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A Correlation Between Solutions of Uncertain Fractional Forward Difference Equations and Their Paths

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We consider the comparison theorems for the fractional forward h-difference equations in the context of discrete fractional calculus. Moreover, we consider the existence and uniqueness theorem for the uncertain fractional forward h-difference equations. After that the relations between the solutions for the uncertain fractional forward h-difference equations with symmetrical uncertain variables and their α-paths are established and verified using the comparison theorems and existence and uniqueness theorem. Finally, two examples are provided to illustrate the relationship between the solutions.

Keywords: uncertain fractional h-difference equations, the comparison theorems, α-paths, existence and uniqueness theorem, discrete fractional calculus

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1. INTRODUCTION

The study of fractional calculus and fractional differential equations has received recent attention from both applied and theoretical disciplines. Indeed, it was observed that the use of them are very useful for modeling many problems in mathematical analysis, medical labs, engineering sciences, and integral inequalities (see for e.g., [1–14]). There is much interesting research on what is usually called integer-order difference equations (see for e.g., [15, 16]). Discrete fractional calculus and fractional difference equations represent a new branch of fractional calculus and fractional differential equations, respectively. Also, for scientists, they represent new areas that have, in their early stages, developed slowly. Some works are dedicated to boundary value problems, initial value problems, chaos, and stability for the fractional difference equations (see for e.g., [17–23]).

Besides the discrete fractional calculus, the uncertain fractional differential and difference equations have been introduced and investigated in order to model the continuous or discrete systems with memory effects and human uncertainty (see for e.g., [24–28]). In Lu and Zhu [27], the relations between uncertain fractional differential equations and the associated fractional differential equations have been created via comparison theorems for fractional differential equations of Caputo type in Lu and Zhu [26]. Lu et al. [28] presented analytic solutions to a type of special linear uncertain fractional difference equation (UFDE) by the Picard iteration method. Moreover, they provided an existence and uniqueness theorem for the solutions by applying the Banach contraction mapping theorem. After that, Mohammed [29] generalized the above work.

Nowadays, discrete fractional calculus shows incredible performance in the fields of physical and mathematical modeling. The motivation behind solving the fractional difference equations relies on fast investigation of the properties within models of fractional sum and difference operators (see for e.g., [20, 30–36]).

Motivated by the aforementioned results, we will try to create a link between uncertain fractional forward h-difference equations (UFFhDEs) and associated fractional forward

h-difference equations (FFhDEs) in the sense of Riemann–Liouville fractional operators via the comparison theorems and existence and uniqueness theorem.

The rest of our article is designed as follows. In section 2, we presented the preliminary definitions and important features that are useful in the accomplishment of this study. In section 3, the comparison theorems of the fractional differences are pointed out. Inverse uncertainty distribution, the existence and uniqueness theorem, the relation between UFFhDEs and associated FFhDEs, and some related examples are pointed out in section 4. Finally, the future scope and concluding remarks are summarized in section 5.

2. PRELIMINARIES

In what follows, we recall some results in discrete fractional calculus that has been developed in the last few years; for more details, we refer to references [24–28, 28, 29, 37, 38] and the related references therein.

Definition 2.1 ([39]). The forward difference operator on $h\mathbb{Z}$ is defined by

$$\Delta_h f(\eta) = \frac{f(\eta+h)-f(\eta)}{h},$$

and the backward difference operator on $h\mathbb{Z}$ is defined by

$$\nabla_h f(\eta) = \frac{f(\eta)-f(\eta-h)}{h}.$$

For $h = 1$, we get the classical forward and backward difference operators $\Delta\psi(\eta) = \psi(\eta+1)-\psi(\eta)$ and $\nabla\psi(\eta) = \psi(\eta) - \psi(\eta-h)$, respectively. The forward jumping operator on $h\mathbb{Z}$ is $\sigma(r) = r+h$ and the backward jumping operator is $\rho(r) = r-h$.

For $a, b \in \mathbb{R}$ with $a < b$, $\frac{b-a}{h} \in \mathbb{N}$ and $0 < h \leq 1$, we use the notations $\mathbb{N}_{a,h} = \{a, a+h, a+2h, ...\}$, ${}_{b,h}\mathbb{N} = \{b, b-h, b-2h, ...\}$.

Definition 2.2 ([39]). Let $\eta, \theta \in \mathbb{R}$ and $0 < h \leq 1$, the delta h-factorial of η is defined by

$$\eta_h^{(\theta)} = \frac{\Gamma\left(\frac{\eta}{h}+1\right)}{\Gamma\left(\frac{\eta}{h}+1-\theta\right)}, \tag{2.1}$$

where we use the convention that division at a pole yields zero and θ is the falling delta h-factorial order of η. It is worth mentioning that $\eta_h^{(\theta)}$ is a function of η for given θ and h.

Definition 2.3 ([37, 38, 40]). Let f be defined on $\mathbb{N}_{a,h}$ for the left case and ${}_{b,h}\mathbb{N}$ for the right case. Then, the left delta h-fractional sum of order $\theta > 0$ is defined by

$$\left({}_a\Delta_h^{-\theta}\psi\right)(\eta) = \int_a^{\sigma(\eta-\theta h)} (\eta-\sigma(\tau))_h^{(\theta-1)}\psi(\tau)\Delta_h\tau$$
$$= \frac{1}{\Gamma(\theta)}\sum_{r=\frac{a}{h}}^{\frac{\eta}{h}-\theta}(\eta-\sigma(rh))_h^{(\theta-1)}\psi(rh)h, \quad \eta \in \mathbb{N}_{a+\theta h,h},$$

and the right delta h-fractional sum is defined by

$$\left({}_h\Delta_b^{-\theta}\psi\right)(\eta) = \int_{\rho(\eta+\theta h)}^{b} (\rho(\tau)-\eta)_h^{(\theta-1)}\psi(\tau)\nabla_h\tau$$
$$= \frac{1}{\Gamma(\theta)}\sum_{r=\frac{\eta}{h}+\theta}^{\frac{b}{h}}(rh-\sigma(\eta))_h^{(\theta-1)}\psi(rh)h,$$
$$\eta \in {}_{b-\theta h,h}\mathbb{N}.$$

Lemma 2.1 ([40]). *Let $\theta, \mu > 0, h > 0$, and p be defined on $\in \mathbb{N}_{a,h}$. We then have*

$$\left({}_{a+\mu h}\Delta_h^{-\theta}\,{}_a\Delta_h^{-\mu}p\right)(\eta) = \left({}_a\Delta_h^{-(\mu+\theta)}p\right)(\eta)$$
$$= \left({}_{a+\theta h}\Delta_h^{-\mu}\,{}_a\Delta_h^{-\theta}p\right)(\eta), \tag{2.2}$$

for all $\eta \in \mathbb{N}_{a+(\theta+\mu)h,h}$.

Lemma 2.2 ([40]). *Let $\theta > 0$ and ψ be defined on $\mathbb{N}_{a,h}$ and ${}_{b,h}\mathbb{N}$, respectively. Then the left and right delta h-fractional differences of order θ are defined by*

$$\left({}_a\Delta_h^{\mu}\psi\right)(\eta) = \left(\Delta_h^m\,{}_a\Delta_h^{-(m-\mu)}\psi\right)(\eta), \tag{2.3}$$

$$\left({}_h\Delta_b^{\mu}\psi\right)(\eta) = (-1)^m\left(\nabla_h^m\,{}_h\Delta_b^{-(m-\mu)}\psi\right)(\eta), \tag{2.4}$$

where $m = [\theta]+1$.

