



New Estimates for the Jensen Gap Using s -Convexity With Applications

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In this article, we use s -convex and Green functions to obtain a bound for the Jensen gap in discrete form and a bound for the Jensen gap in integral form. We present two numerical examples to verify the main results and to examine the tightness of the bounds. Then, as an application of the discrete result, we derive a converse of the Hölder inequality. Based on the integral result, we obtain a bound for the Hermite-Hadamard gap and present a converse of the Hölder inequality in its integral form. Also, we obtain bounds for the Csiszár and Rényi divergences as applications of the discrete result. Finally, we utilize the bound obtained for the Csiszár divergence to deduce new estimates for some other divergences in information theory.

Keywords: Jensen inequality, s -convex function, green function, Csiszár divergence, Hölder inequality

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1. INTRODUCTION

Convex functions and their generalizations play a significant role in scientific observation and calculation of various parameters in modern analysis, especially in the theory of optimization. Moreover, convex functions have some nice properties, such as differentiability, monotonicity, and continuity, which are useful in applications [1–5]. Interest in mathematical inequalities for convex and generalized convex functions has been growing exponentially, and research in this respect has had a significant impact on modern analysis [6–20]. Several mathematical inequalities have been established for s -convex functions in particular [21–28], one of the most important being the Jensen inequality. In this paper, we study the Jensen inequality in a more standard framework for s -convex functions.

Definition 1.1 (s -convexity [29]). For $s > 0$ and a convex subset B of a real linear space S , a function $\Gamma : B \rightarrow \mathbb{R}$ is said to be s -convex if the inequality

$$\Gamma(\kappa_1 \varepsilon_1 + \kappa_2 \varepsilon_2) \leq \kappa_1^s \Gamma(\varepsilon_1) + \kappa_2^s \Gamma(\varepsilon_2) \quad (1.1)$$

holds for all $\varepsilon_1, \varepsilon_2 \in B$ and $\kappa_1, \kappa_2 \geq 0$ with $\kappa_1 + \kappa_2 = 1$.

The function Γ is said to be s -concave if the inequality (1.1) holds in the reverse direction. Obviously, for $s = 1$ an s -convex function becomes a convex function, which shows that s -convexity of a function is a generalization of ordinary convexity of that function.

Lemma 1.2 ([29]). Let B be a convex subset of a real linear space S and let $\Gamma : B \rightarrow \mathbb{R}$ be a convex function. Then the following two statements hold:

- Γ is s -convex for $0 < s \leq 1$ if Γ is non-negative;
- Γ is s -convex for $1 \leq s < \infty$ if Γ is non-positive.

The Green function [30]

$$G_1(t, x) = \begin{cases} \alpha_1 - x, & \alpha_1 \leq x \leq t, \\ \alpha_1 - t, & t \leq x \leq \alpha_2 \end{cases} \quad (1.2)$$

defined on $[\alpha_1, \alpha_2] \times [\alpha_1, \alpha_2]$ and the integral identity

$$\Gamma(t) = \Gamma(\alpha_1) + (t - \alpha_1)\Gamma'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} G_1(t, x)\Gamma''(x) dx \quad (1.3)$$

for the function $\Gamma \in C^2[\alpha_1, \alpha_2]$ will be used to obtain the main results. Note that G_1 is convex and continuous with respect to both variables.

This paper is organized as follows. In section 2 we give a bound for the Jensen gap in discrete form, which pertains to functions for which the absolute value of the second derivative is s -convex. We also derive a bound for the integral version of the Jensen gap. Then we conduct two numerical experiments that provide evidence for the tightness of the bound in the main result. We deduce a converse of the Hölder inequality from the discrete result and a bound for the Hermite-Hadamard gap from the integral result. Moreover, as a consequence of the integral result we obtain a converse of the Hölder inequality in its corresponding integral version. At the beginning of section 3 we present bounds for the Csiszár and Rényi divergences in the discrete case. Finally, we give estimates for the Shannon entropy, Kullback-Leibler divergence, χ^2 divergence, Bhattacharyya coefficient, Hellinger distance, and triangular discrimination as applications of the bound obtained for the Csiszár divergence. Conclusions are presented in the final section.

2. MAIN RESULTS

Using the concept of s -convexity, we derive a bound for the Jensen gap in discrete form, which is presented in the following theorem.

Theorem 2.1. *Suppose $|\Gamma|''$ is s -convex for a function $\Gamma \in C^2[\alpha_1, \alpha_2]$ and that $z_i \in [\alpha_1, \alpha_2]$ and $\kappa_i \in [0, \infty)$ for $i = 1, \dots, n$ with $\sum_{i=1}^n \kappa_i = K > 0$. Then the following inequality holds:*

$$\begin{aligned} & \left| \frac{1}{K} \sum_{i=1}^n \kappa_i \Gamma(z_i) - \Gamma\left(\frac{1}{K} \sum_{i=1}^n \kappa_i z_i\right) \right| \\ & \leq \frac{|\Gamma''(\alpha_1)|}{(s+1)(s+2)(\alpha_2 - \alpha_1)^s} \\ & \quad \left(\frac{1}{K} \sum_{i=1}^n \kappa_i (\alpha_2 - z_i)^{s+2} - \left(\alpha_2 - \frac{1}{K} \sum_{i=1}^n \kappa_i z_i\right)^{s+2} \right) \\ & \quad + \frac{|\Gamma''(\alpha_2)|}{(s+1)(s+2)(\alpha_2 - \alpha_1)^s} \\ & \quad \left(\frac{1}{K} \sum_{i=1}^n \kappa_i (z_i - \alpha_1)^{s+2} - \left(\frac{1}{K} \sum_{i=1}^n \kappa_i z_i - \alpha_1\right)^{s+2} \right). \end{aligned} \quad (2.4)$$

