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Periodic Solutions of Some Classes of One Dimensional Non-autonomous Equation

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In this paper, the periodic solutions of a certain one-dimensional differential equation are investigated for the first order cubic non-autonomous equation. The method used here is the bifurcation of periodic solutions from a fine focus $z = 0$. We aimed to find the maximum number of periodic solutions into which a given solution can bifurcate under perturbation of the coefficients. For classes $C_{3,8}, C_{4,3}, C_{7,5}, C_{7,6}$, eight periodic multiplicities have been found. To investigate the multiplicity >9, the formula for the focal value was not available in the literature. We also succeeded in constructing the formula for η_{10}. By implementing our newly developed formula, we are able to get multiplicity ten for classes $C_{7,3}, C_{9,1}$, which is the highest known to date. A perturbation method has been properly established for making the maximal number of limit cycles for each class. Some examples are also presented to show the implementation of the newly developed method. By considering all of these facts, it can be concluded that the presented methods are new, authentic, and novel.

Keywords: multiplicity, periodic solution, non-autonomous equation, bifurcation method, trigonometric coefficients

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1. INTRODUCTION

On August 8, 1900, David Hilbert presented a set of mathematical problems [1] to the Second International Congress of Mathematicians in Paris. The sixteenth problem he posed was titled the Problem of the Topology of Algebraic Curves and Surfaces. It is stated in two parts. In the first part, Hilbert suggested a thorough investigation of the relative positions of the separate branches of algebraic curves in nth-order vector fields, which is in the area of real algebraic geometry. In the second part, Hilbert asked for a search for the upper bound of the number of limit cycles and their relative locations in polynomial vector fields of order n. This part of the problem is related to ordinary differential equations and dynamical systems. Generally, this part of the problem is what is usually meant when talking about Hilbert's 16th problem.

Limit cycle theory takes a central role in Hilbert's 16th problem. Studying the number of limit cycles for differential equations is the most difficult part of the problem. The phenomenon of the limit cycle was first discovered and introduced by Poincaré in his four-part article, Integral curves defined by differential equations [2–5] published between 1881 and 1886.

At that time, Poincaré also noticed the close relationship between the study of limit cycles and the solutions of the global structural problems of a family of integral curves of differential equations.

His work was later extended by Bendixson to the well-known Poincaré-Bendixson theorem [6] on the limit set of trajectories of dynamical systems in a bounded region. The driving force behind the study of limit cycle theory was the invention of the triode vacuum tube, which was able to produce stable self-excited oscillations of constant amplitude. It was noted that this kind of oscillation phenomenon could not be described by linear differential equations. At the end of the 1920's Van der Pol [7] developed a differential equation to describe the oscillations of constant amplitude in a triode vacuum tube. Limit cycles are common solutions for all types of dynamical systems. They model systems that exhibit self-sustained oscillations. In other words, these systems oscillate even in the absence of external periodic forcing. For a practical example, consider a specific Holling-Tanner predator-prey model [8]. This model appears to match very well with what happens for many predator-prey species in the natural world, for example, house sparrows and sparrow hawks in Europe, muskrat and mink in Central North America, and white-tailed deer and wolf in Ontario, Canada.

Other examples of self-excited oscillation are the beating of a heart, rhythms in body temperature, hormone secretion, chemical reactions that oscillate spontaneously, and vibrations in bridges and airplane wings. Due to the wide occurrence of limit cycles in science and technology, limit cycle theory has also been extensively studied by physicists, and more recently by chemists, biologists, and economists [9–16].

We consider the differential equation of the form

$$\dot{z} = \gamma(t)z^3 + \delta(t)z^2 + \upsilon(t)z \tag{1}$$

where independent variable t and coefficients γ, δ, υ are real-valued functions but $z \in \mathbb{C}$. To find the maximum count of periodic solutions we use the complexified form of the equation (1) ; for more details, see [17–20]. Also consider that $\exists\ \beta \in \mathbb{R}$ such that:

