



An Efficient Computational Method for the Time-Space Fractional Klein-Gordon Equation

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In this paper, we present a computational method to solve the fractional Klein-Gordon equation (FKGE). The proposed technique is the grouping of orthogonal polynomial matrices and collocation method. The benefit of the computational method is that it reduces the FKGE into a system of algebraic equations which makes the problem straightforward and easy to solve. The main reason for using this technique is its high accuracy and low computational cost compared to other methods. The main solution behaviors of these equations are due to fractional orders, which are explained graphically. Numerical results obtained by the proposed computational method are also compared with the exact solution. The results obtained by the suggested technique reveals that the method is very useful for solving FKGE.

Keywords: fractional Klein-Gordon equation, fractional derivative, numerical solution, Chebyshev polynomials, operational matrices

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INTRODUCTION

The standard Klein-Gordon equation (KGE) is written as

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + v = h(x, t), \quad x \geq 0, t \geq 0 \quad (1)$$

where v indicates an unknown function in variables x and t , and $h(x, t)$ stands for the source term. Due to the non-local nature and real-life applications of fractional derivatives, the fractional extension of this equation is very useful [1–12]. The fractional extension of this model handles the initial and boundary conditions of the model very accurately. The non-integer derivative helps in understanding the complete memory effect of the system. A broad literature of models with fractional derivatives can be found in [13–17]. Therefore, motivated by our ongoing research work into this special branch of mathematics (namely, fractional calculus), we study non-integer KGE by changing integer order derivative in both time and space using the Liouville-Caputo derivative of fractional order in the following manner:

$$\begin{aligned} \frac{\partial^\beta v(x,t)}{\partial t^\beta} - \frac{\partial^\gamma v(x,t)}{\partial x^\gamma} + v(x,t) \\ = h(x,t), \quad 1 < \beta \leq 2, \quad 1 < \gamma \leq 2 \end{aligned} \quad (2)$$

having the initial conditions:

$$v(x,0) = h_1(x), \quad \frac{\partial v(x,0)}{\partial t} = h_2(x), \quad \text{for } 0 \leq x, t \leq 1 \quad (3)$$

and boundary conditions:

$$v(0,t) = g_1(t), \quad v(1,t) = g_2(t) \quad (4)$$

The KGE is used in science, plasma (especially in quantum field theory), optical fibers, and dispersive wave-phenomena. Due to the great importance of KGE, many authors have studied it using various numerical and analytical schemes [18–27], each with their own limitations and shortcomings. The operational matrix method [28–38] is also applied to solve problems in fractional calculus. There are several other numerical and analytical methods which have been used to solve non-linear problems pertaining to fractional calculus, which can be found in [39, 40]. Some other applications of orthogonal polynomials-based solutions can be found in [41, 42].

In this paper, we present a computational technique which is a combination of the operational matrix and collocation method. We have used Chebyshev polynomials as a basis function for the construction of operational matrices of differentiations and integrations. In our proposed method, first the unknown function and their derivatives are approximated by taking finite dimensional approximations. Then, by using these approximations along with operational matrices of differentiations and integrations in the FKGE, we obtain a system of equations. Finally, by collocating this system, we get an approximate solution for the FKGE. The efficiency and accuracy of the used technique is shown by making a comparison amongst the results derived by our technique, exact solutions, and numerical results by some existing methods.

SOME BASIC DEFINITIONS

In this paper, we use non-integer order integrals and derivatives in the Riemann-Liouville and Caputo sense, respectively, which are given as:

Definition 2.1: The Riemann-Liouville non-integer integral operator of order α is presented as

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \quad x > 0,$$

$$I^0 f(x) = f(x).$$

Definition 2.2: The Liouville-Caputo non-integer derivative of order β are defined as [1–3]

$$D^\beta f(x) = I^{l-\beta} D^l f(x) = \frac{1}{\Gamma(l-\beta)} \int_0^x (x-t)^{l-\beta-1} \frac{d^l f(t)}{dt^l} dt,$$

$l-1 < \beta < l, x > 0$ and l is a natural number.

