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New Solitary Wave Solutions for Variants of (3+1)-Dimensional Wazwaz-Benjamin-Bona-Mahony Equations

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We solve distinct forms of (3+1)-Dimensional Wazwaz-Benjamin-Bona-Mahony [(3+1)-Dimensional WBBM] equations by employing the method of Sardar-subequation. When parameters involving this approach are taken to be special values, we can obtain the solitary wave solutions (sws) which is concluded from other approaches such as the functional variable method, the trail equation method, the first integral method and so on. We obtain new and general solitary wave solutions in terms of generalized hyperbolic and trigonometric functions. The results demonstrate the power of the proposed method for the determination of sws of non-linear evolution equations (NLEs).

Keywords: Sardar-subequation method, (3+1)-dimensional WBBM equations, solitary wave solutions, NLEs, simulations

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INTRODUCTION

Exact solutions and solitary wave solutions of NLEs have a high importance in non-linearity theory. It is a well-known fact that numerous complex events in various fields of engineering application and non-linear science such as chemistry, mathematical physics, mechanics, hydrodynamics, biology, cosmology, and mechanics are explained by NLEs. Exact solutions produce corporal information to describe the physical behavior of system connected with these NLEs. In recent years, several efficient methods including method of extended tanh [1, 2], tanh-coth [3, 4], Hirota's direct [5, 6], sine-cosine [7, 8], extended direct algebraic [9, 10], extended trial approach [11, 12], Exp [-φ(ξ)]-Expansion [13, 14], a new auxiliary equation [15, 16], Jacobi elliptic ansatz [17, 18], generalized Bernoulli sub-ODE [19, 20], functional variable [21, 22], sub equation [23, 24], and so on [25–39], have been established for efficient solutions of NLEs.

In this paper, we utilize of an effective and efficient technique for manufacturing a range of solitary wave solutions for the variants of the (3+1)-dimensional WBBM equations.

The Benjamin-Bona-Mahony (BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1}$$

proposed in [40] as a model to study the approximately the unidirectional propagation of small-amplitude long waves on certain non-linear dispersive systems as a good alternative to the KdV [41]. It is used in modeling surface waves of long wavelength in liquids; it covers hydro-magnetic waves in cold plasma, acoustic waves in anharmonic crystals, and acoustic gravity waves in compressible fluids. Many efforts have been offered to modified forms of this equation such as

$$u_t + u_x + u^2 u_x - u_{xxt} = 0, \tag{2}$$

known as the modified Benjamin-Bona-Mahoney (mBBM) equation [42]. Furthermore, as higher dimensional samples convert to be more realistic; various modifications to Equation (2) have been proposed in the literature among which the (3+1)-dimensional mBBM equation given in Equations (3–5) by Wazwaz [43]. Using the tanh/sech method, Wazwaz obtained soliton, kink, and periodic solutions for the following three different types of Equation (2)

$$u_t + u_x + u^2 u_y - u_{xzt} = 0, \tag{3}$$

$$u_t + u_z + u^2 u_x - u_{xyt} = 0, \tag{4}$$

and

$$u_t + u_y + u^2 u_z - u_{xxt} = 0. \tag{5}$$

Based on these ideas, this paper is organized as follows. Section The Sardar-Subequation Method introduces a brief description of the Sardar-sub-equation method. Section The (3+1)-Dimensional WBBM Equation discusses the application of the Sardar-subequation method to variants of (3+1)-dimensional WBBM equations represent by (3)–(5). The graphical presentation for the acquired solution is given in section Exact Solutions of the (3+1)-Dimensional WBBM Equation. We complete the paper with conclusions part.

THE SARDAR-SUBEQUATION METHOD

We consider the following NLEs

$$G(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{6}$$

where $u = u(x, t)$ is an unknown function and G is a polynomial in u and its partial derivatives.

To solve (Equation 6), we take the traveling wave transformation (twt)

$$u(x, t) = U(\eta), \qquad \eta = x - ct, \tag{7}$$

where $c \neq 0$ is constants to be determined later.

By using (7), Equation (6) is turned into following ODE w.r.t. η

$$P(U, U', U'', U''', \ldots) = 0, \tag{8}$$

in which $U = U(\eta)$, $U' = \frac{dU}{d\eta}$, $U'' = \frac{d^2U}{d\eta^2}, \ldots$.

Suppose that the Equation (8) has a solution of the form

$$U(\eta) = \sum_{i=0}^{s} \varpi_i \varphi^i(\eta), \tag{9}$$

where $\varpi_i, (i = 0, 1, \ldots, s)$ are coefficients to be determined with $(\varpi_s \neq 0)$ and $\varphi(\eta)$ satisfies the ODE in the form

$$\left(\varphi'(\eta)\right)^2 = \rho + a\varphi^2(\eta) + \varphi^4(\eta), \tag{10}$$

where a and ρ are real constants. The solutions of ODE (10) are

Case I: If $a > 0$ and $\rho = 0$ then

$$\varphi_1^{\pm}(\eta) = \pm\sqrt{-pqa}\,\mathrm{sech}_{pq}\left(\sqrt{a}\eta\right),$$
$$\varphi_2^{\pm}(\eta) = \pm\sqrt{pqa}\,\mathrm{csch}_{pq}\left(\sqrt{a}\eta\right),$$

where

$$\mathrm{sech}_{pq}(\eta) = \frac{2}{pe^{\eta} + qe^{-\eta}}, \quad \mathrm{csch}_{pq}(\eta) = \frac{2}{pe^{\eta} - qe^{-\eta}}.$$

Case II: If $a < 0$ and $\rho = 0$ then

$$\varphi_3^{\pm}(\eta) = \pm\sqrt{-pqa}\,\sec_{pq}\left(\sqrt{-a}\eta\right),$$
$$\varphi_4^{\pm}(\eta) = \pm\sqrt{-pqa}\,\csc_{pq}\left(\sqrt{-a}\eta\right),$$

where

$$\sec_{pq}(\eta) = \frac{2}{pe^{i\eta} + qe^{-i\eta}}, \quad \csc_{pq}(\eta) = \frac{2i}{pe^{i\eta} - qe^{-i\eta}}.$$

