neither a *red* copy of G_1 nor a *blue* copy of G_2 in G^1. Let $E(w) = \{uv \in E(G) | u = w \text{ or } v = w\}$. Since $d(v_1|G^0) < \delta(G_1) + \delta(G_2) - 1$, we can divide $E(v_1)$ into E_1, E_2 with $|E_1| < \delta(G_1), |E_2| < \delta(G_2)$. Let f be a coloring of G obtained by assigning *red* to E_1, *blue* to E_1 based on g.

Case 1: If there is a *red* copy of G_1 as a subgraph of G under f, then $v_1 \in V(G_1)$. Since $|E_1| < \delta(G_1)$, then $d(v_1|G_1) < \delta(G_1)$, a contraction.

Case 2: If there is a *blue* copy of G_2 as a subgraph of G under f, then $v_1 \in V(G_2)$. Since $|E_2| < \delta(G_2)$, then $d(v_1|G_2) < \delta(G_2)$, a contraction.

There is neither a *red* copy of G_1 nor a *blue* copy of G_2 in G under f. Therefore, $G \nrightarrow (G_1, G_2)$.

The contrapositive of the Lemma 3 for $t = 1$ produces the following corollary:

Corollary 1. *For any graphs G_1 and G_2, if G is any graph with the smallest size for which $G \to (G_1, G_2)$, then $\delta(G) \geq \delta(G_1) + \delta(G_2) - 1$.*

Lemma 4. *For any graphs G_1 and G_2, if G is any graph with order n and size m such that $G \not\to (G_1, G_2)$, then for any graph G' with order at most n and size $m - 1 < \frac{n(n-1)}{2}$, we have $G' \not\to (G_1, G_2)$.*

Proof: First, we have G' is not a complete graph, then there are two vertices u, v with $uv \notin E(G')$. Now, we insert the edge uv to obtain a graph G'' based on G'. Then G'' is a graph with m edges and n vertices and so $G'' \not\to (G_1, G_2)$. Therefore, there is a $red - blue$ coloring f of G'' such that there is neither a *red* copy of G_1 nor a *blue* copy of G_2 in G'' under f. Then, there is also neither a *red* copy of G_1 nor a *blue* copy of G_2 in G' under $f|_{G'}$. Then $G' \not\to (G_1, G_2)$.

By applying the Lemma 1 and the Corollary 1, we only need to consider the connected graphs, and then propose the following algorithm (*FindSizeRamseynumber*) to find the size Ramsey number of G_1 and G_2. We will use the software *nauty* [17] to generate non-isomorphic graphs with necessary properties. If G_1 and G_2 are k-connected graphs, we further apply the Lemma 2 to reduces the number of graphs needed to be processed. For testing if $G \to (G_1, G_2)$, we applying the backtracking procedure proposed in [18].

Procedure Find(m,n,G_1,G_2);
input: m, n be integers;
graphs G_1 and G_2.

TABLE 1 | Exact values $\hat{R}(G_1, G_2)$ of the size Ramsey numbers of some paths and cycles.

G_1	G_2	(n, m)	$\#A(n, m)$	$\#B(n, m)$	result
C_4	C_5	(7,19)	2	1	$\hat{R}(C_4, C_5) = 19$
C_6	C_6	(8,26)	2	1	$\hat{R}(C_6, C_6) = 26$
P_4	C_5	(7,14)	59	1	$\hat{R}(P_4, C_5) = 14$
P_4	P_5	(6,10)	14	4	$\hat{R}(P_4, P_5) = 10$
P_4	P_6	(7,14)	64	30	$\hat{R}(P_4, P_6) = 14$
P_5	P_5	(6,11)	9	3	$\hat{R}(P_5, P_5) = 11$
P_3	P_5	(5,7)	4	2	$\hat{R}(P_3, P_5) = 7$
P_3	P_6	(6,8)	22	1	$\hat{R}(P_3, P_6) = 8$

begin
generate the family $\mathcal{G}$ of all the non-isomorphic connected graphs with size m and
order n with minimum degree $\delta(G_1) + \delta(G_2) - 1$; (Apply Lemma 1 and Corollary 1);
foreach G in $\mathcal{G}$
if $(G \to (G_1, G_2))$
return true;
end if
end for
return true;
end.
Algorithm FindSizeRamseynumber(G_1,G_2);

input: graphs G_1 and G_2.
begin
1 : Find a graph G such $G \to (G_1, G_2)$;
2 : $m = |E(G)| - 1$;
3 : $n = \min\{\lfloor \frac{2m}{\delta(G_1)+\delta(G_2)-1} \rfloor, m+1\}$;
4 : **while Find**(m, n, G_1, G_2) **do**;
5 : $n = n - 1$;
6 : **if** $m > \frac{n(n-1)}{2}$ **do**
7 : $m = m - 1$;
8 : $n = \min\{\lfloor \frac{2m}{\delta(G_1)+\delta(G_2)-1} \rfloor, m+1\}$;
9 : **end if**
10: **end while**
11: **return** $m + 1$.
end.

3. RESULTS

Example 1. $\hat{R}(C_4, C_5) = 19$.

Proof: Consider $G_1 = C_4$, $G_2 = C_5$. By Algorithm *FindSizeRamseynumber*, we first find the graph H satisfying $H \to (C_4, C_5)$ (line 1). Therefore, $\hat{R}(C_4, C_5) \leq 19$. Then, we consider the edge number less than 19 (i.e., $m \leq 18$, by line 2), and the order of graph at most $\min\{\lfloor \frac{2m}{\delta(G_1)+\delta(G_2)-1} \rfloor, m+1\} \leq 12$. Now, the procedure will check if there is no graph G with minimum degree 3, size at most and order from 7 to 12 satisfying $G \to (C_4, C_5)$ (line 3-10). In this case, by applying Procedure *Find*, we find that there is no such graph. Therefore, $\hat{R}(C_4, C_5) \geq 19$.

By applying Algorithm *FindSizeRamseynumber*, we obtain many size Ramsey numbers presented in **Table 1**, where $\#A(n, m)$ denote the number of non-isomorphic connected graphs with minimum degree $\delta(G_1) + \delta(G_2) - 1$ with size m and order n, and $\#B(n, m)$ denote the number of such graphs G with $G \to (G_1, G_2)$. An application of the algorithm can be used in some other graph problems, see [19].

4. CONCLUSION

It is a very hard task to determine the size Ramsey number even for small graphs. Faudree and Sheehan gave a table of the size Ramsey numbers for graphs with order not more than four [3]. Su and Shao [18] provide upper bounds for the size Ramsey numbers of some paths and cycles. Until now, very limited results on the size Ramsey numbers are known. In this paper, we have developed some computational techniques to determine many of those size Ramsey numbers. There are numerous variants of the Ramsey numbers such as ordered Ramsey numbers, size Ramsey numbers and zero-sum Ramsey numbers, see [20]. It is also very difficult to compute each variant of these Ramsey numbers. In order to compute some possible Ramsey numbers, we need to obtain the structure of the graphs by studying their mathematical properties. So, the approach of this paper may be considered to compute some challenging Ramsey numbers.

AUTHOR CONTRIBUTIONS

All authors contributed equally in completing the current work.

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