Chebyshev polynomial of the third kind of degree i on $[0, 1]$ is given as,

$$H_i(t) = \sum_{k=0}^i (-1)^{i-k} \frac{\Gamma(i+\frac{3}{2})\Gamma(i+k+1)}{\Gamma(k+\frac{3}{2})\Gamma(i+1)(i-k)!k!} t^k \quad (5)$$

The orthogonal property of these polynomials is given as:

$$\int_0^1 H_n(t) H_m(t) w(t) dt = \begin{cases} \frac{\pi}{2}, & n = m \\ 0, & n \neq m \end{cases} \quad (6)$$

where, $w(t) = \sqrt{\frac{t}{1-t}}$, is a weight function and n and m are the degrees of polynomials.

A function $g(x,t) \in L^2_{w(t)}([0,1] \times [0,1])$ can be approximated as

$$g(x,t) \cong \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} c_{i_1, i_2} H_{i_1, i_2}(x,t) = C^T \theta_{n_1, n_2}(x,t) \quad (7)$$

where, $C = [c_{0,0}, \dots, c_{0,n_2}, \dots, c_{n_1,1}, \dots, c_{n_1, n_2}]^T$ and

$$\theta_{n_1, n_2}(x,t) = [H_{0,0}(x,t), \dots, H_{0,n_2}(x,t), \dots, H_{n_1,0}(x,t), \dots, H_{n_1, n_2}(x,t)]^T.$$

For any approximation taking $n_1 = n_2 = n$ then Equation (6), can be written as,

$$g(x,t) \cong \theta_n^T(x) C \theta_n(t) \quad (8)$$

The matrix C in Equation (8), is given as:

$$C = P^{-1} \left(\int_0^1 \int_0^1 \theta_n(x) C \theta_n^T(t) w(x) w(t) dx dt \right) P^{-1} \quad (9)$$

where, $P = \int_0^1 \theta_n(x) \theta_n^T(x) w(x) dx$, is called the matrix of dual.

Theorem 1. If $\theta_n(t) = [H_0, H_1, \dots, H_n]^T$, is Chebyshev vector and we consider $\nu > 0$, then

$$I^\nu H_i(t) = I^{(\nu)} \theta_n(t) \quad (10)$$

where, $I^{(\nu)} = (e(i,j))$, is $(n+1) \times (n+1)$ matrix of integral of non-integer order ν and its entries are given by

$$e(i, j) = \sum_{k=0}^i \sum_{l=0}^j (-1)^{i+j-k-l} \times \frac{\Gamma(\frac{1}{2})\Gamma(i+\frac{3}{2})\Gamma(i+k+1)\Gamma(j+l+1)\Gamma(\nu+k+l+\frac{3}{2})(2j+1)j!}{(i-k)!(j-l)!(l)!\Gamma(k+\frac{3}{2})\Gamma(i+1)\Gamma(\nu+k+1)\Gamma(j+\frac{1}{2})\Gamma(l+\frac{3}{2})\Gamma(k+l+\nu+2)}.$$

Proof. Please see [30, 32, 38].

Theorem 2. If $\theta_n(t) = [H_0, H_1, \dots, H_n]^T$, is Chebyshev vector and we consider $\beta > 0$, then

$$D^\beta H_i(t) = D^{(\beta)} \theta_n(t) \quad (11)$$

where, $D^{(\beta)} = (s(i, j))$, is $(n+1) \times (n+1)$ matrix of differentiation of non-integer order β and its entries are given by

$$s(i, j) = \sum_{k=\lceil \beta \rceil}^i \sum_{l=0}^j (-1)^{i+j-k-l} \times \frac{\Gamma(\frac{1}{2})\Gamma(i+\frac{3}{2})\Gamma(i+k+1)\Gamma(j+l+1)\Gamma(k+l-\beta+\frac{3}{2})(2j+1)j!}{(i-k)!(j-l)!(l)!\Gamma(k+\frac{3}{2})\Gamma(i+1)\Gamma(k-\beta+1)\Gamma(j+\frac{1}{2})\Gamma(l+\frac{3}{2})\Gamma(k+l-\beta+2)}.$$

Proof. Please see [38].