Case III: If $a < 0$ and $\rho = \frac{a^2}{4b}$ then

$$\varphi_5^{\pm}(\eta) = \pm\sqrt{-\frac{a}{2}}\tanh_{pq}\left(\sqrt{-\frac{a}{2}}\eta\right),$$
$$\varphi_6^{\pm}(\eta) = \pm\sqrt{-\frac{a}{2}}\coth_{pq}\left(\sqrt{-\frac{a}{2}}\eta\right),$$
$$\varphi_7^{\pm}(\eta) = \pm\sqrt{-\frac{a}{2}}\left(\tanh_{pq}\left(\sqrt{-2a}\eta\right) \pm i\sqrt{pq}\,\mathrm{sech}_{pq}\left(\sqrt{-2a}\eta\right)\right),$$
$$\varphi_8^{\pm}(\eta) = \pm\sqrt{-\frac{a}{2}}\left(\coth_{pq}\left(\sqrt{-2a}\eta\right) \pm \sqrt{pq}\,\mathrm{csch}_{pq}\left(\sqrt{-2a}\eta\right)\right),$$
$$\varphi_9^{\pm}(\eta) = \pm\sqrt{-\frac{a}{8}}\left(\tanh_{pq}\left(\sqrt{-\frac{a}{8}}\eta\right) + \coth_{pq}\left(\sqrt{-\frac{a}{8}}\eta\right)\right),$$

where

$$\tanh_{pq}(\eta) = \frac{pe^{\eta} - qe^{-\eta}}{pe^{\eta} + qe^{-\eta}}, \quad \coth_{pq}(\eta) = \frac{pe^{\eta} + qe^{-\eta}}{pe^{\eta} - qe^{-\eta}}.$$

Case IV: If $a > 0$ and $\rho = \frac{a^2}{4}$ then

$$\varphi_{10}^{\pm}(\eta) = \pm\sqrt{\frac{a}{2}}\tan_{pq}\left(\sqrt{\frac{a}{2}}\eta\right),$$
$$\varphi_{11}^{\pm}(\eta) = \pm\sqrt{\frac{a}{2}}\cot_{pq}\left(\sqrt{\frac{a}{2}}\eta\right),$$
$$\varphi_{12}^{\pm}(\eta) = \pm\sqrt{\frac{a}{2}}\left(\tan_{pq}\left(\sqrt{2a}\eta\right) \pm \sqrt{pq}\,\sec_{pq}\left(\sqrt{2a}\eta\right)\right),$$
$$\varphi_{13}^{\pm}(\eta) = \pm\sqrt{\frac{a}{2}}\left(\cot_{pq}\left(\sqrt{2a}\eta\right) \pm \sqrt{pq}\,\csc_{pq}\left(\sqrt{2a}\eta\right)\right),$$

$$\varphi_{14}^{\pm}(\eta) = \pm\sqrt{\frac{a}{8}}\left(\tan_{pq}\left(\sqrt{\frac{a}{8}}\eta\right) + \cot_{pq}\left(\sqrt{\frac{a}{8}}\eta\right)\right),$$

where

$$\tan_{pq}(\eta) = -i\frac{pe^{i\eta} - qe^{-i\eta}}{pe^{i\eta} + qe^{-i\eta}},\ \cot_{pq}(\eta) = i\frac{pe^{i\eta} + qe^{-i\eta}}{pe^{i\eta} - qe^{-i\eta}},$$

The procedure starts by determining s by the assistance of the classical balance rule. When s is determined the predicted solution (8) is substituted into (Equation 7). Since we seek a non-zero solution that is $\varphi(\eta) \neq 0$, all the coefficient of power of $\varphi(\eta)$ are equated to zero. Thus, the resultant algebraic system is solved for ϖ_is and c. One should note that $\varpi_s \neq 0$ and $c \neq 0$ both have to be satisfied in the solution sets. Whenever ϖ_is and c are determined the solutions are constructed by using these parameters explicitly. The procedure ends by substitution of $\eta = x - ct$ into the solutions satisfying (Equation 7).

Remark 1. The Sardar-subequation method is considered among those general ones from which, under certain cases, various methods can be deduced such as the functional variable method [44], the first integral method [45], and so on.

THE (3+1)-DIMENSIONAL WBBM EQUATION

To solve the (3+1)-Dimensional (3+1)-Dimensional WBBM equations (3), (4), and (5) by using the Sardar-sub-equation method, we make the following subsection.

The First (3+1)-Dimensional WBBM Equation

We next study the first (3+1)-dimensional WBBM equation (3). The twt $u(x,t) = U(\eta), \eta = kx + \lambda y + \mu z - ct$, reduces (Equation 3) to the following ordinary differential equation

$$(k - c)U' + \lambda U^2 U' + k\,\mu c U''' = 0. \tag{11}$$

Integrating (Equation 11), we get

$$(k - c)U + \frac{\lambda}{3}U^3 + k\,\mu c U'' = 0. \tag{12}$$

Then, Equation (12) can be written as.

$$\Omega_1 U + \Omega_2 U^3 + U'' = 0, \tag{13}$$

where

$$\Omega_1 = \frac{k - c}{k\,\mu c}, \quad \Omega_2 = \frac{\lambda}{3k\,\mu c}. \tag{14}$$

The Second (3+1)-Dimensional WBBM Equation

We next study the second (3+1)-dimensional WBBM equation (4). The twt $u(x,t) = U(\eta), \eta = kx + \lambda y + \mu z - ct$, reduces (Equation 4) to the following ordinary differential equation

$$(\mu - c)U' + kU^2 U' + k\,\lambda c U''' = 0. \tag{15}$$

Integrating (Equation 15), we get

$$(\mu - c)U + \frac{k}{3}U^3 + k\,\lambda c U'' = 0. \tag{16}$$

Then, Equation (16) can be written as.

$$\Omega_1 U + \Omega_2 U^3 + U'' = 0, \tag{17}$$

where.

$$\Omega_1 = \frac{\mu - c}{k\,\lambda c}, \quad \Omega_2 = \frac{1}{3\lambda c}. \tag{18}$$

The Third (3+1)-Dimensional WBBM Equation

We next study the third (3+1)-dimensional WBBM equation (5). By making the traveling wave transformation

$$u(x,y,z,t) = U(\eta), \qquad \eta = kx + \lambda y + \mu z - ct, \tag{19}$$

(Equation 5) becomes

$$(\lambda - c)U' + \mu U^2 U' + k^2 c U''' = 0. \tag{20}$$

Integrating (Equation 20) twice and setting the integration constants to zero yield

$$(\lambda - c)U + \frac{\mu}{3}U^3 + k^2 c U'' = 0. \tag{21}$$

Then, Equation (21) can be written as.