Proof: Using (1.3), we get

$$\frac{1}{K} \sum_{i=1}^n \kappa_i \Gamma(z_i) = \frac{1}{K} \sum_{i=1}^n \kappa_i \left(\Gamma(\alpha_1) + (z_i - \alpha_1)\Gamma'(\alpha_2) \right)$$

$$+ \int_{\alpha_1}^{\alpha_2} G_1(z_i, x)\Gamma''(x) dx \quad (2.5)$$

and

$$\begin{aligned} \Gamma\left(\frac{1}{K} \sum_{i=1}^n \kappa_i z_i\right) &= \Gamma(\alpha_1) + \left(\frac{1}{K} \sum_{i=1}^n \kappa_i z_i - \alpha_1\right)\Gamma'(\alpha_2) \\ & \quad + \int_{\alpha_1}^{\alpha_2} G_1\left(\frac{1}{K} \sum_{i=1}^n \kappa_i z_i, x\right)\Gamma''(x) dx. \end{aligned} \quad (2.6)$$

Equations (2.5) and (2.6) give

$$\begin{aligned} & \frac{1}{K} \sum_{i=1}^n \kappa_i \Gamma(z_i) - \Gamma\left(\frac{1}{K} \sum_{i=1}^n \kappa_i z_i\right) \\ &= \int_{\alpha_1}^{\alpha_2} \left(\frac{1}{K} \sum_{i=1}^n \kappa_i G_1(z_i, x) - G_1\left(\frac{1}{K} \sum_{i=1}^n \kappa_i z_i, x\right) \right) \Gamma''(x) dx. \end{aligned} \quad (2.7)$$

Taking the absolute value of (2.7), we get

$$\begin{aligned} & \left| \frac{1}{K} \sum_{i=1}^n \kappa_i \Gamma(z_i) - \Gamma\left(\frac{1}{K} \sum_{i=1}^n \kappa_i z_i\right) \right| \\ &= \left| \int_{\alpha_1}^{\alpha_2} \left(\frac{1}{K} \sum_{i=1}^n \kappa_i G_1(z_i, x) - G_1\left(\frac{1}{K} \sum_{i=1}^n \kappa_i z_i, x\right) \right) \Gamma''(x) dx \right| \\ &\leq \int_{\alpha_1}^{\alpha_2} \left| \frac{1}{K} \sum_{i=1}^n \kappa_i G_1(z_i, x) - G_1\left(\frac{1}{K} \sum_{i=1}^n \kappa_i z_i, x\right) \right| |\Gamma''(x)| dx. \end{aligned} \quad (2.8)$$

By applying a change of variable $x = t\alpha_1 + (1-t)\alpha_2$ for $t \in [0, 1]$ and using the convexity of $G_1(t, x)$, the inequality (2.8) is transformed to

$$\begin{aligned} & \left| \frac{1}{K} \sum_{i=1}^n \kappa_i \Gamma(z_i) - \Gamma(\bar{z}) \right| \\ & \leq (\alpha_2 - \alpha_1) \int_0^1 \left(\frac{1}{K} \sum_{i=1}^n \kappa_i G_1(z_i, t\alpha_1 + (1-t)\alpha_2) \right. \\ & \quad \left. - G_1(\bar{z}, t\alpha_1 + (1-t)\alpha_2) \right) |\Gamma''(t\alpha_1 + (1-t)\alpha_2)| dt, \end{aligned} \quad (2.9)$$

where $\bar{z} = \frac{1}{K} \sum_{i=1}^n \kappa_i z_i$. The inequality (2.9) leads to the following by using s -convexity of the function $|\Gamma|''$:

$$\begin{aligned} & \left| \frac{1}{K} \sum_{i=1}^n \kappa_i \Gamma(z_i) - \Gamma(\bar{z}) \right| \\ & \leq (\alpha_2 - \alpha_1) \int_0^1 \left(\frac{1}{K} \sum_{i=1}^n \kappa_i G_1(z_i, t\alpha_1 + (1-t)\alpha_2) \right. \\ & \quad \left. - G_1(\bar{z}, t\alpha_1 + (1-t)\alpha_2) \right) \\ & \quad \times \left(t^s |\Gamma''(\alpha_1)| + (1-t)^s |\Gamma''(\alpha_2)| \right) dt. \end{aligned}$$

$$\begin{aligned}
 &= (\alpha_2 - \alpha_1) \int_0^1 \left(\frac{1}{K} \sum_{i=1}^n \kappa_i G_1(z_i, t\alpha_1 + (1-t)\alpha_2) t^s |\Gamma''(\alpha_1)| \right. \\
 &\quad + \frac{1}{K} \sum_{i=1}^n \kappa_i G_1(z_i, t\alpha_1 + (1-t)\alpha_2) (1-t)^s |\Gamma''(\alpha_2)| \\
 &\quad - G_1(\bar{z}, t\alpha_1 + (1-t)\alpha_2) t^s |\Gamma''(\alpha_1)| \\
 &\quad \left. - G_1(\bar{z}, t\alpha_1 + (1-t)\alpha_2) (1-t)^s |\Gamma''(\alpha_2)| \right) dt \\
 &= (\alpha_2 - \alpha_1) \left(|\Gamma''(\alpha_1)| \frac{1}{K} \sum_{i=1}^n \kappa_i \int_0^1 t^s G_1(z_i, t\alpha_1 + (1-t)\alpha_2) dt \right. \\
 &\quad + |\Gamma''(\alpha_2)| \frac{1}{K} \sum_{i=1}^n \kappa_i \int_0^1 (1-t)^s G_1(z_i, t\alpha_1 + (1-t)\alpha_2) dt \\
 &\quad - |\Gamma''(\alpha_1)| \int_0^1 t^s G_1(\bar{z}, t\alpha_1 + (1-t)\alpha_2) dt \\
 &\quad \left. - |\Gamma''(\alpha_2)| \int_0^1 (1-t)^s G_1(\bar{z}, t\alpha_1 + (1-t)\alpha_2) dt \right). \tag{2.10}
 \end{aligned}$$