METHOD OF SOLUTION

In this section, we apply our proposed algorithm to solve a fractional model of KGE. We use equal number basis elements i.e. $n_1 = n_2 = n$, for any approximations of space and time variables. We initially approximate the time derivative of the unknown function as follows:

$$\frac{\partial^\beta v(x, t)}{\partial t^\beta} = \theta_n^T(x) C \theta_n(t) \quad (12)$$

Taking integral of order β with respect to t on both sides of Equation (12), we have

$$v(x, t) = \theta_n^T(x) CI^{(\beta)} \theta_n(t) + \theta_n^T(x) AI^{(1)} \theta_n(t) + \theta_n^T(x) B \theta_n(t) \quad (13)$$

where $I^{(\beta)}$ and $I^{(1)}$ are operational matrices of integration of order β and 1, respectively, and are given by Equation (10) and

$$\frac{\partial v(x, 0)}{\partial t} = h_2(x) = \theta_n^T(x) A \theta_n(t) \quad (14)$$

$$v(x, 0) = h_1(x) = \theta_n^T(x) B \theta_n(t) \quad (15)$$

where A and B are known square matrices and can be calculated using Equation (9).

Taking the differentiation of order γ on both sides of Equation (13), we get

$$\frac{\partial^\gamma v(x, t)}{\partial x^\gamma} = \theta_n^T(x) D^{(\gamma), T} CI^{(\beta)} \theta_n(t) + \theta_n^T(x) D^{(\gamma), T} AI^{(1)} \theta_n(t) + \theta_n^T(x) D^{(\gamma), T} B \theta_n(t) \quad (16)$$

where, $D^{(\gamma)}$ is the operational matrix of differentiation of order γ and is given by Equation (11). Further, the inhomogeneous term can be approximated as

$$h(x, t) = \theta_n^T(x) E \theta_n(t) \quad (17)$$

where E is the known square matrix and can be calculated using Equation (9).

Grouping Equations (12), (13), (16), (17), and (2), we get

$$\begin{aligned} & \theta_n^T(x) C \theta_n(t) - \left(\theta_n^T(x) D^{(\gamma), T} CI^{(\beta)} \theta_n(t) \right. \\ & \left. + \theta_n^T(x) D^{(\gamma), T} AI^{(1)} \theta_n(t) + \theta_n^T(x) D^{(\gamma), T} B \theta_n(t) \right) \\ & + \theta_n^T(x) CI^{(\beta)} \theta_n(t) + \theta_n^T(x) AI^{(1)} \theta_n(t) \\ & + \theta_n^T(x) B \theta_n(t) = \theta_n^T(x) E \theta_n(t) \end{aligned} \quad (18)$$

Equation (18), can be written as

$$C - D^{(\gamma), T} CI^{(\beta)} - D^{(\gamma), T} AI^{(1)} - D^{(\gamma), T} B + CI^{(\beta)} + AI^{(1)} + B = E \quad (19)$$

Equation (19) is a system of equations which is easy to handle using the collocation method to determine the unknown matrix. By making use of the value of C in Equation (13), we can obtain an approximate solution for FLGE.