$$\Omega_1 U + \Omega_2 U^3 + U'' = 0, \tag{22}$$

where

$$\Omega_1 = \frac{\lambda - c}{k^2 c}, \quad \Omega_2 = \frac{\mu}{3k^2 c}. \tag{23}$$

EXACT SOLUTIONS OF THE (3+1)-DIMENSIONAL WBBM EQUATION

Balancing U with U'' in Equations (13, 17, 22), we get $s = 1$. Therefore, the solutions form of Equations (13, 17, 22) has the following expression

$$U(\eta) = \varpi_0 + \varpi_1\,\varphi(\eta), \tag{24}$$

Substituting Equation (24) into (13) [or (17) or (22)] with along (Equation 10) and equating all the coefficients of $\varphi(\eta)$ to zero, we obtain a highly complicated system of algebraic equations. A set of algebraic equations is obtained in ϖ_0, ϖ_1, and a as follows:

$$\begin{aligned}
\varpi^0 &: \quad \lambda_2\Omega_0^3 + \lambda_1\Omega_0 = 0,\\
\varpi^1 &: \quad 3\lambda_2\Omega_0^2\Omega_1 + \lambda_1\Omega_1 + \Omega_1 a = 0,\\
\varpi^1 &: \quad 3\lambda_2\Omega_0\Omega_1^2 = 0,\\
\varpi^1 &: \quad 2\Omega_1 b + \lambda_2\Omega_1^3 = 0,
\end{aligned}$$

Solving these algebraic equations with Maple, we acquire:

$$\varpi_0 = 0, \ \ \varpi_1 = \pm\sqrt{-\frac{2}{\Omega}}, \ \ a = -\Omega_1. \tag{25}$$

The First (3+1)-Dimensional WBBM Equation

Substituting (Equation 14) into (Equation 25), we have

$$\varpi_0 = 0, \ \ \varpi_1 = \pm\sqrt{-\frac{6k\,\mu c}{\Omega}}, \ \ a = \frac{c-k}{k\,\mu c}. \tag{26}$$

The solutions of (3) corresponding to (26), along with solution (10) are

Case I: If $\frac{c-k}{k\,\mu c} > 0$ and $\rho = 0$ then

$$\begin{aligned}
u_{1,1}^{\pm} &= \pm\sqrt{\frac{6pq(c-k)}{\lambda}}\,\mathrm{sech}_{pq}\left(\sqrt{\frac{c-k}{k\,\mu c}}(kx+\lambda y+\mu z-ct)\right),\\
u_{1,2}^{\pm} &= \pm\sqrt{-\frac{6pq(c-k)}{\lambda}}\,\mathrm{csch}_{pq}\left(\sqrt{\frac{c-k}{k\,\mu c}}(kx+\lambda y+\mu z-ct)\right).
\end{aligned}$$

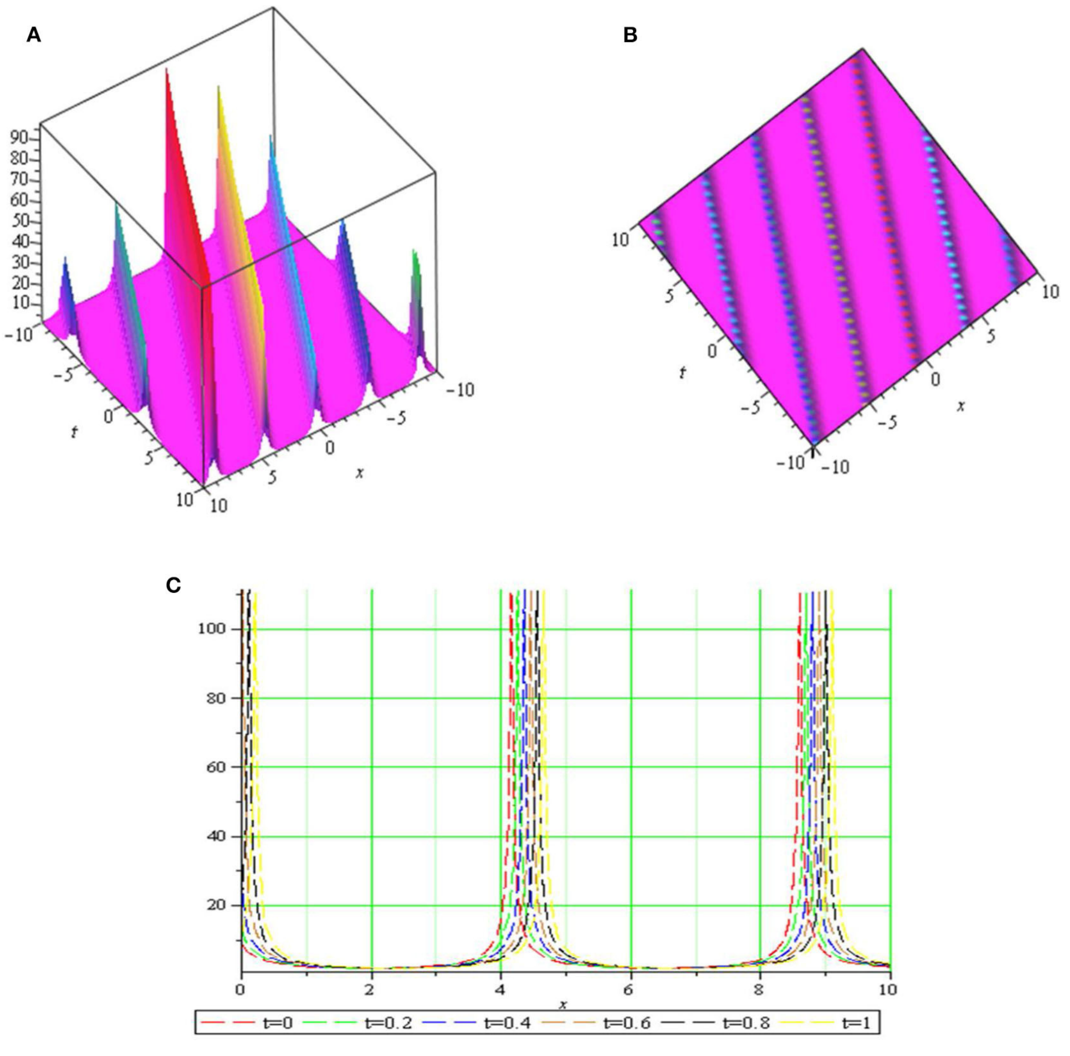

FIGURE 1 | The sws $|u_{1,3}^{+}|$ with parameters $\mu = 1, k = 1, c = 0.5, \lambda = 1, p = 0.98, q = 0.95, y = 0.5, z = 1$ and **(A)** 3D plot, **(B)** contour plot (cp), and **(C)** 2D plot.