Now, by using the change of variable $x = t\alpha_1 + (1-t)\alpha_2$ for $t \in [0, 1]$, we obtain

$$\begin{aligned}
 &\int_0^1 t^s G_1(z_i, t\alpha_1 + (1-t)\alpha_2) dt \\
 &= \frac{1}{(\alpha_2 - \alpha_1)^{s+1}} \left(\frac{(\alpha_2 - z_i)^{s+2}}{(s+1)(s+2)} - \frac{(\alpha_2 - \alpha_1)^{s+2}}{(s+1)(s+2)} \right). \tag{2.11}
 \end{aligned}$$

Upon replacing z_i by \bar{z} in (2.11), we get

$$\begin{aligned}
 &\int_0^1 t^s G_1(\bar{z}, t\alpha_1 + (1-t)\alpha_2) dt \\
 &= \frac{1}{(\alpha_2 - \alpha_1)^{s+1}} \left(\frac{(\alpha_2 - \bar{z})^{s+2}}{(s+1)(s+2)} - \frac{(\alpha_2 - \alpha_1)^{s+2}}{(s+1)(s+2)} \right). \tag{2.12}
 \end{aligned}$$

Also,

$$\begin{aligned}
 &\int_0^1 (1-t)^s G_1(z_i, t\alpha_1 + (1-t)\alpha_2) dt \\
 &= \frac{1}{(\alpha_2 - \alpha_1)^{s+1}} \left(\frac{(z_i - \alpha_1)^{s+2}}{(s+1)(s+2)} - \frac{(z_i - \alpha_1)(\alpha_2 - \alpha_1)^{s+1}}{(s+1)} \right). \tag{2.13}
 \end{aligned}$$

Upon replacing z_i by \bar{z} in (2.13), we get

$$\begin{aligned}
 &\int_0^1 (1-t)^s G_1(\bar{z}, t\alpha_1 + (1-t)\alpha_2) dt \\
 &= \frac{1}{(\alpha_2 - \alpha_1)^{s+1}} \left(\frac{(\bar{z} - \alpha_1)^{s+2}}{(s+1)(s+2)} - \frac{(\bar{z} - \alpha_1)(\alpha_2 - \alpha_1)^{s+1}}{(s+1)} \right). \tag{2.14}
 \end{aligned}$$

The result (2.4) is then obtained by substituting the values from (2.11)–(2.14) into (2.10).

Remark 2.2. If we use the Green function G_2, G_3 , or G_4 instead of G_1 in Theorem 2.1, where G_2, G_3 , and G_4 are given in [30], we obtain the same result (2.4).

In the following theorem, we give a bound for the Jensen gap in integral form.

Theorem 2.3. Suppose $|\Gamma''|$ is an s -convex function for $\Gamma \in C^2[\alpha_1, \alpha_2]$, and let ξ_1 and ξ_2 be real-valued functions defined on $[c_1, c_2]$ with $\xi_1(y) \in [\alpha_1, \alpha_2]$ for all $y \in [c_1, c_2]$ and such that $\xi_2, \xi_1 \xi_2$, and $(\Gamma \circ \xi_1)\xi_2$ are all integrable functions on $[c_1, c_2]$. Then the inequality

$$\begin{aligned}
 &\left| \frac{\int_{c_1}^{c_2} (\Gamma \circ \xi_1)(y)\xi_2(y) dy}{\xi} - \Gamma \left(\frac{\int_{c_1}^{c_2} \xi_1(y)\xi_2(y) dy}{\xi} \right) \right| \\
 &\leq \frac{|\Gamma''(\alpha_1)|}{(s+1)(s+2)(\alpha_2 - \alpha_1)^s} \left\{ \frac{\int_{c_1}^{c_2} \xi_2(y) (\alpha_2 - \xi_1(y))^{s+2} dy}{\xi} \right. \\
 &\quad \left. - \left(\alpha_2 - \frac{\int_{c_1}^{c_2} \xi_2(y)\xi_1(y) dy}{\xi} \right)^{s+2} \right\} \\
 &\quad + \frac{|\Gamma''(\alpha_2)|}{(s+1)(s+2)(\alpha_2 - \alpha_1)^s} \left\{ \frac{\int_{c_1}^{c_2} \xi_2(y) (\xi_1(y) - \alpha_1)^{s+2} dy}{\xi} \right. \\
 &\quad \left. - \left(\frac{\int_{c_1}^{c_2} \xi_2(y)\xi_1(y) dy}{\xi} - \alpha_1 \right)^{s+2} \right\} \tag{2.15}
 \end{aligned}$$

holds provided that $\int_{c_1}^{c_2} \xi_2(y) dy := \xi > 0$ when $\xi_2(y) \in [0, \infty)$ for all $y \in [c_1, c_2]$.

Proof: Using the same procedure as in the proof of Theorem 2.1, (2.15) can be obtained.

Example 1. Let $\Gamma(y) = \frac{4}{15}y^{\frac{5}{2}}$, $\xi_1(y) = y^2$, and $\xi_2(y) = 1$ for all $y \in [0, 1]$. Then $\Gamma''(y) = y^{\frac{1}{2}} > 0$ for all $y \in [0, 1]$. This shows that Γ is a convex function while $|\Gamma''|$ is $\frac{1}{2}$ -convex. Also, $\xi_1(y) \in [0, 1]$ for all $y \in [0, 1]$ and we have $[\alpha_1, \alpha_2] = [c_1, c_2] = [0, 1]$. Now, the left-hand side of inequality (2.15) gives $\int_0^1 \Gamma(\xi_1(y)) dy - \Gamma(\int_0^1 \xi_1(y) dy) = 0.0444 - 0.0171 = 0.0273 = E_1$, which shows how sharp the Jensen inequality is. The right-hand side of (2.15) gives 0.0274, which is very close to the true discrepancy E_1 . That is, from inequality (2.15) we have

$$0.0273 < 0.0274. \tag{2.16}$$

The difference $0.0274 - 0.0273 = 0.0001$ between the two sides of (2.16) shows that the bound for the Jensen gap given by inequality (2.15) is very close to the true value.