NUMERICAL EXPERIMENTS AND DISCUSSION

Example 1. Firstly, we take the time fractional KGE [26] given as

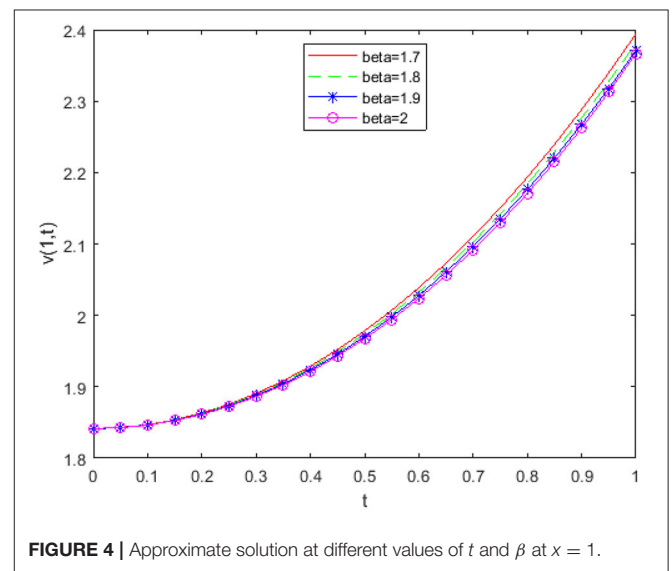
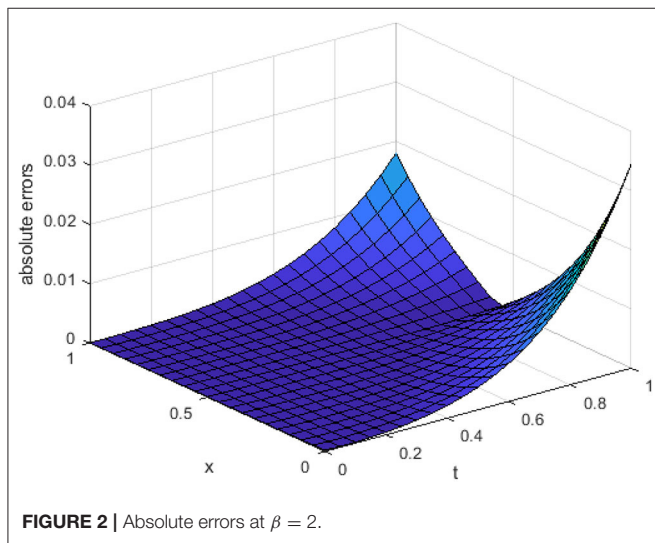
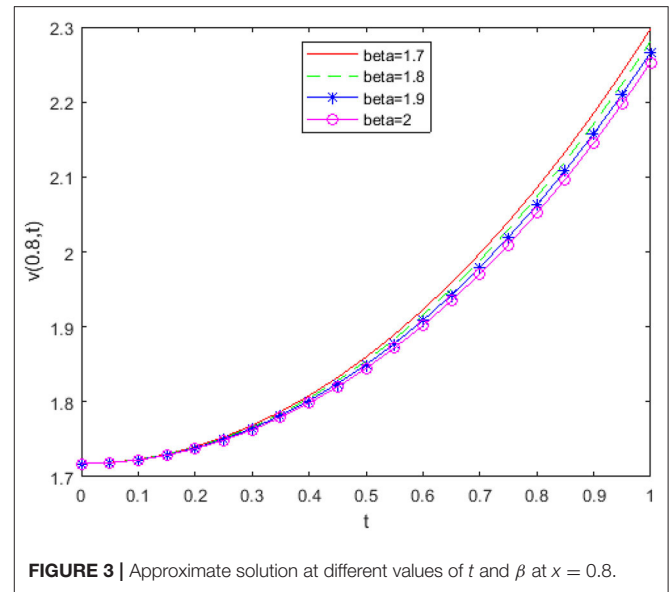
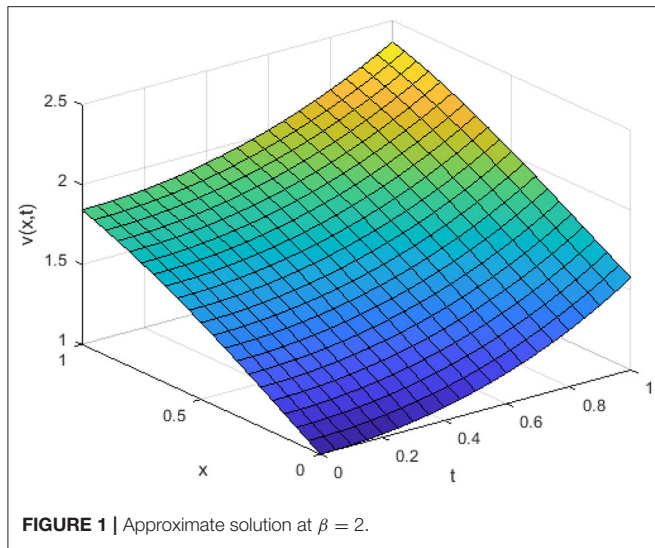
$$\frac{\partial^\beta v(x, t)}{\partial t^\beta} - \frac{\partial^2 v(x, t)}{\partial x^2} - v(x, t) = 0, \quad 1 < \beta \leq 2, \text{ having the ICs:}$$

$v(x, 0) = 1 + \sin(x)$, $\frac{\partial v(x, 0)}{\partial t} = 0$, for $0 \leq x, t \leq 1$, and boundary conditions:

$$v(0, t) = \cosh(t), \quad v(1, t) = \sin(1) + \cosh(t).$$

The exact solution is $v(x, t) = \sin(x) + \cosh(t)$.