Lemma 2.3 ([40]). *Let ψ be defined on $\mathbb{N}_{a,h}$, then, for any $\theta > 0$, we have*

$$\left({}_a\Delta_h^{-\theta}\,\Delta_h\psi\right)(\eta) = \Delta_h\,{}_a\Delta_h^{-\theta}\psi(\eta) - \frac{(\eta-a)_h^{(\theta-1)}}{\Gamma(\theta)}\psi(a). \tag{2.5}$$

Lemma 2.4 ([40]). *Let $\theta > 0, \mu > 0$, and $h > 0$, and we then have*

$${}_{a+\mu h}\Delta_h^{\theta}(\eta-a)_h^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\theta+1)}(\eta-a)_h^{(\theta+\mu)},$$
$${}_h\Delta_{b-\mu h}^{\theta}(b-\eta)_h^{(\theta)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\theta+1)}(b-\eta)_h^{(\theta+\mu)}.$$

Lemma 2.5 ([40]). *Let $\theta \in \mathbb{R}$ and q be any positive integer, then*

$$\left({}_a\Delta_h^{-\theta}\,\Delta_h^q\psi\right)(\eta) = \left(\Delta_h^q\,{}_a\Delta_h^{-\theta}\psi\right)(\eta)$$
$$-\sum_{k=0}^{q-1}\frac{(\eta-a)_h^{(\nu-q+k)}}{\Gamma(\nu-q+k+1)}\Delta_h^k\psi(a), \tag{2.6}$$

for $\eta \in \mathbb{N}_{a+\theta\,h,h}$.

Lemma 2.6 ([38]). *Suppose that $\frac{\mu}{h}, \frac{\mu}{h}+\theta \in \mathbb{R}\setminus\{..., -2, -1\}$, then we have*

$${}_a\Delta_h^{-\theta}(\eta-a+\mu)_h^{\left(\frac{\mu}{h}\right)} = \frac{\Gamma\left(\frac{\mu}{h}+1\right)}{\Gamma\left(\frac{\mu}{h}+\theta+1\right)}(\eta-a+\mu)_h^{\left(\frac{\mu}{h}+\theta\right)},$$

for each $\eta \in \mathbb{N}_{a+\theta\,h,h}$.

Lemma 2.7. *Let ψ be defined on $\mathbb{N}_{a,h}$ and m be a positive integer with $0 < m-1 < \mu \le m$. The definition of the fractional h-difference (2.3) is then equivalent to*

$$\left({}_a\Delta_h^{-\mu}\psi\right)(\eta) = \begin{cases} \frac{1}{\Gamma(-\mu)}\sum_{r=\frac{a}{h}}^{\frac{\eta}{h}+\mu}(\eta-\sigma(rh))_h^{(-\mu-1)}\psi(rh)h, & m-1<\mu<m, \\ {}_a\Delta_h^m p(\eta), & \mu=m, \end{cases}$$

for $\eta \in \mathbb{N}_{a,h}$.

Motivated by the definition of nth order forward sum for uncertain sequence ξ_η, we define the θth order forward sum for uncertain sequence ξ_η as follows:

Definition 2.4. Let θ be a positive real number, $a \in \mathbb{R}$, and ξ_η be an uncertain sequence indexed by $\eta \in \mathbb{N}_{a,h}$. Then,

$${}_a\Delta_h^{-\theta}\xi_\eta = \frac{1}{\Gamma(\theta)}\sum_{r=\frac{a}{h}}^{\frac{\eta}{h}-\theta}(\eta-\sigma(rh))_h^{(\theta-1)}\xi_{rh}\,h$$

is called the θth order forward fractional sum of uncertain sequence ξ_η, where $\sigma(r) = r + h$.

Definition 2.5. The fractional Riemann–Liouville-like forward difference for uncertain sequence ξ_η is defined by

$${}_a\Delta_h^{\mu}\xi_\eta = \Delta_h^n\left({}_a\Delta_h^{-(n-\mu)}\xi_\eta\right),$$

where $\theta > 0$ and $0 \le n-1 < \mu \le n$, n represents a positive integer.

3. THE COMPARISON THEOREMS

Consider the following FFhDEs:

$${}_{(\theta-n)h}\Delta_h^{\theta}\psi(\eta) = g(\eta+(\theta-n)h, \psi(\eta+(\theta-n)h)), \quad (3.1)$$

subject to the initial conditions

$${}_{(\theta-n)h}\Delta_h^{\theta-n+i}\psi(\eta)\Big|_{t=0} = \psi_i, \quad i = 0,1,...,n-1, \quad (3.2)$$

where ${}_{(\theta-n)h}\Delta_h^{\theta}$ denotes a fractional Riemann–Liouville forward h-difference with $0 \le n-1 < \theta \le n$, g is a real-valued function defined on $[0,\infty)\times\mathbb{R}$, $\eta \in \mathbb{N}_{0,h}$, and $\psi_i \in \mathbb{R}$ for $i = 0,1,...,n-1$.

Now, by applying the operator ${}_0\Delta_h^{-\theta}$ to Equation (3.1), then the initial value problem (3.1) and (3.2) is equivalent to the following fractional sum equation:

$$\psi(\eta) = \sum_{i=0}^{n-1}\frac{(\eta)_h^{(\theta-n+i)}}{\Gamma(\theta-n+i+1)}\psi_i + \frac{1}{\Gamma(\theta)}\sum_{r=0}^{\frac{\eta}{h}-\theta}\left(\eta-\sigma(rh)\right)_h^{(\theta-1)}g(r+(\theta-n)h, \psi(r+(\theta-n)h))h, \quad (3.3)$$

where we have used Lemma 2.1, Lemma 2.5, and the fact that $\Delta_h^n\Delta_h^{-n}\psi(\eta) = \psi(\eta)$.

First, a comparison theorem for Riemann–Liouville fractional h-difference equations with $\theta \in (0,1]$ will be presented.

Theorem 3.1. *Suppose $g(\eta,\psi)$ and $k(\eta,\psi)$ are two real-value functions defined on $[0,\infty]\times\mathbb{R}$. Function k is Lipschitz continuous in y with Lipschitz constant L_k that has $0 < L_k \le h^{-\theta}\theta$. If $\psi_1(\eta)$ and $\psi_2(\eta)$ are, respectively, unique solutions of the following IVPs*

$$\begin{cases} {}_{(\theta-1)h}\Delta_h^{\theta}\psi(\eta) = g(\eta+(\theta-1)h, \psi(\eta+(\theta-1)h)), & \eta \in \mathbb{N}_0, \\ {}_{(\theta-1)h}\Delta_h^{\theta-1}\psi(\eta)\Big|_{t=0} = \mathbf{X}_0, \end{cases} \quad (3.4)$$

and

$$\begin{cases} {}_{(\theta-1)h}\Delta_h^{\theta}\psi(\eta) = k(\eta+(\theta-1)h, \psi(\eta+(\theta-1)h)), & \eta \in \mathbb{N}_0, \\ {}_{(\theta-1)h}\Delta_h^{\theta-1}\psi(\eta)\Big|_{t=0} = \psi_0. \end{cases} \quad (3.5)$$

1. *if $g(\eta,\psi) \le k(\eta,\psi)$, then $\psi_1(\eta) \le \psi_2(\eta)$ for each $\eta \in \mathbb{N}_{(\theta-1)h,h}$,*
2. *if $g(\eta,\psi) > k(\eta,\psi)$, then $\psi_1(\eta) > \psi_2(\eta)$ for each $\eta \in \mathbb{N}_{\theta\,h,h}$.*

Proof: **(1)** Assume that the condition $\psi_1(\eta) \le \psi_2(\eta)$ is not valid; there thus exists $\eta_0 \in \mathbb{N}_{(\theta-1)h,h}$ such that $\psi_1(\eta_0) > \psi_2(\eta_0)$. Let $\eta_1 = \min\left\{\eta \in \mathbb{N}_{(\theta-1)h,h};\ \psi_1(\eta) > \psi_2(\eta)\right\}$ and $\mathbf{X}(\eta) = \psi_1(\eta) - \psi_2(\eta)$. Then, we have

$$\mathbf{X}(\eta_1) > 0, \quad (3.6)$$
$$\mathbf{X}(\eta) \le 0, \quad \eta \in \mathbb{N}_{(\theta-1)h,h}\cap[0,\eta_1-h]. \quad (3.7)$$

Considering the fractional sum equations equivalent to IVPs (3.4) and (3.5), we have

$$\psi_1(\theta\,h) = \theta\,h^{\nu-1}\psi_0 + h^{\theta}g((\theta-1)h, \mathbf{X}_0),$$
$$\psi_2(\theta\,h) = \theta\,h^{\nu-1}\psi_0 + h^{\theta}k((\theta-1)h, \mathbf{X}_0).$$