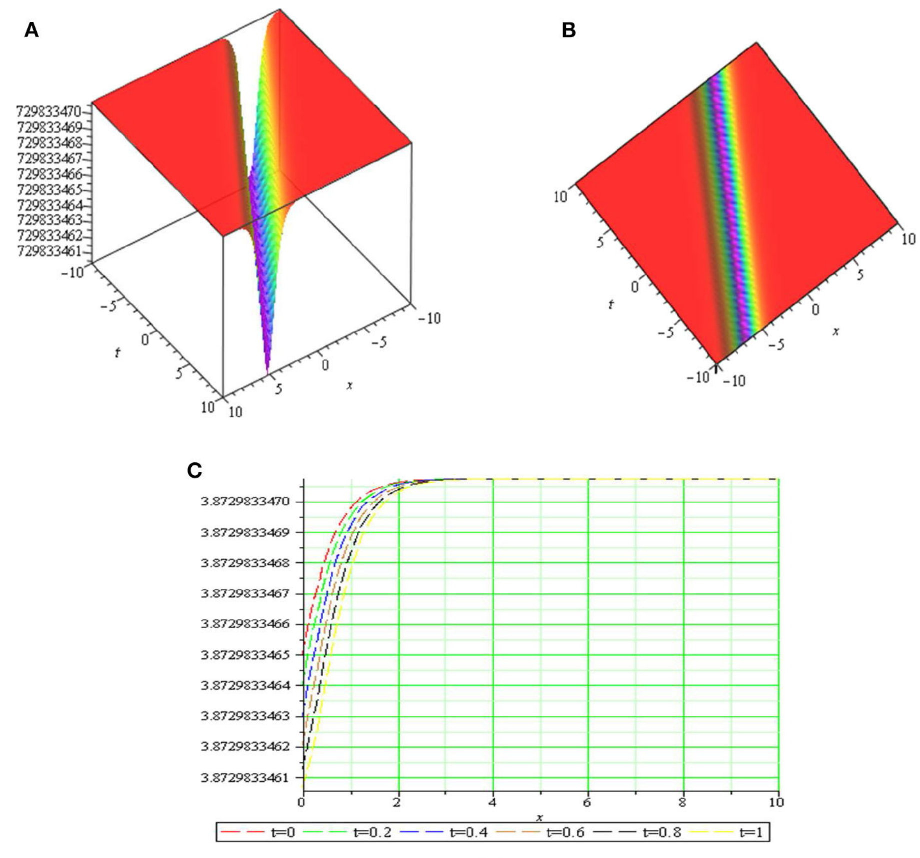

FIGURE 2 | The sws $|u_{1,7}^{-}|$ with parameters $\mu = 3, k = 2.5, c = 1.5, \lambda = 0.2, p = 1.2, q = 1.1, y = 1.5, z = 0.5$ and **(A)** 3D plot, **(B)** cp, and **(C)** 2D plot.

Case II: If $\frac{c-k}{k\,\mu c} < 0$ and $\rho = 0$ then

$$u_{1,3}^{\pm} = \pm\sqrt{\frac{6pq(c-k)}{\lambda}}\,\sec_{pq}\left(\sqrt{\frac{c-k}{k\,\mu c}}(kx+\lambda y+\mu z-ct)\right),$$

$$u_{1,4}^{\pm} = \pm\sqrt{\frac{6pq(c-k)}{\lambda}}\csc_{pq}\left(\sqrt{\frac{c-k}{k\,\mu c}}(kx+\lambda y+\mu z-ct)\right).$$

Case III: If $\frac{c-k}{k\,\mu c} < 0$ and $\rho = \left(\frac{c-k}{2k\,\mu c}\right)^2$ then

$$u_{1,5}^{\pm} = \pm\sqrt{\frac{3(c-k)}{\lambda}}\tanh_{pq}\left(\sqrt{-\frac{c-k}{2k\,\mu c}}(kx+\lambda y+\mu z-ct)\right),$$

$$u_{1,6}^{\pm} = \pm\sqrt{\frac{3(c-k)}{\lambda}}\coth_{pq}\left(\sqrt{-\frac{c-k}{2k\,\mu c}}(kx+\lambda y+\mu z-ct)\right),$$

$$u_{1,7}^{\pm} = \pm\sqrt{\frac{3(c-k)}{\lambda}}\left(\tanh_{pq}\left(\sqrt{-\frac{2(c-k)}{k\mu c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.\pm i\sqrt{pq}\,\mathrm{sech}_{pq}\left(\sqrt{-\frac{2(c-k)}{k\,\mu c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

$$u_{1,8}^{\pm} = \pm\sqrt{\frac{3(c-k)}{\lambda}}\left(\coth_{pq}\left(\sqrt{-\frac{2(c-k)}{k\,\mu c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.\pm\sqrt{pq}\,\mathrm{csch}_{pq}\left(\sqrt{-\frac{2(c-k)}{k\,\mu c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

$$u_{1,9}^{\pm} = \pm\sqrt{\frac{3(c-k)}{4\lambda}}\left(\tanh_{pq}\left(\sqrt{-\frac{c-k}{8k\,\mu c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.+\coth_{pq}\left(\sqrt{-\frac{c-k}{8k\,\mu c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

Case IV: If $\frac{c-k}{k\,\mu c} > 0$ and $\rho = \left(\frac{c-k}{2k\,\mu c}\right)^2$ then

$$u_{1,10}^{\pm} = \pm\sqrt{-\frac{3(c-k)}{\lambda}}\tan_{pq}\left(\sqrt{\frac{c-k}{2k\,\mu c}}(kx+\lambda y+\mu z-ct)\right),$$

$$u_{1,11}^{\pm} = \pm\sqrt{-\frac{3(c-k)}{\lambda}}\cot_{pq}\left(\sqrt{\frac{c-k}{2k\,\mu c}}(kx+\lambda y+\mu z-ct)\right),$$