Example 2. Let $\Gamma(y) = \frac{100}{231}y^{\frac{21}{10}}$, $\xi_1(y) = y$, and $\xi_2(y) = 1$ for all $y \in [0, 1]$. Then $\Gamma''(y) = y^{\frac{1}{10}} > 0$ for all $y \in [0, 1]$, which shows that Γ is a convex function while $|\Gamma''|$ is s -convex with $s = \frac{1}{10}$. Also, $\xi_1(y) \in [0, 1]$ for all $y \in [0, 1]$ and we have $[\alpha_1, \alpha_2] = [c_1, c_2] = [0, 1]$. Therefore, from the left-hand side of inequality (2.15) we

obtain $\int_0^1 \Gamma(\xi_1(y)) dy - \Gamma(\int_0^1 \xi_1(y) dy) = 0.1396 - 0.1010 = 0.0386 = E_2$, which shows that the Jensen inequality is quite sharp. The right-hand side of (2.15) gives 0.0387, a value very close to the true discrepancy E_2 . Finally, from inequality (2.15) we have

$$0.0386 < 0.0387. \tag{2.17}$$

The difference $0.0387 - 0.0386 = 0.0001$ between the two sides of (2.17) provides further evidence of the tightness of the bound for the Jensen gap given by inequality (2.15).

As an application of Theorem 2.1, we derive a converse of the Hölder inequality, stated in the following proposition.

Proposition 2.4. Let $q_2 > 1$ and $q_1 \notin (2, 3)$ be such that $\frac{1}{q_1} + \frac{1}{q_2} = 1$, and let $s \in (0, 1)$. Also, let $[\alpha_1, \alpha_2]$ be a positive interval and let (d_1, \dots, d_n) and (b_1, \dots, b_n) be two positive n -tuples such that $\frac{\sum_{i=1}^n d_i b_i}{\sum_{i=1}^n b_i^{q_2}}$, with $d_i b_i^{-\frac{q_2}{q_1}} \in [\alpha_1, \alpha_2]$ for $i = 1, \dots, n$. Then

$$\begin{aligned} & \left(\sum_{i=1}^n d_i^{q_1} \right)^{\frac{1}{q_1}} \left(\sum_{i=1}^n b_i^{q_2} \right)^{\frac{1}{q_2}} - \sum_{i=1}^n d_i b_i \\ & \leq \left[\frac{q_1(q_1 - 1)}{(s + 1)(s + 2)(\alpha_2 - \alpha_1)^s} \right] \alpha_1^{q_1 - 2} \\ & \left(\frac{\sum_{i=1}^n b_i^{q_2} (\alpha_2 - d_i b_i^{-\frac{q_2}{q_1}})^{s+2}}{\sum_{i=1}^n b_i^{q_2}} - \left(\alpha_2 - \frac{\sum_{i=1}^n d_i b_i}{\sum_{i=1}^n b_i^{q_2}} \right)^{s+2} \right) \\ & + \alpha_2^{q_1 - 2} \left(\frac{\sum_{i=1}^n b_i^{q_2} (d_i b_i^{-\frac{q_2}{q_1}} - \alpha_1)^{s+2}}{\sum_{i=1}^n b_i^{q_2}} \right. \\ & \left. - \left(\frac{\sum_{i=1}^n d_i b_i}{\sum_{i=1}^n b_i^{q_2}} - \alpha_1 \right)^{s+2} \right) \right]^{\frac{1}{q_1}} \sum_{i=1}^n b_i^{q_2}. \end{aligned} \tag{2.18}$$

Proof: Let $\Gamma(x) = x^{q_1}$ for $x \in [\alpha_1, \alpha_2]$; then $\Gamma''(x) = q_1(q_1 - 1)x^{q_1 - 2} > 0$ and $|\Gamma''|'(x) = q_1(q_1 - 1)(q_1 - 2)(q_1 - 3)x^{q_1 - 4} > 0$, which shows that Γ and $|\Gamma''|$ are convex functions. The function $|\Gamma''|$ is also non-negative, so by Lemma 1.2 it is also an s -convex function for $s \in (0, 1]$. Thus, using (2.4) with $\Gamma(x) = x^{q_1}$, $\kappa_i = b_i^{q_2}$, and $z_i = d_i b_i^{-\frac{q_2}{q_1}}$, we derive

$$\begin{aligned} & \left(\left(\sum_{i=1}^n d_i^{q_1} \right) \left(\sum_{i=1}^n b_i^{q_2} \right)^{q_1 - 1} - \left(\sum_{i=1}^n d_i b_i \right)^{q_1} \right)^{\frac{1}{q_1}} \\ & \leq \left[\frac{q_1(q_1 - 1)}{(s + 1)(s + 2)(\alpha_2 - \alpha_1)^s} \right] \alpha_1^{q_1 - 2} \\ & \left(\frac{\sum_{i=1}^n b_i^{q_2} (\alpha_2 - d_i b_i^{-\frac{q_2}{q_1}})^{s+2}}{\sum_{i=1}^n b_i^{q_2}} - \left(\alpha_2 - \frac{\sum_{i=1}^n d_i b_i}{\sum_{i=1}^n b_i^{q_2}} \right)^{s+2} \right) \\ & + \alpha_2^{q_1 - 2} \left(\frac{\sum_{i=1}^n b_i^{q_2} (d_i b_i^{-\frac{q_2}{q_1}} - \alpha_1)^{s+2}}{\sum_{i=1}^n b_i^{q_2}} \right. \\ & \left. - \left(\frac{\sum_{i=1}^n d_i b_i}{\sum_{i=1}^n b_i^{q_2}} - \alpha_1 \right)^{s+2} \right) \right]^{\frac{1}{q_1}} \sum_{i=1}^n b_i^{q_2}. \end{aligned} \tag{2.19}$$