Subtracting these and then making use of $h^{\theta} > 0$ for $h > 0, \theta \in (0,1]$, and $g(\eta,\psi) \le k(\eta,\psi)$, we get

$$\psi_1(\theta h) - \psi_2(\theta h) = h^{\nu}\left(g((\theta-1)h, \mathbf{X}_0) - k((\theta-1)h, \mathbf{X}_0)\right) \le 0.$$

This verifies that $\eta_1 > \theta h$. From this and since $\eta_1 \in \mathbb{N}_{(\theta-1)h,h}$, we can write $\eta_1 = (\theta+\ell)h, l = 1, 2, ...$ By Lemma 2.6, we then get

$$\begin{aligned}
&{}_{(\theta-1)h}\Delta_h^{\theta}\mathbf{X}(\eta_1 - \theta h)\\
&= \frac{1}{\Gamma(-\theta)} \sum_{r=\theta-1}^{\frac{\eta_1}{h}} \left(\eta_1 - \theta h - \sigma(rh)\right)_h^{(-\theta-1)} \mathbf{X}(rh)h\\
&= \frac{1}{\Gamma(-\theta)} \sum_{r=\theta-1}^{\theta+\ell} \left(\ell h - \sigma(rh)\right)_h^{(-\theta-1)} \mathbf{X}(rh)h\\
&= h^{-\theta}\mathbf{X}((\theta+\ell)h) - \theta h^{-\theta}\mathbf{X}((\theta+\ell-1)h)\\
&+ \frac{1}{\Gamma(-\theta)} \sum_{r=\theta-1}^{\theta+\ell-2} \left(\ell h - \sigma(rh)\right)_h^{(-\theta-1)} \mathbf{X}(rh)h.
\end{aligned}$$

That is,

$$\begin{aligned}
h^{-\theta}\mathbf{X}((\theta+\ell)h) &= {}_{(\theta-1)h}\Delta_h^{\theta}\mathbf{X}(\eta_1 - \theta h) + \theta h^{-\theta}\mathbf{X}((\theta+\ell-1)h)\\
&- \frac{1}{\Gamma(-\theta)} \sum_{r=\theta-1}^{\theta+\ell-2} \left(\ell h - \sigma(rh)\right)_h^{(-\theta-1)} \mathbf{X}(rh)h.
\end{aligned} \tag{3.8}$$

Now, by using the Lipschitz continuity of k in y, $g(\eta, x) \leq k(\eta, x)$, and (3.7), we get

$$\begin{aligned}
{}_{(\theta-1)h}\Delta_h^{\theta}\mathbf{X}(\eta_1 - \theta h) &= {}_{(\theta-1)h}\Delta_h^{\theta}\psi_1(\eta_1 - \theta h)\\
&- {}_{(\theta-1)h}\Delta_h^{\theta}\psi_2(\eta_1 - \theta h)\\
&= g(\eta_1 - h, \psi_1(\eta_1 - h))\\
&- k(\eta_1 - h, \psi_2(\eta_1 - h))\\
&\leq k(\eta_1 - h, \psi_1(\eta_1 - h))\\
&- k(\eta_1 - h, \psi_2(\eta_1 - h))\\
&\leq -L_k\left(\psi_1(\eta_1 - h) - \psi_2(\eta_1 - h)\right)\\
&\leq -L_k\mathbf{X}(\eta_1 - h).
\end{aligned}$$

Denoting $\omega(\eta_1 - h) := {}_{(\theta-1)h}\Delta_h^{\theta}\mathbf{X}(\eta_1 - \theta h) + L_k\mathbf{X}(\eta_1 - h)$, it follows that

$$\omega((\theta+\ell-1)h) \leq 0. \tag{3.9}$$

This gives

$${}_{(\theta-1)h}\Delta_h^{\theta}\mathbf{X}(\eta_1 - \theta h) = -L_k\mathbf{X}((\theta+\ell-1)h) + \omega((\theta+\ell-1)h).$$

Thus, Equation (3.8) becomes

$$\begin{aligned}
h^{-\theta}\mathbf{X}((\theta+\ell)h) &= \left(\theta h^{-\theta} - L_k\right)\mathbf{X}((\theta+\ell-1)h) + \omega((\theta+\ell-1)h)\\
&- \frac{1}{\Gamma(-\theta)} \sum_{r=\theta-1}^{\theta+\ell-2} \left(\ell h - \sigma(rh)\right)_h^{(-\theta-1)} \mathbf{X}(rh)h.
\end{aligned} \tag{3.10}$$

We write $r = \nu - 1 + i, \ i = 0, 1, ..., \ell - 1$ to obtain

$$\begin{aligned}
\frac{\left(\ell h - \sigma(rh)\right)_h^{(-\theta-1)}}{\Gamma(-\theta)} &= \frac{\left(\ell h - (\theta+i)h\right)_h^{(-\theta-1)}}{\Gamma(-\theta)}\\
&= h^{-\theta-1}\frac{\Gamma(\ell - i + 1 - \theta)}{\Gamma(-\theta)\Gamma(\ell - i + 2)}\\
&= h^{-\theta-1}\frac{(\ell - i - \theta)(\ell - i - 1 - \theta)\cdots(-\theta)\Gamma(-\theta)}{\Gamma(-\theta)\Gamma(\ell - i + 2)}\\
&= h^{-\theta-1}\frac{(-\theta)(-\theta+1)\cdots(\ell - i - 1 - \theta)(\ell - i - \theta)}{\Gamma(\ell - i + 2)}\\
&= h^{-\theta-1}\frac{(-\theta)(-\theta+1)\cdots(-\theta-1+c)(-\theta+c)}{\Gamma(c+1)},
\end{aligned}$$

where $c = \ell - i$.

Since $\theta \in (0, 1]$ and $h^{-\theta-1} > 0$, so

$$\frac{\left(\ell h - \sigma(rh)\right)_h^{(-\theta-1)}}{\Gamma(-\theta)} \leq 0. \tag{3.11}$$

Considering $L_k < \theta h^{-\theta}$, $h^{-\theta} > 0$ and Equations (3.9)–(3.11), it follows that

$$h^{-\theta}\mathbf{X}((\theta+\ell)h) \leq 0.$$

This implies that $\mathbf{X}(\eta_1) \leq 0$, which contradicts with (3.6).

(2) By the same technique of (1), we assume that the condition $\psi_1(\eta) > \psi_2(\eta)$ is not valid. There thus exists $\eta_2 \in \mathbb{N}_{\theta h,h}$, such that $\psi_1(\eta_2) \leq \psi_2(\eta_2)$. Let $\eta_3 = \min\left\{\eta \in \mathbb{N}_{\theta h,h};\ \psi_1(\eta) \leq \psi_2(\eta)\right\}$ and $z(\eta) = \psi_2(\eta) - \psi_1(\eta)$. We then have

$$z(\eta_3) \geq 0, \tag{3.12}$$

$$z(\eta) < 0, \quad \eta \in \mathbb{N}_{\theta h,h} \cap [0, \eta_3 - h]. \tag{3.13}$$

Considering the fractional sum equations equivalent to IVPs (3.4) and (3.5), $h^{\theta} > 0$ and $g(\eta, \psi) > k(\eta, \psi)$, we find $\psi_1(\theta h) > \psi_2(\theta h)$. That is; $\eta_3 > \theta h$. If we write $\eta_3 = (\theta+\ell)h, l = 1, 2, ...,$ then, by Lemma 2.6, we get

$$\begin{aligned}
&{}_{(\theta-1)h}\Delta_h^{\theta}z(\eta_3 - \theta h)\\
&= \frac{1}{\Gamma(-\theta)} \sum_{r=\theta-1}^{\frac{\eta_3}{h}} \left(\eta_3 - \theta h - \sigma(rh)\right)_h^{(-\theta-1)} z(rh)h\\
&= h^{-\theta}z((\theta+\ell)h) - \theta h^{-\theta}z((\theta+\ell-1)h)\\
&+ \frac{1}{\Gamma(-\theta)} \sum_{r=\theta-1}^{\theta+\ell-2} \left(\ell h - \sigma(rh)\right)_h^{(-\theta-1)} z(rh)h,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
h^{-\theta}z((\theta+\ell)h) &= {}_{(\theta-1)h}\Delta_h^{\theta}z(\eta_3 - \theta h) + \theta h^{-\theta}z((\theta+\ell-1)h)\\
&- \frac{1}{\Gamma(-\theta)} \sum_{r=\theta-1}^{\theta+\ell-2} \left(\ell h - \sigma(rh)\right)_h^{(-\theta-1)} z(rh)h.
\end{aligned} \tag{3.14}$$