$$z(\beta) = z(0).$$

These solutions are periodic, even if γ, δ, and υ are not themselves periodic. Our fundamental focus is to get the maximum number of periodic solutions of any class of the form (1) in which a solution may bifurcate by perturbing the coefficients. Neto [21] states that for Equation (1), until some coefficients are restricted, we are unable to have an upper bound for the number of focal values. The number of periodic solutions depends upon the multiplicity of the solutions $z = 0$. The multiplicity of $z = 0$, as solution of (1) is also multiplicity of $z = 0$; as a zero of the following displacement function

$$p : r \longrightarrow z(\beta, 0, r) - r \tag{2}$$

described in complex function theory. For $z = 0$, the means of computing multiplicity (μ) is explained in Alwash and Llyod [22], but for the sake of ease, we explained it briefly here. We write $z(t, 0, r) = \sum_{i=1}^{\infty} a_i(t) r^i$ for $0 \le t \le \beta$ where also r lies in neighborhood of $z = 0$, and use it in equation (2); for more detail, see [21, 23–26]. This provides a differential equation for $a_\mu(t)$ with some starting conditions $a_1(0) = 1$ and $a_\mu(0) = 0$ for $i > 1$. Therefore

$$p(r) = (a_1(\alpha) - 1) r + \sum_{i=2}^{\infty} a_i(\beta) r^i \tag{3}$$

The multiplicity (μ) is "$\mu > 1$" if

$$a_1(\beta) = 1$$
$$a_2(\beta) = a_3(\beta) = ... = a_{\mu-1}(\beta) = 0.$$

However, $a_\mu(\beta) \neq 0$. When $a_1(\beta) = 1$ and $a_\mu(\beta) = 0$, $\forall\ \mu > 1$, the origin is center. We can observe from Equation (1) that $\dot{a_1}(t) = a_1(t)\,\upsilon(t)$, where $a_1(t)$ is defined as

$$a_1(t) = e^{\int_0^t \upsilon(s)ds}.$$

In this way, $\mu > 1$ iff

$$\int_0^t \upsilon(s)\, ds = 0 \tag{4}$$

because $a_1(t) = 1$. We are especially interested in the situation where $z = 0$ has multiple solutions, so we consider that (4) holds. By using the following transformation

$$\xi = z e^{-\int_0^t \upsilon(s)ds}.$$

(1) takes the form

$$\dot{\xi} = \widehat{\gamma}(t)\xi^3 + \widehat{\delta}(t)\xi^3 \tag{5}$$

where $\widehat{\gamma}(t) = \gamma(t)\, e^{2\int_0^t \upsilon(s)ds}$ and

$$\widehat{\delta}(t) = \delta(t)\, e^{2\int_0^t \upsilon(s)ds}.$$

We can see that $\widehat{\gamma}$ and $\widehat{\delta}$ are periodic if γ, δ, and υ are periodic. By using Lemma (2.6) in Alwash and Llyod [22], consider multiplicity of $z = 0$ as a periodic solution of (1). If, for equation (1), $\mu > 1$, then the multiplicity of $\xi = 0$ as a periodic solution of (5) is also μ. So we consider that $\upsilon(t) \cong 0$ in (1). As a result, equation (1) takes the form

$$\dot{z} = \gamma(t)z^3 + \delta(t)z^2 \tag{6}$$

where γ and δ may be polynomials (i) in t (ii) in *cost* and *sint* (trigonometric functions). The functions $a_i(t)$, for $i > 1$ are calculated by utilizing the relation

$$\dot{a_i} = \gamma \sum_{\substack{j+k+l=i \\ j,k,l \ge 1}} a_j a_k a_l + \delta \sum_{\substack{j+k=i \\ j,k \ge 1}} a_j a_k \tag{7}$$

with $a_1(t) = 1$. Calculation of these functions is tough because of the integration by parts used in it. Assume that $\eta_i = a_i(\beta)$; at that point, $\mu = i$ if $\eta_1 = 1$ and $\eta_k = 0$ for $2 \le k \le i - 2$

but $\eta_i \neq 0$. These $\eta_{i's}$ are known as focal values. For $i \leq 8$ functions $a_i(t)$ and η_i are given in Alwash and Llyod [22]. For $i = 9$ N, Yasmin calculated $a_9(t)$ and η_9 in [27]. For $i = 10$ we have calculated $a_{10}(t)$ and η_{10} in Nawaz [28], also presented in theorem 2.1 and 2.2.