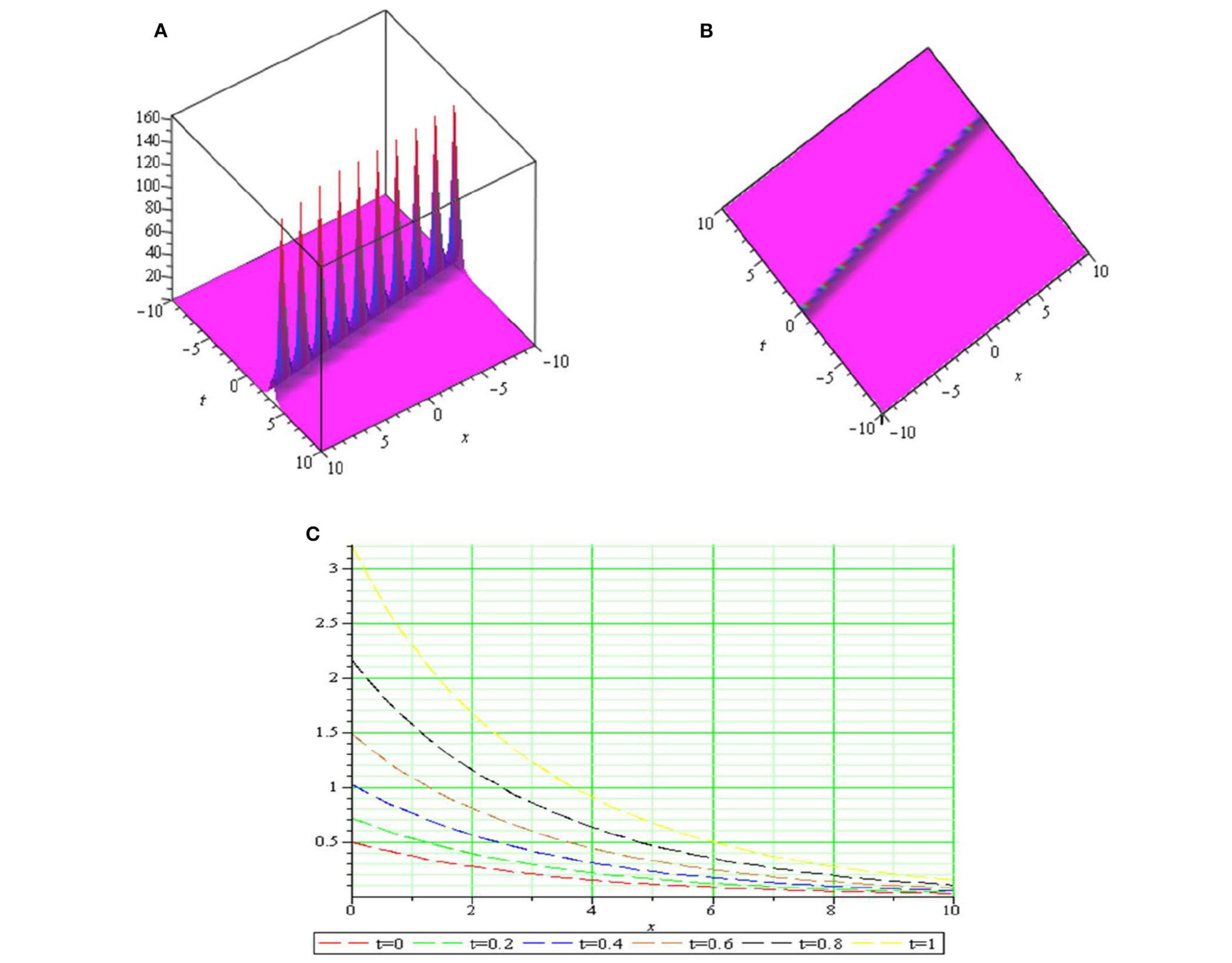

FIGURE 3 | The sws $|u^{+}_{2,2}|$ with $\mu = 0.3, k = 0.5, c = 3, \lambda = 5, p = 1.1, q = 1.2, y = 1, z = 1$ **(A)** 3D plot, **(B)** cp, and **(C)** 2D plot.

$$u^{\pm}_{1,12} = \pm\sqrt{-\frac{3(c-k)}{\lambda}}\left(\tan_{pq}\left(\sqrt{\frac{2(c-k)}{k\,\mu c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.\pm\sqrt{pq}\,\sec_{pq}\left(\sqrt{\frac{2(c-k)}{k\,\mu c}}(kx+\lambda y+\mu z-ct)\right)\right),$$
$$u^{\pm}_{1,14} = \pm\sqrt{-\frac{3(c-k)}{4\lambda}}\left(\tan_{pq}\left(\sqrt{\frac{c-k}{8k\,\mu c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.+\cot_{pq}\left(\sqrt{\frac{c-k}{8k\,\mu c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

The Second (3+1)-Dimensional WBBM Equation

Substituting (Equation 18) into (Equation 25), we have

$$\varpi_0 = 0,\ \ \varpi_1 = \pm\sqrt{-6\lambda c},\ \ a = \frac{c-\mu}{k\,\lambda c}. \tag{27}$$

The solutions of (4) corresponding to (27), along with solution (10) are

Case I: If $\frac{c-\mu}{k\lambda c} > 0$ and $\rho = 0$ then

$$u^{\pm}_{2,1} = \pm\sqrt{\frac{6pq(c-\mu)}{k}}\ \mathrm{sech}_{pq}\left(\sqrt{\frac{c-\mu}{k\,\lambda c}}(kx+\lambda y+\mu z-ct)\right),$$
$$u^{\pm}_{2,2} = \pm\sqrt{-\frac{6pq(c-\mu)}{k}}\ \mathrm{csch}_{pq}\left(\sqrt{\frac{c-\mu}{k\,\lambda c}}(kx+\lambda y+\mu z-ct)\right).$$

Case II: If $\frac{c-\mu}{k\lambda c} < 0$ and $\rho = 0$ then

$$u^{\pm}_{2,3} = \pm\sqrt{\frac{6pq(c-\mu)}{k}}\ \sec_{pq}\left(\sqrt{-\frac{c-\mu}{k\,\lambda c}}(kx+\lambda y+\mu z-ct)\right),$$