Now, by using the Lipschitz continuity of k in y, $g(\eta, z) > k(\eta, z)$, and (3.13), we get

$$\begin{aligned}
{}_{(\theta-1)h}\Delta_h^{\theta} z(\eta_3 - \theta\, h) &= {}_{(\theta-1)h}\Delta_h^{\theta} \psi_1(\eta_3 - \theta\, h) \\
&- {}_{(\theta-1)h}\Delta_h^{\theta} \psi_2(\eta_3 - \theta\, h) \\
&= w(\eta_3 - h, \psi_2(\eta_3 - h)) \\
&- k(\eta_3 - h, \psi_1(\eta_3 - h)) \\
&\leq k(\eta_3 - h, \psi_2(\eta_3 - h)) \\
&- k(\eta_3 - h, \psi_1(\eta_3 - h)) \\
&\leq -L_k \left(\psi_2(\eta_3 - h) - \psi_1(\eta_3 - h)\right) \\
&\leq -L_k z(\eta_3 - h).
\end{aligned}$$

Denoting $w(\eta_3 - h) := {}_{(\theta-1)h}\Delta_h^{\theta} z(\eta_3 - \theta\, h) + L_k z(\eta_3 - h)$, it follows that

$$w((\theta + \ell - 1)h) \leq 0. \tag{3.15}$$

This gives

$${}_{(\theta-1)h}\Delta_h^{\theta} z(\eta_3 - \theta\, h) = -L_k z((\theta + \ell - 1)h) + w((\theta + \ell - 1)h).$$

Equation (3.14) thus becomes

$$\begin{aligned}
h^{-\theta} z((\theta + \ell)h) &= \left(\theta h^{-\theta} - L_k\right) z((\theta + \ell - 1)h) + w((\theta + \ell - 1)h) \\
&- \frac{1}{\Gamma(-\theta)} \sum_{r=\theta-1}^{\theta+\ell-2} \left(\ell\, h - \sigma(rh)\right)_h^{(-\theta-1)} z(rh)h.
\end{aligned} \tag{3.16}$$

Similarly for $\theta \in (0, 1]$ and $h^{-\theta-1} > 0$, we can show that

$$\frac{\left(\ell\, h - \sigma(rh)\right)_h^{(-\theta-1)}}{\Gamma(-\theta)} \leq 0. \tag{3.17}$$

Considering $L_k < \theta\, h^{-\theta}$, $h^{-\theta} > 0$ and Equations (3.15)–(3.17), it follows that

$$h^{-\theta} z((\theta + \ell)h) \leq 0.$$

This implies that $z(\eta_3) \leq 0$, which contradicts with (3.12). The proof of Theorem 3.1 is thus completed. □

In the sequel, we will extend a comparison theorem for Riemann-Liouville fractional h-difference equations of the order θ with $0 \leq n - 1 < \theta \leq n$.

Theorem 3.2. *Suppose $g(\eta, \psi)$, and $k(\eta, \psi)$ are two real-value functions defined on $[0, \infty] \times \mathbb{R}$. Function k is Lipschitz continuous in y with a Lipschitz constant L_k that has $0 < L_k \leq h^{-\theta}\theta$. If $\psi_1(\eta)$ and $\psi_2(\eta)$ are, respectively, unique solutions of the following IVPs*

$$\begin{cases}
{}_{(\theta-n)h}\Delta_h^{\theta} \psi(\eta) = g(\eta + (\theta - n)h, \psi(\eta + (\theta - n)h)), \quad \eta \in \mathbb{N}_0, \\
{}_{(\theta-n)h}\Delta_h^{\theta-n+i} \psi(\eta)\Big|_{t=0} = \psi_i, i = 0, 1, ..., n - 1
\end{cases} \tag{3.18}$$

and

$$\begin{cases}
{}_{(\theta-n)h}\Delta_h^{\theta} \psi(\eta) = k(\eta + (\theta - n)h, \psi(\eta + (\theta - n)h)), \quad \eta \in \mathbb{N}_0, \\
{}_{(\theta-n)h}\Delta_h^{\theta-n+i} \psi(\eta)\Big|_{t=0} = \psi_i, i = 0, 1, ..., n - 1.
\end{cases} \tag{3.19}$$

1. *if $g(\eta, \psi) \leq k(\eta, \psi)$, then $\psi_1(\eta) \leq \psi_2(\eta)$ for each $\eta \in \mathbb{N}_{(\theta-n)h,h}$,*
2. *if $g(\eta, \psi) > k(\eta, \psi)$, then $\psi_1(\eta) > \psi_2(\eta)$ for each $\eta \in \mathbb{N}^h_{(\theta-n+1)h}$.*

Proof: **(1)** For $\mu = \theta - n + 1 \in (0, 1]$ and $\eta \in \mathbb{N}_{0,h}$, we have ${}_{(\theta-n)h}\Delta_h^{\theta} \psi(\eta) = \Delta_h^{n-1}\, {}_{(\mu-1)h}\Delta_h^{\mu} \psi(\eta)$. By using Lemma 2.5, the IVPs (3.18) and (3.19) can be easily converted to the following IVPs, respectively,

$$\begin{cases}
{}_{(\mu-1)h}\Delta_h^{\mu} \psi(\eta) = \frac{1}{\Gamma(n-1)} \sum\limits_{r=0}^{\frac{\eta}{h}-(n-1)} \left(\eta - \sigma(rh)\right)_h^{(\mu-2)} g(r + (\mu - 1)h, \\
\qquad \psi(r + (\mu - 1)h))h + \sum\limits_{i=0}^{n-2} \frac{(\eta)_h^{(i)}}{\Gamma(i+1)} \psi_{i+1}, \\
{}_{(\mu-1)h}\Delta_h^{\mu-1} \psi(\eta)\Big|_{t=0} = \psi_0,
\end{cases} \tag{3.20}$$

and

$$\begin{cases}
{}_{(\mu-1)h}\Delta_h^{\mu} \psi(\eta) = \frac{1}{\Gamma(n-1)} \sum\limits_{r=0}^{\frac{\eta}{h}-(n-1)} \left(\eta - \sigma(rh)\right)_h^{(\mu-2)} k(r + (\mu - 1)h, \\
\qquad \psi(r + (\mu - 1)h))h + \sum\limits_{i=0}^{n-2} \frac{(\eta)_h^{(i)}}{\Gamma(i+1)} \psi_{i+1}, \\
{}_{(\mu-1)h}\Delta_h^{\mu-1} \psi(\eta)\Big|_{t=0} = \psi_0.
\end{cases} \tag{3.21}$$

Denote

$$\begin{aligned}
\bar{g}(\eta, x) &= \frac{1}{\Gamma(n-1)} \sum_{r=0}^{\frac{\eta}{h}-(n-1)} \left(\eta - \sigma(rh)\right)_h^{(n-2)} g(r + (\mu - 1)h, \\
&\psi(r + (\mu - 1)h))h + \sum_{i=0}^{n-2} \frac{(\eta)_h^{(i)}}{\Gamma(i+1)} \psi_{i+1},
\end{aligned}$$

and

$$\begin{aligned}
\bar{k}(\eta, x) &= \frac{1}{\Gamma(n-1)} \sum_{r=0}^{\frac{\eta}{h}-(n-1)} \left(\eta - \sigma(rh)\right)_h^{(n-2)} k(r + (\mu - 1)h, \\
&\psi(r + (\mu - 1)h))h + \sum_{i=0}^{n-2} \frac{(\eta)_h^{(i)}}{\Gamma(i+1)} \psi_{i+1},
\end{aligned}$$

for $\eta \in \mathbb{N}_{(\theta-1)h,h}$. These give

$$\bar{g}(\eta,x)-\bar{k}(\eta,x)=\frac{1}{\Gamma(n-1)}\sum_{r=0}^{\frac{\eta}{h}-(n-1)}\left(\eta-\sigma(rh)\right)_h^{(n-2)} \times\Big[g(r+(\mu-1)h,\psi(r+(\mu-1)h))-k(r+(\mu-1)h, \psi(r+(\mu-1)h))\Big]h. \quad (3.22)$$