By using the inequality $x^\gamma - y^\gamma \leq (x - y)^\gamma$ for $0 \leq y \leq x$ and $\gamma \in [0, 1]$ with $x = \left(\sum_{i=1}^n d_i^{q_1} \right) \times \left(\sum_{i=1}^n b_i^{q_2} \right)^{q_1 - 1}$, $y = \left(\sum_{i=1}^n d_i b_i \right)^{q_1}$, and $\gamma = \frac{1}{q_1}$, we obtain

$$\begin{aligned} & \left(\sum_{i=1}^n d_i^{q_1} \right)^{\frac{1}{q_1}} \left(\sum_{i=1}^n b_i^{q_2} \right)^{\frac{1}{q_2}} - \sum_{i=1}^n d_i b_i \\ & \leq \left(\left(\sum_{i=1}^n d_i^{q_1} \right) \left(\sum_{i=1}^n b_i^{q_2} \right)^{q_1 - 1} - \left(\sum_{i=1}^n d_i b_i \right)^{q_1} \right)^{\frac{1}{q_1}}. \end{aligned} \tag{2.20}$$

The inequality (2.18) follows from (2.19) and (2.20).

In the following proposition, we provide a converse of the Hölder inequality in integral form as an application of Theorem 2.3.

Proposition 2.5. Let $q_2 > 1$ and $q_1 \notin (2, 3)$ be such that $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Also, let $\zeta_1, \zeta_2 : [c_1, c_2] \rightarrow \mathbb{R}^+$ be two functions such that $\zeta_1^{q_1}(y)$, $\zeta_2^{q_2}(y)$, and $\zeta_1(y)\zeta_2(y)$ are integrable on $[c_1, c_2]$ with $\zeta_1(y)\zeta_2^{-\frac{q_2}{q_1}}(y) \in [\alpha_1, \alpha_2]$ when $[\alpha_1, \alpha_2] \subset \mathbb{R}$. Then the inequality

$$\begin{aligned} & \left(\int_{c_1}^{c_2} \zeta_1^{q_1}(y) dy \right)^{\frac{1}{q_1}} \left(\int_{c_1}^{c_2} \zeta_2^{q_2}(y) dy \right)^{\frac{1}{q_2}} - \int_{c_1}^{c_2} \zeta_1(y)\zeta_2(y) dy \\ & \leq \left[\frac{q_1(q_1 - 1)}{(s + 1)(s + 2)(\alpha_2 - \alpha_1)^s} \right] \alpha_1^{q_1 - 2} \\ & \left(\frac{1}{\int_{c_1}^{c_2} \zeta_2^{q_2}(y) dy} \int_{c_1}^{c_2} \zeta_2^{q_2}(y) (\alpha_2 - \zeta_1(y)\zeta_2^{-\frac{q_2}{q_1}}(y))^{s+2} dy \right. \\ & \left. - \left(\alpha_2 - \frac{1}{\int_{c_1}^{c_2} \zeta_2^{q_2}(y) dy} \int_{c_1}^{c_2} \zeta_1(y)\zeta_2(y) dy \right)^{s+2} \right) \\ & + \alpha_2^{q_1 - 2} \left(\frac{1}{\int_{c_1}^{c_2} \zeta_2^{q_2}(y) dy} \int_{c_1}^{c_2} \zeta_2^{q_2}(y) (\zeta_1(y)\zeta_2^{-\frac{q_2}{q_1}}(y) - \alpha_1)^{s+2} dy \right. \\ & \left. - \left(\frac{1}{\int_{c_1}^{c_2} \zeta_2^{q_2}(y) dy} \int_{c_1}^{c_2} \zeta_1(y)\zeta_2(y) dy - \alpha_1 \right)^{s+2} \right) \right]^{\frac{1}{q_1}} \int_{c_1}^{c_2} \zeta_2^{q_2}(y) dy \end{aligned} \tag{2.21}$$

holds for $s \in (0, 1]$.

Proof: Using (2.15) with $\Gamma(x) = x^{q_1}$ for $x \in [\alpha_1, \alpha_2]$, $\xi_2(y) = \zeta_2^{q_2}(y)$, and $\xi_1(y) = \zeta_1(y)\zeta_2^{-\frac{q_2}{q_1}}(y)$ and following the procedure of Proposition 2.4, we deduce (2.21).

As an application of Theorem 2.3, in the following corollary we establish a bound for the Hermite-Hadamard gap.

Corollary 2.6. Let $\psi \in C^2[c_1, c_2]$ be a function such that $|\psi''|$ is s -convex; then

$$\begin{aligned} & \left| \frac{1}{c_2 - c_1} \int_{c_1}^{c_2} \psi(y) dy - \psi\left(\frac{c_1 + c_2}{2}\right) \right| \\ & \leq \frac{(c_2 - c_1)^2}{(s + 1)(s + 2)} \left(|\psi''(c_1)| + |\psi''(c_2)| \right) \left(\frac{1}{s + 3} - \frac{1}{2^{s+2}} \right). \end{aligned} \tag{2.22}$$

Proof: The inequality (2.22) can be obtained by using (2.15) with $\psi = \Gamma$, $[\alpha_1, \alpha_2] = [c_1, c_2]$, $\xi_2(y) = 1$, and $\xi_1(y) = y$ for $y \in [c_1, c_2]$.