In **Figure 1**, we have shown the three-dimensional trajectory of the approximate solution obtained by our used technique for



integer KGE. In **Figure 2**, we have shown absolute errors by our proposed method for integer order KGE at $n = 4$.

From **Figure 2** it is detected that absolute errors are very low, showing good agreement between the exact and approximate solution. In **Figure 3**, we have plotted fractional order KGE by changing the values of β and t at $x = 0.8$. In **Figure 4**, we have plotted fractional order KGE by changing the values of β and t at $x = 1$.

From **Figures 3, 4**, it can be seen that the solution changes consistently from fractional order to integer solution, showing the consistency of the proposed algorithm for time fractional order models.

Example 2. Secondly, taking the space fractional KGE [26] given as

$$\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\partial^\gamma v(x,t)}{\partial x^\gamma} = h(x,t), \quad 1 < \gamma \leq 2, \text{ having the ICs:}$$

$$v(x,0) = x^\gamma(1-x), \quad \frac{\partial v(x,0)}{\partial t} = x^\gamma(x-1), \text{ for } 0 \leq x, t \leq 1, \text{ and boundary conditions:}$$

$$v(0,t) = 0, \quad v(1,t) = 0, \text{ with source function } h(x,t) = x^\gamma(1-x)\exp(-t) - [\Gamma(\gamma+1) - \Gamma(\gamma+2)x]\exp(-t) \text{ and the exact solution } v(x,t) = x^\gamma(1-x)\exp(-t).$$

In **Figure 5**, we have shown the three-dimensional trajectory of the approximate solution obtained by our proposed method for integer KGE. In **Figures 6–8**, we have shown absolute errors by our proposed method for integer order KGE at different values of $n = 3, 5, \text{ and } 7$, respectively.

From **Figures 6–8**, it is detected that absolute errors are very low and show good agreement between the exact and approximate solution. It is also observed that absolute errors decrease when increasing the basis elements. In **Figure 9**, we have plotted fractional order KGE by changing the values of γ and x at

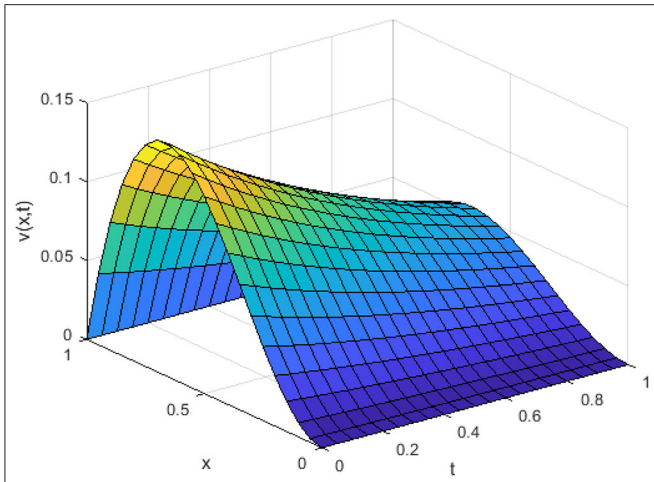


FIGURE 5 | Approximate solution at $\gamma = 2$.

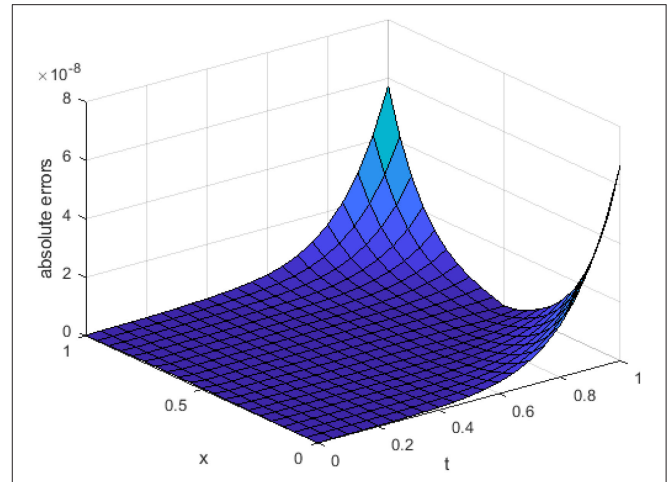


FIGURE 8 | Absolute errors at $n = 7$ and $\gamma = 2$.