Since $g(\eta,\psi)\leq k(\eta,\psi)$ and

$$\frac{\left(\eta-\sigma(rh)\right)_h^{(n-2)}}{\Gamma(n-1)}=\frac{\left(\eta-(r+1)h\right)_h^{(n-2)}}{\Gamma(n-1)} = h^{-n-2}\frac{\Gamma\left(\frac{\eta}{h}-r\right)}{\Gamma(n-1)\Gamma\left(\frac{\eta}{h}-r-n+2\right)} = h^{-n-2}\frac{\Gamma(c)}{\Gamma(n-1)\Gamma(c-n+2)},$$
$$\text{where } c=\frac{\eta}{h}-r, r=0,1,...,\frac{\eta}{h}-n+1>0,$$

it follows from (3.22) that $\bar{g}(\eta,\psi)\leq\bar{k}(\eta,\psi)$ for $\eta\in\mathbb{N}_{(\theta-1)h,h}$. Then, by applying Theorem 3.1 for the above findings, we get $\psi_1(\eta)\leq\psi_2(\eta)$ for $\eta\in\mathbb{N}_{(\theta-n)h,h}$. Hence, the proof of the first item is completed.

(2) Analogously, we can obtain the proof of this item, and thus our proof is completely done. □

4. INVERSE UNCERTAINTY DISTRIBUTION

In this section, we make a link between the solution for an UFFhDE and the solution for the associated FFhDE; we firstly define a symmetrical uncertain variable and α-path for an UFFhDE in view of Lu and Zhu [27]. After that, we state and verify a theorem that demonstrates a link between solution for the UFFhDE with symmetrical uncertain variables and its α-path via the comparison theorems in section 3. To understand the theory of inverse uncertainty distribution, we advise the readers to read [41] carefully.

First, we recall the inverse uncertainty distribution theory:

Definition 4.1 ([41])**.** An uncertainty distribution Ψ is called regular if it is a continues and strictly increasing function and satisfies

$$\lim_{x\to-\infty}\Psi(x)=0,\quad \lim_{x\to+\infty}\Psi(x)=1. \quad (4.1)$$

Definition 4.2 ([41])**.** Let ξ be an uncertain variable with a regular uncertainty distribution Ψ. Then, the inverse function Ψ^{-1} is called the inverse uncertainty distribution of ξ.

Example 4.1. From definition 4.2, we deduce that

(i) the inverse uncertainty distribution of a linear uncertain variable $\mathcal{L}(a,b)$ is given by

$$\Psi^{-1}(\theta)=(1-\theta)a+\theta\, b; \quad (4.2)$$

(ii) the inverse uncertainty distribution of a normal uncertain variable $\mathcal{N}(e,\sigma)$ is given by

$$\Psi^{-1}(\theta)=e+\frac{\sqrt{3}\,\sigma}{\pi}\ln\left(\frac{\theta}{1-\theta}\right); \quad (4.3)$$

(iii) and the inverse uncertainty distribution of a normal uncertain variable $\mathcal{LOGN}(e,\sigma)$ is given by

$$\Psi^{-1}(\theta)=\exp(e)+\left(\frac{\theta}{1-\theta}\right)^{\frac{\sqrt{3}\sigma}{\pi}}. \quad (4.4)$$

Definition 4.3 ([41])**.** We say that an uncertain variable ξ is symmetrical if

$$\Psi(x)+\Psi(-x)=1, \quad (4.5)$$

where $\Psi(x)$ is a regular uncertainty distribution of ξ.

Remark 4.1. *From definition 4.3, we can deduce that the symmetrical uncertain variable has the inverse uncertainty distribution $\Psi^{-1}(\theta)$, which satiates*

$$\Psi^{-1}(\theta)+\Psi^{-1}(1-\theta)=0. \quad (4.6)$$

Example 4.2. From definition 4.3, we deduce the following:

1. the linear uncertain variable $\mathcal{L}(-a,a)$ is symmetrical for any positive real number a.
2. The normal uncertain variable $\mathcal{N}(0,1)$ is symmetrical.

Consider the following UFFhDE with Riemann-Liouville-like forward difference:

$$_{(\theta-n)h}\Delta_h^{\theta}\mathbf{X}(\eta)=\mathbf{F}(\eta+(\theta-n)h,\mathbf{X}(\eta+(\theta-n)h)) +\mathbf{G}(\eta+(\theta-n)h,\mathbf{X}(\eta+(\theta-n)h))\xi_{\eta+(\theta-n)h}, \quad (4.7)$$

subject to the crisp initial conditions

$$_{(\theta-n)h}\Delta_h^{\theta-n-k}\mathbf{X}(\eta)\Big|_{t=0}=\mathbf{X}_k,\quad k=0,1,...,n-1, \quad (4.8)$$

where $_{(\theta-n)h}\Delta_h^{\theta}$ denotes a fractional Riemann–Liouville forward h-difference with $0\leq n-1<\theta\leq n$, M,N are two real-valued functions defined on $[0,\infty)\times\mathbb{R}$, $\eta\in\mathbb{N}_{0,h}\cap[0,Th]$, $\mathbf{X}_k\in\mathbb{R}$ for $k=0,1,...,n-1$, and $\xi_{(\theta-n)h},\xi_{(\theta-n+1)h},\cdots,\xi_{\eta+(\theta-n)h}$ are i.i.d. uncertain variables with symmetrical uncertainty distribution $\mathcal{L}(a,b)$.

Definition 4.4 ([41])**.** An UFFhDE (4.7) with crisp initial conditions (4.8) is said to have an α-path if it is the solution of the corresponding FFhDE

$$_{(\theta-n)h}\Delta_h^{\theta}\mathbf{X}(\eta)=\mathbf{F}(\eta+(\theta-n)h,\mathbf{X}(\eta+(\theta-n)h)) +|\mathbf{G}(\eta+(\theta-n)h,\mathbf{X}(\eta+(\theta-n)h))|\Psi^{-1}(\theta) \quad (4.9)$$

with the same initial conditions (4.8), where $\Psi^{-1}(\theta)$ is the inverse uncertainty distribution of uncertain variables ξ_η for $\eta\in\mathbb{N}_{(\theta-n)h,h}\cap[0,Th]$.

Theorem 4.1. *Let $\eta \in \mathbb{N}_{0,h} \cap [0, Th], n \in \mathbb{N}, \lambda \in (0,1)$ and $\theta \in (0,1]$. The linear UFFhDE:*

$$_{(\theta-n)h}\Delta_h^{\theta}\mathbf{X}(\eta) = \lambda\,\mathbf{X}(\eta + (\theta - n)h) + \lambda\,\xi_{\eta+(\theta-n)h},$$

with the initial conditions

$$_{(\theta-n)h}\Delta_h^{\theta-n-i}\mathbf{X}(\eta)\Big|_{t=0} = \mathbf{X}_i, \quad i = 0,1,...,n-1,$$

has a solution

$$\mathbf{X}(\eta) = \mathbf{X}_i F_{\mu,\lambda;h}(\eta) + \xi_\eta, \quad , \; i = 0,1,...,n-1,$$

where ξ_η is an uncertain sequence with the uncertainty distribution $\mathcal{L}\left(a \cdot e_{\theta,\lambda;h}(\eta), b \cdot e_{\theta,\lambda;h}(\eta)\right)$, and

$$F_{\theta,\lambda;h}(\eta) = \sum_{k=0}^{\infty}\lambda^k \sum_{i=0}^{n-1} \frac{(\eta + k(\theta-n)h)_h^{((k+1)\theta\, h - nh + i)}}{\Gamma((k+1)\theta - n + i + 1)},$$

and

$$e_{\theta,\lambda;h}(\eta) = \sum_{k=1}^{\infty}\lambda^k \frac{(\eta + (k-1)(\theta-n)h)_h^{(k\theta)}}{\Gamma(k\theta+1)}.$$

Proof: By making the use of Lemma 2.5, we can easily prove this theorem by the similar technique of [29, Theorem 3.1], so it is omitted. □

Example 4.3. Consider the following UFFhDE:

$$_{(\theta-1)h}\Delta_h^{\theta}\mathbf{X}(\eta) = \lambda\,\mathbf{X}(\eta + (\theta-1)h) + \lambda\,\xi_{\eta+(\theta-1)h},$$
$$\eta \in \mathbb{N}_{0,h} \cap [0, Th], \lambda \in (0,1), \theta \in (0,1], \quad (4.10)$$

where $\xi_{(\theta-1)h}, \xi_{\theta\, h}, \ldots, \xi_{\eta+(\theta-1)h}$ are i.i.d linear uncertain variable $\mathcal{L}(-2,2)$, which has the inverse uncertainty distribution $\Psi^{-1}(\theta) = 4\theta - 2$ by (4.2).