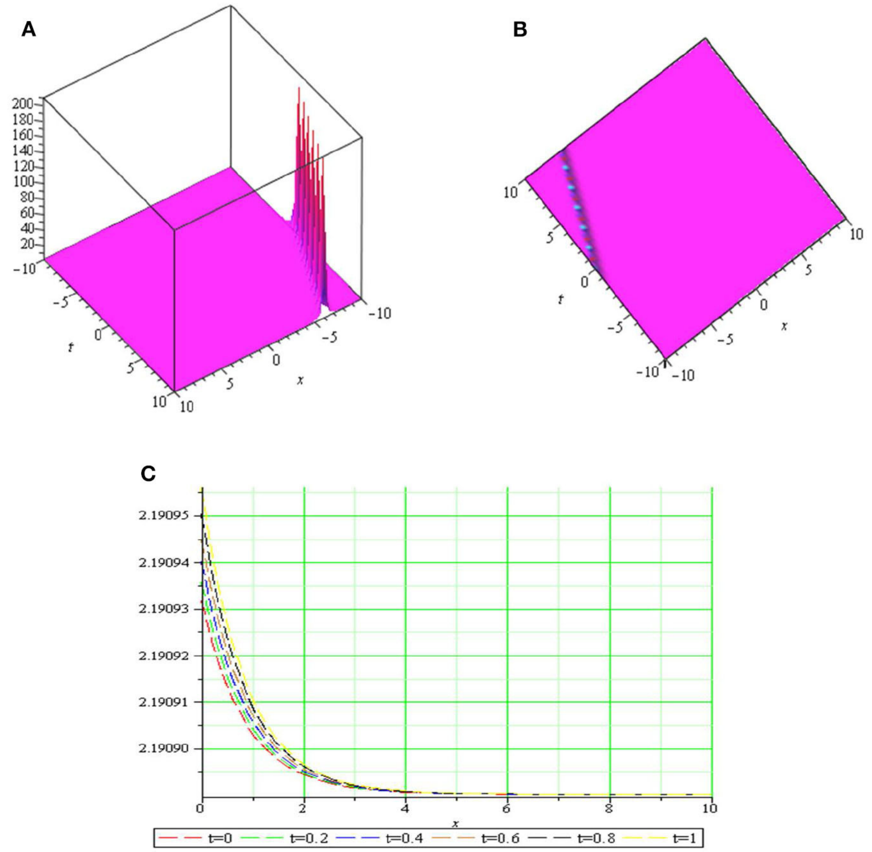

FIGURE 4 | The sws $|u_{2,6}^{-}|$ with $\mu = 1, k = 0.5, c = 0.2, \lambda = 3, p = 0.97, q = 0.95, y = 1.5, z = 0.5$ and **(A)** 3D plot, **(B)** cp, and **(C)** 2D plot.

$$u_{2,4}^{\pm} = \pm\sqrt{\frac{6pq(c-\mu)}{k}}\csc_{pq}\left(\sqrt{-\frac{c-\mu}{k\lambda c}}(kx+\lambda y+\mu z-ct)\right),$$

Case III: If $\frac{c-\mu}{k\lambda c} < 0$ and $\rho = \left(\frac{c-\mu}{2k\lambda c}\right)^2$ then

$$u_{2,5}^{\pm} = \pm\sqrt{\frac{3(c-\mu)}{k}}\tanh_{pq}\left(\sqrt{-\frac{c-\mu}{2k\lambda c}}(kx+\lambda y+\mu z-ct)\right),$$

$$u_{2,6}^{\pm} = \pm\sqrt{\frac{3(c-\mu)}{k}}\coth_{pq}\left(\sqrt{-\frac{c-\mu}{2k\lambda c}}(kx+\lambda y+\mu z-ct)\right),$$

$$u_{2,7}^{\pm} = \pm\sqrt{\frac{3(c-\mu)}{k}}\left(\tanh_{pq}\left(\sqrt{-\frac{2(c-\mu)}{k\lambda c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.\pm i\sqrt{pq}\,\mathrm{sech}_{pq}\left(\sqrt{-\frac{2(c-\mu)}{k\lambda c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

$$u_{2,8}^{\pm} = \pm\sqrt{\frac{3(c-\mu)}{k}}\left(\coth_{pq}\left(\sqrt{-\frac{2(c-\mu)}{k\lambda c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.\pm\sqrt{pq}\,\mathrm{csch}_{pq}\left(\sqrt{-\frac{2(c-\mu)}{k\lambda c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

$$u_{2,9}^{\pm} = \pm\sqrt{\frac{3(c-\mu)}{4k}}\left(\tanh_{pq}\left(\sqrt{-\frac{c-\mu}{8k\lambda c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.+\coth_{pq}\left(\sqrt{-\frac{c-\mu}{8k\lambda c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

Case IV: If $\frac{c-\mu}{k\lambda c} > 0$ and $\rho = \left(\frac{c-\mu}{2k\lambda c}\right)^2$ then

$$u_{2,10}^{\pm} = \pm\sqrt{-\frac{3(c-\mu)}{k}}\tan_{pq}\left(\sqrt{\frac{c-\mu}{2k\lambda c}}(kx+\lambda y+\mu z-ct)\right),$$

$$u_{2,11}^{\pm} = \pm\sqrt{-\frac{3(c-\mu)}{k}}\cot_{pq}\left(\sqrt{\frac{c-\mu}{2k\lambda c}}(kx+\lambda y+\mu z-ct)\right),$$

$$u_{2,12}^{\pm} = \pm\sqrt{-\frac{3(c-\mu)}{k}}\left(\tan_{pq}\left(\sqrt{\frac{2(c-\mu)}{k\lambda c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.\pm\sqrt{pq}\,\sec_{pq}\left(\sqrt{\frac{2(c-\mu)}{k\lambda c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