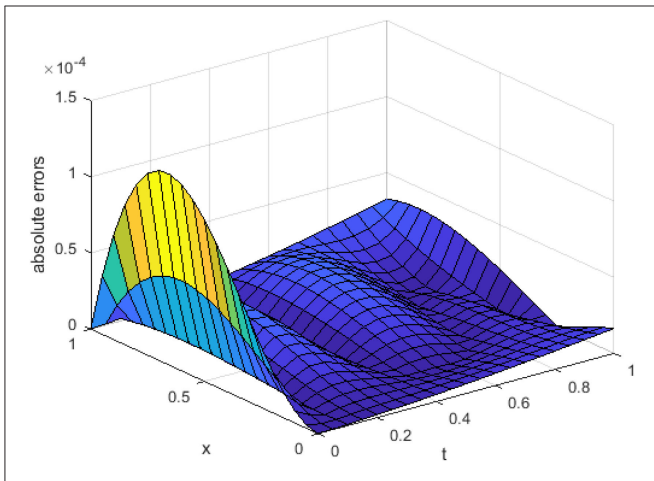


FIGURE 6 | Absolute errors at $n = 3$ and $\gamma = 2$.

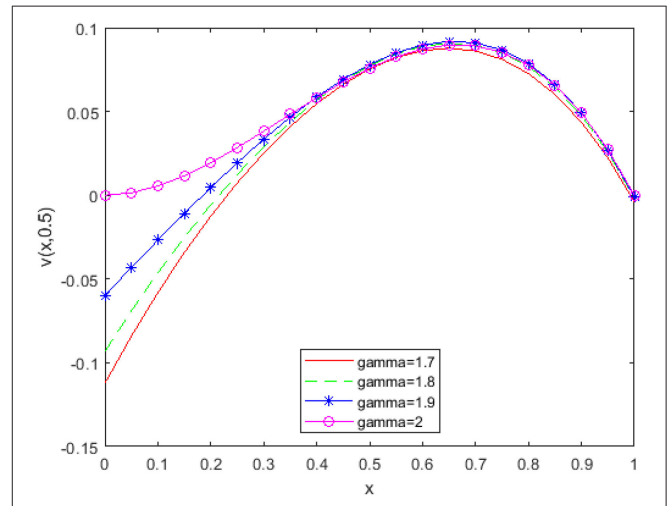


FIGURE 9 | Approximate solution at different values of x and γ at $t = 0.5$.

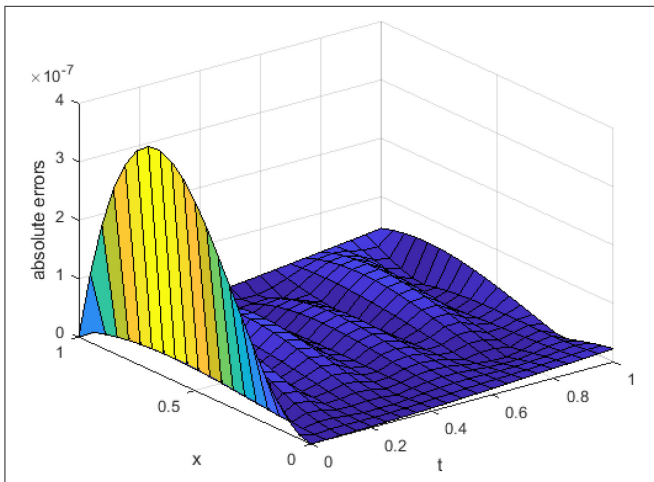


FIGURE 7 | Absolute errors at $n = 5$ and $\gamma = 2$.