3. APPLICATIONS TO INFORMATION THEORY

Definition 3.1 (Csiszár f -divergence [31]). Let $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$ and $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{R}_+^n$ with $\frac{t_i}{r_i} \in [\alpha_1, \alpha_2]$ ($i = 1, \dots, n$) for $[\alpha_1, \alpha_2] \subset \mathbb{R}$. For a function $f: [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$, the Csiszár f -divergence functional is defined as

$$\bar{D}_c(\mathbf{t}, \mathbf{r}) = \sum_{i=1}^n r_i f\left(\frac{t_i}{r_i}\right).$$

Theorem 3.2. Let $f \in C^2[\alpha_1, \alpha_2]$ be a function such that $|f''|$ is s -convex. Then for $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$ and $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{R}_+^n$ the inequality

$$\begin{aligned} & \left| \frac{1}{\sum_{i=1}^n r_i} \bar{D}_c(\mathbf{t}, \mathbf{r}) - f\left(\frac{\sum_{i=1}^n t_i}{\sum_{i=1}^n r_i}\right) \right| \\ & \leq \frac{|f''(\alpha_1)|}{(s+1)(s+2)(\alpha_2 - \alpha_1)^s} \left\{ \frac{1}{\sum_{i=1}^n r_i} \sum_{i=1}^n r_i \left(\alpha_2 - \frac{t_i}{r_i}\right)^{s+2} \right. \\ & \quad \left. - \left(\alpha_2 - \frac{\sum_{i=1}^n t_i}{\sum_{i=1}^n r_i}\right)^{s+2} \right\} \\ & + \frac{|f''(\alpha_2)|}{(s+1)(s+2)(\alpha_2 - \alpha_1)^s} \left\{ \frac{1}{\sum_{i=1}^n r_i} \sum_{i=1}^n r_i \left(\frac{t_i}{r_i} - \alpha_1\right)^{s+2} \right. \\ & \quad \left. - \left(\frac{\sum_{i=1}^n t_i}{\sum_{i=1}^n r_i} - \alpha_1\right)^{s+2} \right\} \end{aligned} \tag{3.23}$$

holds provided that $\frac{\sum_{i=1}^n t_i}{\sum_{i=1}^n r_i}, \frac{t_i}{r_i} \in [\alpha_1, \alpha_2]$ for $i = 1, \dots, n$.

Proof: The inequality (3.23) can easily be deduced from (2.4) by taking $\Gamma = f$, $z_i = \frac{t_i}{r_i}$, and $\kappa_i = \frac{r_i}{\sum_{i=1}^n r_i}$.

Definition 3.3 (Rényi divergence [31]). For $\mu \geq 0$ with $\mu \neq 1$ and two positive probability distributions $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{r} = (r_1, \dots, r_n)$, the Rényi divergence is defined as

$$D_{\text{re}}(\mathbf{t}, \mathbf{r}) = \frac{1}{\mu - 1} \log \left(\sum_{i=1}^n t_i^\mu r_i^{1-\mu} \right).$$

Corollary 3.4. Let $0 < s \leq 1$ and $[\alpha_1, \alpha_2] \subseteq \mathbb{R}^+$. Then for positive probability distributions $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{r} = (r_1, \dots, r_n)$, the inequality

$$\begin{aligned} & D_{\text{re}}(\mathbf{t}, \mathbf{r}) - \frac{1}{\mu - 1} \sum_{i=1}^n t_i \log \left(\frac{t_i}{r_i}\right)^{\mu-1} \\ & \leq \frac{1}{(\mu - 1)\alpha_1^2(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \end{aligned}$$

$$\begin{aligned} & \times \left\{ \sum_{i=1}^n t_i \left(\alpha_2 - \left(\frac{t_i}{r_i}\right)^{\mu-1}\right)^{s+2} - \left(\alpha_2 - \sum_{i=1}^n v_i^\mu w_i^{1-\mu}\right)^{s+2} \right\} \\ & + \frac{1}{(\mu - 1)\alpha_2^2(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \\ & \times \left\{ \sum_{i=1}^n t_i \left(\left(\frac{t_i}{r_i}\right)^{\mu-1} - \alpha_1\right)^{s+2} - \left(\sum_{i=1}^n v_i^\mu w_i^{1-\mu} - \alpha_1\right)^{s+2} \right\}. \end{aligned} \tag{3.24}$$

holds provided that $\sum_{i=1}^n r_i \left(\frac{t_i}{r_i}\right)^\mu, \left(\frac{t_i}{r_i}\right)^{\mu-1} \in [\alpha_1, \alpha_2]$ for $i = 1, \dots, n$ with $\mu > 1$.

Proof: Let $\Gamma(x) = -\frac{1}{\mu-1} \log x$ for $x \in [\alpha_1, \alpha_2]$. Then $\Gamma''(x) = \frac{1}{(\mu-1)x^2} > 0$ and $|\Gamma''|'(x) = \frac{6}{(\mu-1)x^4} > 0$, which shows that Γ and $|\Gamma''|$ are convex functions with $|\Gamma''| \geq 0$; so by Lemma 1.2 the function $|\Gamma''|$ is s -convex for $s \in (0, 1]$. Therefore, using (2.4) with $\Gamma(x) = -\frac{1}{\mu-1} \log x$, $\kappa_i = t_i$, and $z_i = \left(\frac{t_i}{r_i}\right)^{\mu-1}$, we derive (3.24).