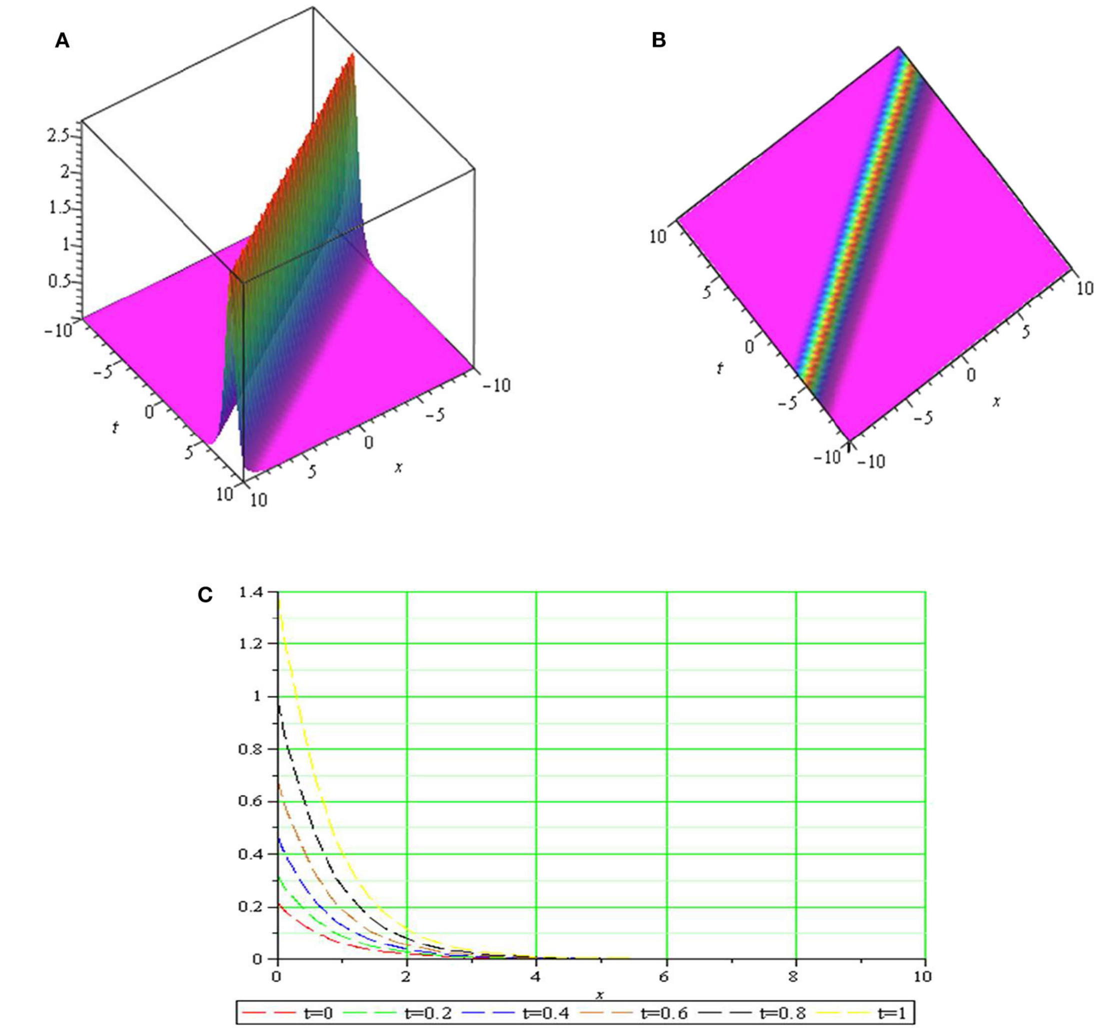

FIGURE 5 | The sws $|u_{3,1}^{+}|$ with parameters $\mu = 2, k = 1, c = 1.5, \lambda = -1, \rho = 1, q = 1, y = 1.5, z = 2$ and **(A)** 3D plot, **(B)** cp, and **(C)** 2D plot.

$$u_{2,14}^{\pm} = \pm\sqrt{-\frac{3(c-\mu)}{4k}}\left(\tan_{pq}\left(\sqrt{\frac{c-\mu}{8k\lambda c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.+\cot_{pq}\left(\sqrt{\frac{c-\mu}{8k\lambda c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

The Third (3+1)-Dimensional WBBM Equation

Substituting (Equation 23) into (Equation 25), we have

$$\varpi_0 = 0,\ \ \varpi_1 = \pm\sqrt{-\frac{6k^2c}{\mu}},\ \ a = \frac{c-\lambda}{k^2c}. \qquad (28)$$

The solutions of (5) corresponding to (28), along with solution (10) are

Case I: If $\frac{c-\lambda}{k^2c} > 0$ and $\rho = 0$ then

$$u_{3,1}^{\pm} = \pm\sqrt{\frac{6pq(c-\lambda)}{\mu}}\ \mathrm{sech}_{pq}\left(\sqrt{\frac{c-\lambda}{k^2c}}(kx+\lambda y+\mu z-ct)\right),$$
$$u_{3,2}^{\pm} = \pm\sqrt{-\frac{6pq(c-\lambda)}{\mu}}\ \mathrm{csch}_{pq}\left(\sqrt{\frac{c-\lambda}{k^2c}}(kx+\lambda y+\mu z-ct)\right).$$

Case II: If $\frac{c-\lambda}{k^2c} < 0$ and $\rho = 0$ then

$$u_{3,3}^{\pm} = \pm\sqrt{\frac{6pq(c-\lambda)}{\mu}}\ \sec_{pq}\left(\sqrt{-\frac{c-\lambda}{k^2c}}(kx+\lambda y+\mu z-ct)\right),$$
$$u_{3,4}^{\pm} = \pm\sqrt{\frac{6pq(c-\lambda)}{\mu}}\csc_{pq}\left(\sqrt{-\frac{c-\lambda}{k^2c}}(kx+\lambda y+\mu z-ct)\right),$$