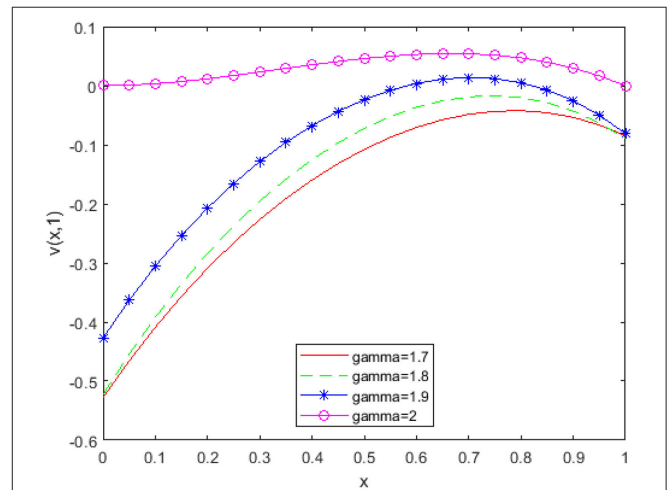


FIGURE 10 | Approximate solution at different values of x and γ at $t = 1$.

TABLE 1 | Comparison of absolute errors by our method and the method in [26] at $t = 1$, Example 2.

x	n = 3		n = 5	
	Present method	Method in [26]	Present method	Method in [26]
0.0	1.6402e-05	1.1234e-03	2.2794e-08	8.1245e-04
0.1	1.0593e-05	2.7894e-03	1.4125e-08	6.8754e-04
0.2	3.4033e-06	4.4561e-03	2.6052e-09	4.9541e-04
0.3	4.5760e-06	1.7418e-03	1.0543e-08	2.4875e-04
0.4	1.2753e-05	7.8527e-03	2.4098e-08	8.5154e-04
0.5	2.0535e-05	5.9634e-03	3.6840e-08	4.0092e-04
0.6	2.7330e-05	6.8527e-03	4.7547e-08	6.1457e-04
0.7	3.2547e-05	3.1237e-03	5.4998e-08	6.9541e-04
0.8	3.5593e-05	1.7595e-03	5.7972e-08	7.1478e-04
0.9	3.5876e-05	3.0030e-03	5.5247e-08	2.0854e-04
1.0	3.2804e-05	0.0129e-03	4.5603e-08	0.0034e-04

TABLE 2 | Comparison approximate and exact solution at $\gamma = 2$ and $n = 5$, Example 2.

(x, t)	Exact solution	Present method	Absolute errors
(0.1, 0.1)	0.00814353	0.00814354	5.9041e-09
(0.2, 0.2)	0.02619938	0.02619938	4.3087e-09
(0.3, 0.3)	0.04667154	0.04667152	2.0611e-08
(0.4, 0.4)	0.06435072	0.06435071	1.2501e-08
(0.5, 0.5)	0.07581633	0.07581635	2.4796e-08
(0.6, 0.6)	0.07902887	0.07902889	2.3619e-08
(0.7, 0.7)	0.07299803	0.07299801	2.7123e-08
(0.8, 0.8)	0.05751410	0.05751405	4.9922e-08
(0.9, 0.9)	0.03293214	0.03293211	2.3894e-08

$t = 0.5$. In **Figure 10**, we have plotted fractional order KGE by changing the values of γ and x at $t = 1$.

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From **Figures 9, 10**, it can be seen that the solution changes consistently from fractional order to integer solution, showing the consistency of the proposed algorithm for space fractional order models. In **Table 1**, we have compared absolute errors by our method and the method used in [26] and observed that our used technique is more accurate in comparison to the technique used in [26].

In **Table 2**, we have compared our solution with the exact solution for different values of x and t at $\gamma = 2$.

CONCLUDING REMARKS

The key benefit of the used algorithm is that it works for both time and space FKGE. Using the proposed algorithm, we can derive an approximate solution for FKGE when the analytical solutions are not possible. It is also easy for computational purposes because FKGE is reduced into algebraic equations. We can apply this method together for time and space fractional, which reduces the time period of computation. Integer and fractional order behavior of KGE is shown. The outcomes of the present study are very helpful for scientists and engineers working in the mathematical modeling of natural phenomena. In a nutshell, we can say that with the aid of this scheme we can examine FKGE for use in quantum field theory, plasma, optical fibers, and dispersive wave-phenomena.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

All authors have worked equally on this manuscript and have read and approved the final manuscript.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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