Definition 3.5 (Shannon entropy [31]). Let $\mathbf{r} = (r_1, \dots, r_n)$ be a positive probability distribution; then the Shannon entropy is defined as

$$E_s(\mathbf{r}) = - \sum_{i=1}^n r_i \log r_i.$$

Corollary 3.6. Let $[\alpha_1, \alpha_2] \subseteq \mathbb{R}^+$, and let $\mathbf{r} = (r_1, \dots, r_n)$ be a positive probability distribution such that $\frac{1}{r_i} \in [\alpha_1, \alpha_2]$ for $i = 1, \dots, n$ with $0 < s \leq 1$. Then

$$\begin{aligned} \log n - E_s(\mathbf{r}) & \leq \frac{1}{\alpha_1^2(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \\ & \left\{ \sum_{i=1}^n r_i \left(\alpha_2 - \frac{1}{r_i}\right)^{s+2} - (\alpha_2 - n)^{s+2} \right\} \\ & + \frac{1}{\alpha_2^2(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \\ & \left\{ \sum_{i=1}^n r_i \left(\frac{1}{r_i} - \alpha_1\right)^{s+2} - (n - \alpha_1)^{s+2} \right\}. \end{aligned} \tag{3.25}$$

Proof: Let $f(x) = -\log x$ for $x \in [\alpha_1, \alpha_2]$. Then $f''(x) = \frac{1}{x^2} > 0$ and $|f''|'(x) = \frac{6}{x^4} > 0$, which shows that f and $|f''|$ are convex functions. Also, $|f''|$ is non-negative and so by Lemma 1.2 we conclude that it is s -convex for $s \in (0, 1]$. Therefore, using (3.23) with $f(x) = -\log x$ and $(t_1, \dots, t_n) = (1, \dots, 1)$, we get (3.25).

Definition 3.7 (Kullback-Leibler divergence [31]). For two positive probability distributions $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{r} = (r_1, \dots, r_n)$, the Kullback-Leibler divergence is defined as

$$D_{\text{kl}}(\mathbf{t}, \mathbf{r}) = \sum_{i=1}^n t_i \log \frac{t_i}{r_i}.$$

Corollary 3.8. Let $0 < s \leq 1$ and $0 < \alpha_1 < \alpha_2$, and let $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{r} = (r_1, \dots, r_n)$ be positive probability distributions such that $\frac{t_i}{r_i} \in [\alpha_1, \alpha_2]$ for $i = 1, \dots, n$. Then

$$D_{kl}(\mathbf{t}, \mathbf{r}) \leq \frac{1}{\alpha_1(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \left\{ \sum_{i=1}^n r_i \left(\alpha_2 - \frac{t_i}{r_i} \right)^{s+2} - (\alpha_2 - 1)^{s+2} \right\} + \frac{1}{\alpha_2(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \left\{ \sum_{i=1}^n r_i \left(\frac{t_i}{r_i} - \alpha_1 \right)^{s+2} - (1 - \alpha_1)^{s+2} \right\}. \quad (3.26)$$

Proof: Let $f(x) = x \log x$ for $x \in [\alpha_1, \alpha_2]$. Then $f''(x) = \frac{1}{x} > 0$ and $|f'''(x)| = \frac{2}{x^3} > 0$, which shows that f and $|f''|$ are convex functions. Also, $|f''| \geq 0$, and so Lemma 1.2 guarantees the s -convexity of $|f''|$ for $s \in (0, 1]$. Therefore, using (3.23) with $f(x) = x \log x$, we get (3.26).

Definition 3.9 (χ^2 divergence [31]). The χ^2 divergence $D_{\chi^2}(\mathbf{t}, \mathbf{r})$ for two positive probability distributions $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{r} = (r_1, \dots, r_n)$ is defined as

$$D_{\chi^2}(\mathbf{t}, \mathbf{r}) = \sum_{i=1}^n \frac{(t_i - r_i)^2}{r_i}.$$

Corollary 3.10. Let $0 < s \leq 1$ and $0 < \alpha_1 < \alpha_2$, and let $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{r} = (r_1, \dots, r_n)$ be positive probability distributions such that $\frac{t_i}{r_i} \in [\alpha_1, \alpha_2]$ for $i = 1, \dots, n$. Then

$$D_{\chi^2}(\mathbf{t}, \mathbf{r}) \leq \frac{2}{(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \left\{ \sum_{i=1}^n r_i \left(\alpha_2 - \frac{t_i}{r_i} \right)^{s+2} - (\alpha_2 - 1)^{s+2} \right\} + \frac{2}{(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \left\{ \sum_{i=1}^n r_i \left(\frac{t_i}{r_i} - \alpha_1 \right)^{s+2} - (1 - \alpha_1)^{s+2} \right\}. \quad (3.27)$$

Proof: Let $f(x) = (x - 1)^2$ for $x \in [\alpha_1, \alpha_2]$. Then $f''(x) = 2 > 0$ and $|f'''(x)| = 0$, which shows that f and $|f''|$ are convex functions. Also, the function $|f''|$ is non-negative, and so Lemma 1.2 confirms its s -convexity for $s \in (0, 1]$. Therefore, using (3.23) with $f(x) = (x - 1)^2$, we obtain (3.27).

Definition 3.11 (Bhattacharyya coefficient [31]). For two positive probability distributions $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{r} = (r_1, \dots, r_n)$, the Bhattacharyya coefficient is defined as

$$C_b(\mathbf{t}, \mathbf{r}) = \sum_{i=1}^n \sqrt{t_i r_i}.$$

Corollary 3.12. Let $0 < s \leq 1$ and $[\alpha_1, \alpha_2] \subseteq \mathbb{R}^+$, and let $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{r} = (r_1, \dots, r_n)$ be two positive probability distributions such that $\frac{t_i}{r_i} \in [\alpha_1, \alpha_2]$ for $i = 1, \dots, n$. Then

$$1 - C_b(\mathbf{t}, \mathbf{r}) \leq \frac{1}{4\alpha_1^{\frac{3}{2}}(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \left\{ \sum_{i=1}^n r_i \left(\alpha_2 - \frac{t_i}{r_i} \right)^{s+2} - (\alpha_2 - 1)^{s+2} \right\} + \frac{1}{4\alpha_2^{\frac{3}{2}}(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \left\{ \sum_{i=1}^n r_i \left(\frac{t_i}{r_i} - \alpha_1 \right)^{s+2} - (1 - \alpha_1)^{s+2} \right\}. \quad (3.28)$$

Proof: Let $f(x) = -\sqrt{x}$ for $x \in [\alpha_1, \alpha_2]$. Then $f''(x) = \frac{1}{4x^{\frac{3}{2}}} > 0$ and $|f'''(x)| = \frac{15}{16x^{\frac{7}{2}}} > 0$, which shows that f and $|f''|$ are convex functions. Also, $|f''| \geq 0$ implies its s -convexity for $s \in (0, 1]$ by Lemma 1.2. Therefore, using (3.23) with $f(x) = -\sqrt{x}$, we obtain (3.28).