In section 2, we write formulas with which we can calculate the highest focal value, and we also implement stopping criteria defined in Alwash and Llyod [22]. Some required conditions and the method of perturbation are described in section 3. Section 2 and 3 are mainly concerned with the calculation of focal values, which we will utilize in section 4. In section 4, we consider polynomial coefficients for equation (6) and calculate the focal values. Some examples are given in section 5. In the last section, 6, we make discussions and conclusions.

2. CALCULATION OF FOCAL VALUES η_{10}

In the following theorem (2.1), some functions $a_2, a_3, ..., a_{10}$ are given that are obtained from Equation (7) and are helpful in calculating the periodic solutions.

Theorem 2.1. *For the equation* (7), *conclusive functions* $a_2, a_3, ..., a_8$ *are given in Alwash and Llyod [22], and* a_9, a_{10} *are described below:*

$$
\begin{aligned}
a_9 = {} & \bar\delta^8 + 7\bar\delta^6\bar\gamma + \overline{\bar\delta^6\gamma} + 6\bar\delta^5\overline{\bar\delta\gamma} + 2\overline{\bar\delta^5\gamma}\bar\delta + 5\bar\delta^4\overline{\bar\delta^2\gamma} \\
& + 3\overline{\bar\delta^4\gamma\bar\gamma} + 3\overline{\bar\delta^3\gamma}\bar\delta^2 + 5\overline{\bar\delta^4\gamma\bar\gamma} + \frac{39}{2}\bar\delta^4\bar\gamma^2 - 2\bar\delta^3\overline{\delta\bar\gamma^2} \\
& + 24\bar\delta^3\overline{\bar\delta\gamma\bar\gamma} + 6\overline{\bar\delta^3\gamma\bar\gamma\bar\delta} - 10\overline{\bar\delta^3\gamma\overline{\bar\delta\gamma}} + 12\bar\delta\bar\gamma\overline{\bar\delta^3\gamma} \\
& + 4\overline{\bar\gamma\delta\overline{\bar\delta^3\gamma}} + 4\overline{\bar\delta^3\gamma}\bar\delta^3 + \frac{43}{6}\bar\delta^2\bar\gamma^3 + 4\overline{\bar\gamma^3\delta\bar\delta} + 4\bar\delta^2\overline{\bar\gamma\gamma\bar\delta^2} \\
& - 10\overline{\delta\bar\delta\bar\gamma\overline{\gamma\bar\delta^2}} + \frac{15}{2}\bar\gamma^2\overline{\bar\delta^2\gamma} + 2\bar\delta^2\overline{\bar\delta^2\gamma} - 4\overline{\bar\delta^3\gamma}\bar\delta^3 + \frac{43}{6}\bar\delta^2\bar\gamma^3 \\
& + 4\overline{\bar\gamma^3\delta\bar\delta} + 4\bar\delta^2\overline{\bar\gamma\gamma\bar\delta^2} - 10\overline{\delta\bar\delta\bar\gamma\overline{\gamma\bar\delta^2}} + \frac{15}{2}\bar\gamma^2\overline{\bar\delta^2\gamma} \\
& + 2\bar\delta^2\overline{\bar\delta^2\gamma} - 2\bar\delta^4\bar\gamma + 8\overline{\bar\gamma\delta\bar\delta^3} + 2\overline{\bar\delta\bar\delta^2\gamma\overline{\bar\delta\gamma}} \\
& + 26\overline{\bar\delta\gamma\bar\delta^2\gamma\bar\delta} + 6\overline{\bar\delta^2\gamma\bar\gamma} - 6\overline{\bar\delta^2\gamma\bar\gamma} + 12\bar\delta^2\overline{\bar\delta\gamma\bar\gamma} \\
& + 16\overline{\bar\delta^2\gamma}\delta\bar\delta\bar\gamma - 16\overline{\bar\delta^3\gamma\overline{\bar\delta\gamma}} + 9\bar\delta^2\overline{(\bar\delta\gamma)^2} + 9\overline{(\bar\delta\gamma)^2\bar\gamma} - \\
& \overline{\delta\bar\gamma^3}\bar\delta + \frac{35}{8}\bar\gamma^4 - 6\bar\delta\bar\gamma\overline{\delta\bar\gamma^2} + 8\overline{\delta\bar\delta^4\gamma\bar\gamma} - 2\overline{\gamma\overline{\bar\delta\bar\delta^4\gamma}} + \frac{1}{2}\bar\delta^4\overline{\bar\delta\gamma} \\
& + 2\overline{\overline{\bar\delta\bar\delta\gamma}\bar\delta^3\gamma} + \overline{\delta\overline{\bar\delta\gamma}\overline{\delta\bar\gamma^2}} + \bar\delta\left(\overline{\bar\delta^2\gamma}\right)^2.
\end{aligned}
$$