By Theorem 4.1, the associated FFhDE of (4.10) with its initial condition

$$_{(\theta-1)h}\Delta_h^{\theta}\mathbf{X}(\eta) = \lambda\,\mathbf{X}(\eta + (\theta-1)h) + \lambda\,\Psi^{-1}(\theta),$$
$$_{(\theta-1)h}\Delta_h^{\theta-1}\mathbf{X}(\eta)\Big|_{t=0} = \mathbf{X}_0$$

has a solution

$$\mathbf{X}(\eta) = \mathbf{X}_0 \sum_{k=0}^{\infty}\lambda^k \frac{(\eta + k(\theta-1)h)_h^{((k+1)\theta-1)}}{\Gamma((k+1)\theta)} + \sum_{k=1}^{\infty}\lambda^k \frac{(\eta + (k-1)(\theta-1)h)_h^{(k\theta)}}{\Gamma(k\theta+1)}(4\theta-2).$$

The UFFhDE (4.10) has an α-path

$$\mathbf{X}_\eta^{\theta} = \mathbf{X}_0 \sum_{k=0}^{\infty}\lambda^k \frac{(\eta + k(\theta-1)h)_h^{((k+1)\theta-1)}}{\Gamma((k+1)\theta)} + \sum_{k=1}^{\infty}\lambda^k \frac{(\eta + (k-1)(\theta-1)h)_h^{(k\theta)}}{\Gamma(k\theta+1)}(4\theta-2).$$

with the initial condition $_{(\theta-1)h}\Delta_h^{\theta-1}\mathbf{X}(\eta)\Big|_{t=0} = \mathbf{X}_0$.

Example 4.4. Consider the following UFFhDE:

$$_{(\theta-2)h}\Delta_h^{\theta}\mathbf{X}(\eta) = q\,\mathbf{X}(\eta + (\theta-2)h) + q\,\xi_{\eta+(\theta-1)h},$$
$$\eta \in \mathbb{N}_{0,h} \cap [0, Th], q \in (0,1), \theta \in (0,1], \quad (4.11)$$

where $\xi_{(\theta-2)h}, \xi_{(\theta-1)\,h}, \ldots, and \xi_{\eta+(\theta-2)h}$ are the i.i.d normal uncertain variable $\mathcal{N}(0,1)$, which has the inverse uncertainty distribution $\Psi^{-1}(\theta) = \frac{\sqrt{3}}{\pi}\ln\left(\frac{\theta}{1-\theta}\right)$ by (4.2).

By Theorem 4.1, the associated FFhDE of (4.11) with its initial condition

$$_{(\theta-2)h}\Delta_h^{\theta}\mathbf{X}(\eta) = q\mathbf{X}(\eta + (\theta-2)h) + q\Psi^{-1}(\theta),$$
$$_{(\theta-2)h}\Delta_h^{\theta-2+i}\mathbf{X}_i(\eta)\Big|_{t=0} = \mathbf{X}_i, \quad i = 0,1$$

has a solution

$$\mathbf{X}(\eta) = \sum_{k=0}^{\infty} q^k \sum_{i=0}^{1}\mathbf{X}_i \frac{(\eta + k(\theta-2)h)_h^{((k+1)\theta\, h - 2h + i)}}{\Gamma((k+1)\theta - 1 + i)} + \frac{\sqrt{3}}{\pi}\ln\left(\frac{\theta}{1-\theta}\right)\sum_{k=1}^{\infty} q^k \frac{(\eta + (k-1)(\theta-2)h)_h^{(k\theta)}}{\Gamma(k\theta+1)}.$$

The UFFhDE (4.11) has an α-path

$$\mathbf{X}_\eta^{\theta} = \sum_{k=0}^{\infty} q^k \sum_{i=0}^{1}\mathbf{X}_i \frac{(\eta + k(\theta-2)h)_h^{((k+1)\theta\, h - 2h + i)}}{\Gamma((k+1)\theta - 1 + i)} + \frac{\sqrt{3}}{\pi}\ln\left(\frac{\theta}{1-\theta}\right)\sum_{k=1}^{\infty} q^k \frac{(\eta + (k-1)(\theta-2)h)_h^{(k\theta)}}{\Gamma(k\theta+1)}.$$

with the initial condition $_{(\theta-2)h}\Delta_h^{\theta-2+i}\mathbf{X}_i(\eta)\Big|_{t=0} = \mathbf{X}_i$, i=0,1.

In the following theorem, we make a relationship between uncertain fractional forward h-difference equations (UFFhDEs) and fractional h-difference equations (FFhDEs) based on the comparison theorems in section 3.

Theorem 4.2. *If $\mathbf{X}_\eta$ and $\mathbf{X}_\eta^{\theta}$ are the unique solution and α-path of UFFhDE (4.7) with the initial conditions (4.8), respectively. Assume that $\mathbf{F} + |\mathbf{G}|\Psi^{-1}(\theta)$ is a Lipschitz continues function in x with a Lipschitz constant L_k that has $0 < L_k < \theta\, h^{-\theta}$. Assume that ξ_η is the i.i.d. symmetrical uncertain variable for $\eta \in \mathbb{N}_{(\theta-(n-1))h,h}^{h} \cap [0, Th]$, then*

(i) $\mathbf{X}_\eta \le \mathbf{X}_\eta^{\theta}$ *if* $\xi_\eta(\gamma) \le \Psi^{-1}(\theta)$ *for* $\eta \in \mathcal{D}^+$ *and* $\xi_\eta(\gamma) \ge \Psi^{-1}(1-\theta)$ *for* $\eta \in \mathcal{D}^-$*, where*

$$\mathcal{D}^+ = \left\{\eta \in \mathbb{N}_{(\theta-(n-1))h,h} \cap [0, Th];\; \mathbf{G}(\eta, x) \ge 0\right\},$$

and

$$\mathcal{D}^- = \left\{\eta \in \mathbb{N}_{(\theta-(n-1))h,h} \cap [0, Th];\; \mathbf{G}(\eta, x) < 0\right\},$$

(ii) $\mathbf{X}_\eta > \mathbf{X}_\eta^\theta$ *if* $\xi_\eta(\gamma) > \Psi^{-1}(\theta)$ *for* $\eta \in \mathcal{D}^+$ *and* $\xi_\eta(\gamma) < \Psi^{-1}(1-\theta)$ *for* $\eta \in \mathcal{D}^-$.