Definition 3.13 (Hellinger distance [31]). The Hellinger distance $D_h^2(\mathbf{t}, \mathbf{r})$ between two positive probability distributions $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{r} = (r_1, \dots, r_n)$ is defined as

$$D_h^2(\mathbf{t}, \mathbf{r}) = \frac{1}{2} \sum_{i=1}^n (\sqrt{t_i} - \sqrt{r_i})^2.$$

Corollary 3.14. Let $0 < \alpha_1 < \alpha_2$ and $0 < s \leq 1$, and let $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{r} = (r_1, \dots, r_n)$ be positive probability distributions such that $\frac{t_i}{r_i} \in [\alpha_1, \alpha_2]$ for $i = 1, \dots, n$. Then

$$D_h^2(\mathbf{t}, \mathbf{r}) \leq \frac{1}{4\alpha_1^{\frac{3}{2}}(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \left\{ \sum_{i=1}^n r_i \left(\alpha_2 - \frac{t_i}{r_i} \right)^{s+2} - (\alpha_2 - 1)^{s+2} \right\} + \frac{1}{4\alpha_2^{\frac{3}{2}}(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \left\{ \sum_{i=1}^n r_i \left(\frac{t_i}{r_i} - \alpha_1 \right)^{s+2} - (1 - \alpha_1)^{s+2} \right\}. \quad (3.29)$$

Proof: Let $f(x) = \frac{1}{2}(1 - \sqrt{x})^2$ for $x \in [\alpha_1, \alpha_2]$. Then $f''(x) = \frac{1}{4x^{\frac{3}{2}}} > 0$ and $|f'''(x)| = \frac{15}{16x^{\frac{7}{2}}} > 0$, which shows that f and $|f''|$ are convex functions. Also, $|f''| \geq 0$, and so from Lemma 1.2 we conclude its s -convexity for $s \in (0, 1]$. Therefore, using (3.23) with $f(x) = \frac{1}{2}(1 - \sqrt{x})^2$, we deduce (3.29).

Definition 3.15 (Triangular discrimination [31]). For two positive probability distributions $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{r} = (r_1, \dots, r_n)$, the

triangular discrimination is defined as

$$D_{\Delta}(\mathbf{t}, \mathbf{r}) = \sum_{i=1}^n \frac{(t_i - r_i)^2}{t_i + r_i}.$$

Corollary 3.16. Let $0 < s \leq 1$ and $0 < \alpha_1 < \alpha_2$, and let $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{r} = (r_1, \dots, r_n)$ be positive probability distributions such that $\frac{t_i}{r_i} \in [\alpha_1, \alpha_2]$ for $i = 1, \dots, n$. Then

$$D_{\Delta}(\mathbf{t}, \mathbf{r}) \leq \frac{8}{(\alpha_1 + 1)^3(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \left\{ \sum_{i=1}^n r_i \left(\alpha_2 - \frac{t_i}{r_i} \right)^{s+2} - (\alpha_2 - 1)^{s+2} \right\} + \frac{8}{(\alpha_2 + 1)^3(\alpha_2 - \alpha_1)^s(s+1)(s+2)} \left\{ \sum_{i=1}^n r_i \left(\frac{t_i}{r_i} - \alpha_1 \right)^{s+2} - (1 - \alpha_1)^{s+2} \right\}. \quad (3.30)$$

Proof: Let $f(x) = \frac{(x-1)^2}{(x+1)^2}$ for $x \in [\alpha_1, \alpha_2]$. Then $f''(x) = \frac{8}{(x+1)^3} > 0$ and $|f'''(x)| = \frac{96}{(x+1)^5} > 0$, which shows that f and $|f''|$ are convex functions. Also, $|f''|$ is non-negative, and thus s -convexity of the function $|f''|$ for $s \in (0, 1]$ follows from Lemma 1.2. Therefore, using (3.23) with $f(x) = \frac{(x-1)^2}{(x+1)^2}$, we get (3.30).

Remark 3.17. Analogously, bounds for various divergences in integral form can be derived as applications of Theorem 2.3.

4. CONCLUSION

The Jensen inequality has numerous applications in engineering, economics, computer science, information theory, and coding;

it has been derived for convex and generalized convex functions. This paper presents a novel approach to bounding the Jensen gap. Some bounds are obtained for the Jensen gap via s -convex functions. Numerical experiments not only confirm the sharpness of the Jensen inequality but also provide evidence for the tightness of the bound given in (2.15) for the Jensen gap. These experiments also show that the bound in (2.15) gives very close estimates for the Jensen gap even when the functions are not convex. The bounds are used to obtain new estimates for the Hermite-Hadamard and Hölder inequalities. Furthermore, based on the main results, various divergences are estimated. These estimates for divergences can be applied to signal processing, magnetic resonance image analysis, image segmentation, pattern recognition, and other areas. The ideas in this paper can also be used with other inequalities and for some other classes of convex functions.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary materials, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

MA gave the main idea. MA and SK worked on Main Results while Y-MC worked on Introduction. All authors checked carefully the whole manuscript and approved.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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