and

$$
\begin{aligned}
a_{10} = {} & \bar\delta^9 - \frac{23}{2}\overline{\bar\delta^7\gamma} - \frac{1235}{6}\overline{\bar\delta^5\bar\gamma\gamma} + 3\overline{\bar\delta^5\gamma\bar\gamma} + 111\bar\gamma\bar\delta^4\overline{\bar\delta\gamma} \\
& - 444\overline{\bar\gamma\delta\bar\delta^3\overline{\bar\delta\gamma}} + 20\bar\gamma\overline{\delta\bar\delta^4\gamma} - 12\overline{\bar\gamma\delta\overline{\bar\delta^4\gamma}} + \frac{214}{3}\bar\gamma\bar\delta^3\overline{\bar\delta^2\gamma} \\
& + 3\gamma\bar\delta^7 - 160\overline{\bar\gamma\delta\bar\delta^2\overline{\bar\delta^2\gamma}} + \frac{15}{2}\bar\gamma^2\overline{\bar\delta^3\gamma} - \frac{970}{3}\overline{\gamma\bar\gamma^2\bar\delta^3} \\
& + 30\bar\gamma\bar\delta^2\overline{\bar\delta^3\gamma} - 68\overline{\bar\gamma\delta\overline{\bar\delta\bar\delta^3\gamma}} + 9\bar\gamma\overline{\bar\delta^3\gamma\bar\gamma} + \frac{1015}{9}\bar\gamma^3\bar\delta^3 \\
& - 237\overline{\delta\bar\delta^2\bar\gamma^3} + 8\bar\gamma\bar\delta^7 - \frac{11}{2}\bar\gamma\bar\delta^2\overline{\delta\bar\gamma^2} + 26\overline{\bar\gamma\delta\bar\delta\overline{\delta\bar\gamma^2}} \\
& + \frac{319}{2}\bar\gamma^2\bar\delta^2\overline{\bar\delta\gamma} - 174\overline{\delta\bar\delta\bar\gamma^2\overline{\bar\delta\gamma}} - 90\overline{\bar\gamma\gamma\overline{\bar\delta\bar\delta^2\gamma}} + 24\bar\gamma\overline{\bar\delta^2\gamma\delta\bar\gamma} \\
& + 40\bar\gamma\overline{\bar\delta\gamma\bar\gamma\bar\delta^2} - 24\bar\gamma\overline{\delta\gamma\bar\gamma\bar\delta^2} + 3\bar\gamma\overline{\bar\delta^2\gamma\overline{\bar\delta\gamma}} - 154\overline{\bar\gamma\gamma\bar\delta^2\overline{\bar\delta\gamma}} \\
& - 24\overline{\gamma\bar\gamma^2\bar\delta^2\delta} + 70\bar\gamma\bar\delta\left(\overline{\bar\delta\gamma}\right)^2 + 42\overline{\bar\gamma\delta\left(\overline{\bar\delta\gamma}\right)^2} - 70\overline{\bar\gamma\bar\delta^3\gamma^2} \\
& - \frac{3}{2}\overline{\bar\gamma\delta\bar\gamma^3} - 21\overline{\delta\bar\gamma^4} + \overline{\delta\overline{\bar\delta\gamma}\overline{\delta\bar\gamma^2}} - \frac{15}{4}\bar\gamma^2\overline{\delta\bar\gamma^2} + \frac{169}{4}\bar\gamma^4\bar\delta \\
& + 24\gamma\bar\gamma^2\overline{\bar\delta\bar\delta^2\gamma} - 24\overline{\bar\gamma^2\delta\overline{\bar\delta^2\gamma}} + 10\bar\gamma^3\overline{\bar\delta\gamma} + \frac{9}{2}\bar\delta^4\overline{\bar\delta^3\gamma} \\