Proof: First, we let $\xi_\eta(\gamma) \le \Psi^{-1}(\theta)$ for $\eta \in \mathcal{D}^+$. Then $\eta \in \mathbb{N}_{(\theta-(n-1))h,h} \cap [0, Th]$ and $\mathbf{G}(\eta, x) \ge 0$. Therefore,

$$\mathbf{G}(\eta,x)\xi_\eta(\gamma) \le |\mathbf{G}(\eta,x)|\Psi^{-1}(\theta). \tag{4.12}$$

Moreover, if $\xi_\eta(\gamma) \ge \Psi^{-1}(1-\theta)$ for $\eta \in \mathcal{D}^-$, we have $\eta \in \mathbb{N}_{(\theta-(n-1))h,h} \cap [0, Th]$ and $\mathbf{G}(\eta, x) < 0$. Since ξ_η is symmetrical, we have $\Psi^{-1}(\theta) + \Psi^{-1}(1-\theta) = 0$. Thus,

$$\begin{aligned}\mathbf{G}(\eta,x)\xi_\eta(\gamma) &\le \mathbf{G}(\eta,x)\Psi^{-1}(1-\theta) = -\mathbf{G}(\eta,x)\Psi^{-1}(\theta)\\ &= |\mathbf{G}(\eta,x)|\Psi^{-1}(\theta).\end{aligned} \tag{4.13}$$

Since $\mathbf{X}_\eta(\gamma)$ and $\mathbf{X}_\eta^\theta$ are the unique solution and α-path of UFFhDE (4.7) with the initial conditions (4.8), respectively, we have

$$\begin{aligned}{}_{(\theta-n)h}\Delta_h^\theta \mathbf{X}(\eta) &= \mathbf{F}(\eta+(\theta-n)h, \mathbf{X}(\eta+(\theta-n)h))\\ &\quad + \mathbf{G}(\eta+(\theta-n)h, \mathbf{X}(\eta+(\theta-n)h))\xi_{\eta+(\theta-n)h}(\gamma),\end{aligned} \tag{4.14}$$

$$\begin{aligned}{}_{(\theta-n)h}\Delta_h^\theta \mathbf{X}(\eta) &= \mathbf{F}(\eta+(\theta-n)h, \mathbf{X}(\eta+(\theta-n)h))\\ &\quad + |\mathbf{G}(\eta+(\theta-n)h, \mathbf{X}(\eta+(\theta-n)h))|\Psi^{-1}(\theta).\end{aligned} \tag{4.15}$$

Hence, by use of Theorem 3.2 with (4.12)–(4.15), we get the proof of item (i). The proof of the second item (ii) is similar to (i). Thus, the proof of Theorem 4.2 is completed. □

Theorem 4.3 (Existence and Uniqueness)**.** *Assume that* $\mathbf{F}(\eta, x)$ *and* $\mathbf{G}(\eta, x)$ *satisfy the Lipschitz condition*

$$|\mathbf{F}(\eta,x)-\mathbf{F}(\eta,\psi)| + |\mathbf{G}(\eta,x)-\mathbf{G}(\eta,\psi)| \le L|x-y|, \tag{4.16}$$

and there is a positive number L that satisfies the following inequality:

$$L < h^{-\theta-1}\frac{\Gamma(\theta+1)\,\Gamma(T+1-\theta)}{\Gamma(T+1)\,(Q+1)}, \tag{4.17}$$

where $Q = |a| \vee |b|$. *Then UFFhDE (4.7) with the initial conditions (4.8) has a unique solution* $\mathbf{X}(\eta)$ *for* $\eta \in \mathbb{N}_{\theta\,h,h} \cap [0, Th]$.

Proof: Proof of this theorem is similar to the existence and uniqueness theorem [29, Theorem 3.2], and it is therefore omitted. □

Example 4.5. Consider the following UFFhDE:

$${}_{-1}\Delta_2^{0.5}\mathbf{X}(\eta) = \frac{\sin \mathbf{X}(\eta-1)}{50+(\eta-1)^2} + \xi_{\eta-1}, \quad \eta \in \mathbb{N}_0^2 \cap [0,8], \tag{4.18}$$

where $\xi_{-1}, \xi_1, \xi_3, \xi_5, \xi_7$ are 5 i.i.d. linear uncertain variables with linear uncertainty distribution $\mathcal{L}(-2, 2)$.

In this example $h = 2, \theta = 0.5, T = 4$,

$$|\mathbf{F}(\eta,x)-\mathbf{F}(\eta,\psi)| + |\mathbf{G}(\eta,x)-\mathbf{G}(\eta,\psi)| \le \frac{1}{50}|x-y| = 0.02|x-y|,$$

and

$$\begin{aligned}h^{-\theta-1}\frac{\Gamma(\theta+1)\,\Gamma(T+1-\theta)}{\Gamma(T+1)\,(Q+1)} &= 2^{-1.5}\frac{\Gamma(0.5+1)\,\Gamma(4+1-0.5)}{3\Gamma(4+1)}\\ &\approx 0.05 > 0.02.\end{aligned}$$

Thus, the existence and uniqueness Theorem 4.3 confirms that UFFhDE (4.18) has a unique solution.

Now, since

$$\mathbf{F}(\eta,x) + |\mathbf{G}(\eta,x)|\Psi^{-1}(\theta) = \frac{\sin x}{50+(\eta-1)^2} + 4\theta - 2,$$

we deduce that $\mathbf{F}(\eta,x) + |\mathbf{G}(\eta,x)|\Psi^{-1}(\theta)$ is Lipschitz continues in x with Lipschitz constant $L = 0.02 < 0.35 = \theta\, h^{-\theta}$.

We see that $\mathbf{G}(\eta, x) = 1 > 0$, and, from example 4.2, we see $\mathcal{L}(-2, 2)$ is symmetrical. Hence, by Theorem 4.2, we deduce the following link between unique solution and α-path of UFFhDE (4.18):

(i) $\mathbf{X}_\eta \le \mathbf{X}_\eta^\theta$ if $\xi_\eta \le 4\theta - 2$,
(ii) $\mathbf{X}_\eta > \mathbf{X}_\eta^\theta$ if $\xi_\eta > 4\theta - 2$.

Example 4.6. Consider the following UFFhDE:

$${}_{-\frac{3}{8}}\Delta_{\frac{1}{2}}^{\frac{1}{4}}\mathbf{X}(\eta) = 0.025\,\mathbf{X}^2\left(\eta-\frac{3}{8}\right) + \xi_{\eta-\frac{3}{8}}, \quad \eta \in \mathbb{N}_0^{\frac{1}{2}} \cap \left[0, \frac{3}{2}\right], \tag{4.19}$$

where $\xi_{-\frac{3}{8}}, \xi_{\frac{1}{8}}, \xi_{\frac{5}{8}}, \xi_{\frac{9}{8}}$ are 4 i.i.d. linear uncertain variables with linear uncertainty distribution $\mathcal{L}(-3, 3)$.

In this example $h = 0.5, \theta = 0.25, T = 3$,

$$\begin{aligned}|\mathbf{F}(\eta,x)-\mathbf{F}(\eta,\psi)| + |\mathbf{G}(\eta,x)-\mathbf{G}(\eta,\psi)| &\le 0.025|x+y||x-y|\\ &= 0.1|x-y|, \quad \text{for } x \in [-2,2],\end{aligned}$$

and

$$\begin{aligned}h^{-\theta-1}\frac{\Gamma(\theta+1)\,\Gamma(T+1-\theta)}{\Gamma(T+1)\,(Q+1)} &= \left(\frac{1}{2}\right)^{-\frac{5}{4}}\frac{\Gamma(0.25+1)\,\Gamma(3+1-0.25)}{4\Gamma(3+1)}\\ &\approx 0.4 > 0.1.\end{aligned}$$

Thus, the existence and uniqueness Theorem 4.3 confirms that UFFhDE (4.19) has a unique solution.

Now, since

$$\mathbf{F}(\eta,x) + |\mathbf{G}(\eta,x)|\Psi^{-1}(\theta) = 0.025x^2 + 6\theta - 3,$$

we deduce that $\mathbf{F}(\eta, x) + |\mathbf{G}(\eta, x)|\Psi^{-1}(\theta)$ is Lipschitz, continued in x with Lipschitz constant $L = 0.1 < 0.3 = \theta\, h^{-\theta}$.

We see that $\mathbf{G}(\eta, x) = 1 > 0$, and, from example 4.2, we see $\mathcal{L}(-3, 3)$ is symmetrical. Hence, by use of Theorem 4.2, we deduce that $\mathbf{X}_\eta \leq \mathbf{X}_\eta^\theta$ if $\xi_\eta \leq 6\theta - 3$ and $\mathbf{X}_\eta > \mathbf{X}_\eta^\theta$ if $\xi_\eta > 6\theta - 3$. This is a link between unique solution and α-path of UFFhDE (4.19).

5. CONCLUSIONS

We have considered the fractional forward h-difference equations and uncertain fractional forward h-difference equations in the context of discrete fractional calculus. The comparison theorems and existence and uniqueness theorem for the FFhDEs and UFFhDEs have been found. From a theoretical point of view, we have created a strong relationship between the solutions for UFFhDEs with the symmetrical uncertain variables and the solutions for associated UFFhDEs (namely the α-path of UFFhDEs).