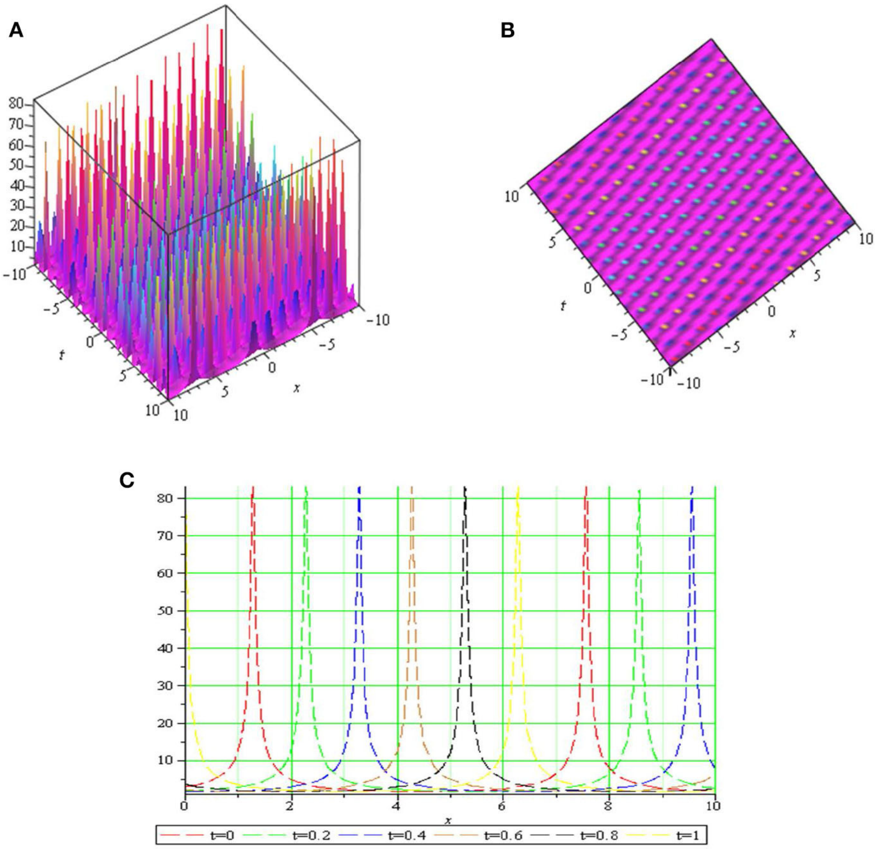

FIGURE 6 | The sws $\left|u_{3,14}^{-}\right|$ with parameters $\mu = 0.5, k = 0.2, c = 1, \lambda = 0.5, p = 0.97, q = 0.95, y = 1, z = 1$ and **(A)** 3D plot, **(B)** cp, and **(C)** 2D plot.

Case III: If $\frac{c-\lambda}{k^2c} < 0$ and $\rho = \left(\frac{c-\lambda}{2k^2c}\right)^2$ then

$$u_{3,5}^{\pm} = \pm\sqrt{\frac{3(c-\lambda)}{\mu}}\tanh_{pq}\left(\sqrt{-\frac{c-\lambda}{k^2c}}(kx+\lambda y+\mu z-ct)\right),$$

$$u_{3,6}^{\pm} = \pm\sqrt{\frac{3(c-\lambda)}{\mu}}\coth_{pq}\left(\sqrt{-\frac{c-\lambda}{k^2c}}(kx+\lambda y+\mu z-ct)\right),$$

$$u_{3,7}^{\pm} = \pm\sqrt{\frac{3(c-\lambda)}{\mu}}\left(\tanh_{pq}\left(\sqrt{-\frac{2(c-\lambda)}{k^2c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.\pm i\sqrt{pq}\,\mathrm{sech}_{pq}\left(\sqrt{-\frac{2(c-\lambda)}{k^2c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

$$u_{3,8}^{\pm} = \pm\sqrt{\frac{3(c-\lambda)}{\mu}}\left(\coth_{pq}\left(\sqrt{-\frac{2(c-k)}{k\mu c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.\pm\sqrt{pq}\,\mathrm{csch}_{pq}\left(\sqrt{-\frac{2(c-\lambda)}{k^2c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

$$u_{3,9}^{\pm} = \pm\sqrt{\frac{3(c-\lambda)}{4\mu}}\left(\tanh_{pq}\left(\sqrt{-\frac{c-\lambda}{8k^2c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.+\coth_{pq}\left(\sqrt{-\frac{c-\lambda}{8k^2c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

Case IV: If $\frac{c-\lambda}{k^2c} > 0$ and $\rho = \left(\frac{c-\lambda}{2k^2c}\right)^2$ then

$$u_{3,10}^{\pm} = \pm\sqrt{-\frac{3(c-\lambda)}{\mu}}\tan_{pq}\left(\sqrt{\frac{c-\lambda}{k^2c}}(kx+\lambda y+\mu z-ct)\right),$$

$$u_{3,11}^{\pm} = \pm\sqrt{-\frac{3(c-\lambda)}{\mu}}\cot_{pq}\left(\sqrt{\frac{c-\lambda}{k^2c}}(kx+\lambda y+\mu z-ct)\right),$$

$$u_{3,12}^{\pm} = \pm\sqrt{-\frac{3(c-\lambda)}{\mu}}\left(\tan_{pq}\left(\sqrt{\frac{2(c-\lambda)}{k^2c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.\pm\sqrt{pq}\,\sec_{pq}\left(\sqrt{\frac{2(c-\lambda)}{k^2c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

$$u_{3,14}^{\pm} = \pm\sqrt{-\frac{3(c-\lambda)}{4\mu}}\left(\tan_{pq}\left(\sqrt{\frac{c-\lambda}{8k^2c}}(kx+\lambda y+\mu z-ct)\right)\right.$$
$$\left.+\cot_{pq}\left(\sqrt{\frac{c-\lambda}{8k^2c}}(kx+\lambda y+\mu z-ct)\right)\right),$$

GRAPHICAL PRESENTATION

Graph is a strong tool for relationship describing clarity the solutions of the challenges. Therefore, some graphs of the solutions are given in each subsection above with different parameters. The graphs easily have shown the solitary wave form of the modulus solution $u^{+}_{1,3}$,$u^{-}_{1,7}$,$u^{+}_{2,2}$,$u^{-}_{2,6}$,$u^{+}_{3,1}$, and $u^{-}_{3,14}$ in **Figures 1–6** [(a) 3D plot, (b) cp, and (c) 2D plot], respectively. We see in **Figures 1**, **6** that the absolute value of $u^{+}_{1,3}$ and $u^{-}_{3,14}$ are singular periodic wave solutions and, in **Figures 2**, **5** that the absolute value of $u^{-}_{1,7}$ and $u^{+}_{3,1}$ are bright solitary wave solutions and in **Figures 3**, **4** that the absolute value of $u^{+}_{2,2}$ and $u^{-}_{2,5}$ are singular solitary wave solutions.

CONCLUSIONS

We have proposed a method, namely Sardar-subequation method to solve NLEs with the help of Maple Software. Distinct forms of (3+1)-dimensional WBBM equations are handled to display the effectiveness of the suggested method. From our results, some sws are obtained including the generalized hyperbolic and trigonometric function solutions. As far as we know, for the first time, we describe and introduce Sardar-subequation method which is a new method for solving NLEs. Thus, all the solutions distinct forms of (3+1)-dimensional WBBM equations are new, which cannot be found in literature to our best knowledge. We can also see that the approach used in this letter is very effective, powerful and convenient and can be steadily applied to NLEs. We will extend the proposed method for some fractional models [46–48] in a future work.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

HR, MI, and DB contributed conception and design of the study. HR organized the database. MI performed the statistical analysis. DB wrote the first draft of the manuscript. HR, MI, and DB wrote sections of the manuscript. All authors contributed to manuscript revision, read, and approved the submitted version.

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