Our presented results are in the sense of Riemann-Liouville fractional operator. It is important to point out the future scope of our results. There is an important task here that the researchers will be able to consider in the future. What is the task? The interested readers can extend the ideas that were presented in this article to the two well-known models of fractional calculus that were defined by operators similar to the Riemann-Liouville fractional operator but with Mittag-Leffler functions in the kernel, namely the Atangana–Baleanu (or briefly AB) [42, 43] and Prabhakar [44] models.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary materials, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

REFERENCES

1. Miller KS, Ross B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. New York, NY: John Wiley & Sons (1993).
2. Podlubny I. *Fractional Differential Equations*. San Diego, CA: Academic Press (1999).
3. Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier B.V. (2006).
4. Diethelm K. *The Analysis of Fractional Differential Equations*. Berlin: Springer (2010).
5. Hamasalh FK, Mohammed PO. Generalized quartic fractional spline approximation function with applications. *Math Sci Lett.* (2016) 5:131–36.
6. Martinez F, Mohammed PO, Valdes JEN. Non-conformable fractional Laplace transform. *Kragujevac J Math.* (2022) 46:341–54.
7. Mohammed PO. Some integral inequalities of fractional quantum type. *Malaya J Mat.* (2016) 4:93–9.
8. Mohammed PO. Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function. *Math Meth Appl Sci.* (2019) 1–11. doi: 10.1002/mma.5784
9. Mohammed PO, Abdeljawad T. Modification of certain fractional integral inequalities for convex functions. *Adv Differ Equ.* (2020) 2020:69. doi: 10.1186/s13662-020-2541-2
10. Mohammed PO, Brevik I. A new version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals. *Symmetry.* (2020) 12:610. doi: 10.3390/sym12040610
11. Mohammed PO, Sarikaya MZ. Hermite-Hadamard type inequalities for F-convex function involving fractional integrals. *J Inequal Appl.* (2018) 2018:359. doi: 10.1186/s13660-018-1950-1
12. Mohammed PO, Sarikaya MZ. On generalized fractional integral inequalities for twice differentiable convex functions. *J Comput Appl Math.* (2020) 372:112740. doi: 10.1016/j.cam.2020.112740
13. Mohammed PO, Sarikaya MZ, Baleanu D. On the generalized Hermite-Hadamard inequalities via the tempered fractional integrals. *Symmetry.* (2020) 12:595. doi: 10.3390/sym12040595
14. Qi F, Mohammed PO, Yao JC, Yao YH. Generalized fractional integral inequalities of Hermite–Hadamard type for (α, m)-convex functions. *J Inequal Appl.* (2019) 2019:135. doi: 10.1186/s13660-019-2079-6
15. Agarwal RP. *Difference Equations and Inequalities: Theory, Methods, and Application*. New York, NY: Marcel Dekker (2000).
16. Bohner M, Peterson AC. *Advances in Dynamic Equations on Time Scales*. Boston, MA: Birkhauser (2003).
17. Atici F, Eloe P. A transform method in discrete fractional calculus. *IJDE.* (2007) 2:165–76.
18. Atici F, Eloe P. Initial value problems in discrete fractional calculus. *P Am Math Soc.* (2009) 137:981–89. doi: 10.1090/S0002-9939-08-09626-3
19. Goodrich C. Existence of a positive solution to a system of discrete fractional boundary value problems. *Appl Math Comput.* (2011) 217:4740–53. doi: 10.1016/j.amc.2010.11.029
20. Goodrich C, Peterson A. *Discrete Fractional Calculus*. Berlin: Springer (2015).
21. Wu G, Baleanu D. Discrete chaos in fractional delayed logistic maps. *Nonlin Dyn.* (2015) 80:1697–703. doi: 10.1007/s11071-014-1250-3
22. Wu G, Baleanu D, Luo W. Lyapunov functions for Riemann–Liouville-like fractional difference equations. *Appl Math Comput.* (2017) 314:228–36. doi: 10.1016/j.amc.2017.06.019
23. Wu G, Baleanu D, Zeng S. Finite-time stability of discrete fractional delay systems: Gronwall inequality and stability criterion. *Commun Nonlin Sci.* (2018) 57:299–308. doi: 10.1016/j.cnsns.2017.09.001
24. Zhu Y. Uncertain fractional differential equations and an interest rate model. *Math Meth Appl Sci.* (2015) 38:3359–68. doi: 10.1002/mma.3335
25. Zhu Y. Existence and uniqueness of the solution to uncertain fractional differential equation. *J Uncertain Anal Appl.* (2015) 3:1–11. doi: 10.1186/s40467-015-0028-6
26. Lu Z, Zhu Y. Comparison principles for fractional differential equations with the Caputo derivatives. *Adv Differ Equat.* (2018) 237:1–11. doi: 10.1186/s13662-018-1691-y
27. Lu Z, Zhu Y. Numerical approach for solution to an uncertain fractional differential equation. *Appl Math Comput.* (2019) 343:137–48. doi: 10.1016/j.amc.2018.09.044
28. Lu Q, Zhu Y, Lu Z. Uncertain fractional forward difference equations for Riemann–Liouville type. *Adv Differ Equ.* (2019) 2019:147. doi: 10.1186/s13662-019-2093-5
29. Mohammed PO. A generalized uncertain fractional forward difference equations of Riemann-Liouville Type. *J Math Res.* (2019) 11:43–50. doi: 10.5539/jmr.v11n4p43
30. Atici F, Sengul S. Modeling with fractional difference equations. *J Math Anal Appl.* (2010) 369:1–9. doi: 10.1016/j.jmaa.2010.02.009
31. Abdeljawad T, Baleanu D. Discrete fractional differences with nonsingular discrete Mittag-Leffler kernels. *Adv Differ. Equat.* (2016) 2016:232. doi: 10.1186/s13662-016-0949-5

32. Abdeljawad T. Fractional difference operators with discrete generalized Mittag-Leffler kernels. *Chaos Soliton Fract.* (2019) 126:315–24. doi: 10.1016/j.chaos.2019.06.012
33. Shah K, Jarad F, Abdeljawad T. On a nonlinear fractional order model of dengue fever disease under Caputo-Fabrizio derivative. *Alex Eng J.* (2020) 59:2305–13. doi: 10.1016/j.aej.2020.02.022
34. Abdeljawad T, Baleanu D. Monotonicity results for fractional difference operators with discrete exponential kernels. *Adv Differ Equat.* (2017) 2017:78. doi: 10.1186/s13662-017-1126-1
35. Abdeljawad T. On delta and Nabla Caputo fractional differences and dual identities. *Discrete Dyn Nat Soc.* (2013) 2013:12. doi: 10.1155/2013/406910
36. Abdeljawad T. Dual identities in fractional difference calculus within Riemann. *Adv Differ Equat.* (2017) 2017:36. doi: 10.1186/1687-1847-2013-36
37. NRO Bastos, Ferreira RAC, Torres DFM. Discrete-time fractional variational problems. *Signal Process* (2011) 91:513–24. doi: 10.1016/j.sigpro.2010.05.001
38. Ferreira RAC, Torres DFM. Fractional h-difference equations arising from the calculus of variations. *Appl Anal Discrete Math.* (2011) 5:110–2. doi: 10.2298/AADM110131002F
39. Suwan I, Abdeljawad T, Jarad F. Monotonicity analysis for nabla h-discrete fractional Atangana-Baleanu differences. *Chaos Solit Fract.* (2018) 117:50–9. doi: 10.1016/j.chaos.2018.10.010
40. Abdeljawad T. Different type kernel h–fractional differences and their fractional h–sums. *Chaos Solit Fract.* (2018) 116:146–56. doi: 10.1016/j.chaos.2018.09.022
41. Liu B. *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*. Berlin: Springer (2010).
42. Atangana A, Baleanu D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *Thermal Sci.* (2016) 20:763–9. doi: 10.2298/TSCI160111018A
43. Baleanu D, Fernandez A. On some new properties of fractional derivatives with Mittag-Leffler kernel. *Commun Nonlin Sci.* (2018) 59:444–62. doi: 10.1016/j.cnsns.2017.12.003
44. Prabhakar TR. A singular integral equation with a generalized Mittag Leffler function in the kernel. *Yokohama Math J.* (1971) 19:7–15.

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