& - 74\overline{\gamma\bar\gamma^3\bar\delta} + 8\bar\delta\overline{\delta\bar\delta\bar\gamma^3} - 5\overline{\gamma\bar\delta^6} - 15\overline{\bar\delta^5\gamma^2} \\
& + \frac{34}{3}\bar\gamma\bar\delta^3\overline{\bar\delta^2\gamma} + 2\overline{\bar\delta\bar\delta^6\gamma} + 7\bar\delta^6\overline{\bar\delta\gamma} + 6\bar\delta^5\overline{\bar\delta^2\gamma} - 6\overline{\gamma\overline{\bar\delta\bar\delta^4\gamma}} \\
& + 2\bar\delta^3\overline{\bar\delta^3\gamma} + 10\bar\delta\overline{\gamma\bar\gamma\bar\delta^4} + 26\bar\gamma^2\bar\delta^5 - \frac{5}{2}\bar\delta^4\overline{\delta\bar\gamma^2} + \frac{5}{2}\overline{\delta\bar\delta^4\bar\gamma^2} \\
& + \frac{73}{2}\bar\gamma\bar\delta^4\overline{\bar\delta\gamma} - \frac{127}{2}\overline{\bar\delta^4\gamma\overline{\bar\delta\gamma}} + 9\bar\delta^2\overline{\gamma\bar\gamma\bar\delta^3} - 20\overline{\bar\delta\bar\delta^3\gamma\overline{\bar\delta\gamma}} \\
& + 19\gamma\bar\gamma^2\overline{\bar\delta^3\gamma} - 21\overline{\bar\delta^2\gamma\overline{\bar\delta^3\gamma}} + 8\overline{\bar\delta\bar\gamma\delta\overline{\bar\delta^3\gamma}} - \frac{160}{3}\overline{\gamma\bar\delta^3\overline{\bar\delta^2\gamma}} \\
& - \frac{4}{5}\bar\gamma\bar\delta^5 + 32\overline{\bar\delta^4\gamma\overline{\delta\bar\gamma}} - 20\overline{\bar\delta\bar\gamma\bar\delta\delta\overline{\bar\delta^2\gamma}} + 24\bar\delta\bar\gamma^2\overline{\bar\delta^2\gamma} \\
& + \frac{4}{3}\bar\delta^3\overline{\bar\delta^2\gamma} - \frac{31}{30}\overline{\gamma\bar\delta^5} + 16\overline{\bar\delta\bar\delta^3\bar\gamma\delta} - 16\overline{\bar\gamma\delta\bar\delta^4} + \frac{13}{2}\overline{\bar\delta^2\gamma\overline{\delta\gamma^2}} \\
& + 3\bar\delta^2\overline{\bar\delta^2\gamma\overline{\bar\delta\gamma}} + 42\bar\delta^2\overline{\bar\delta^2\gamma}\overline{\bar\delta\gamma} + 12\bar\gamma\overline{\bar\delta\bar\delta^2\gamma} - 12\overline{\gamma\overline{\bar\delta\bar\delta^2\gamma}} \\
& - 12\overline{\bar\delta\bar\delta^2\bar\gamma\gamma} + 12\bar\delta^3\overline{\bar\delta^2\bar\gamma\gamma} + 32\overline{\delta^2\bar\gamma\overline{\bar\delta\bar\delta^2\gamma}} - 32\overline{\bar\delta\bar\delta^3\gamma\overline{\delta\bar\gamma}} \\
& + 14\bar\delta^3\left(\overline{\bar\delta\gamma}\right)^2 - 28\gamma\bar\delta\left(\overline{\bar\delta\gamma}\right)^2 - \frac{3}{2}\bar\delta^2\overline{\delta\bar\gamma^3} + \frac{1}{2}\bar\delta^4\overline{\bar\delta\gamma} \\
& + 12\bar\delta\overline{\delta\bar\gamma^2}\overline{\delta\bar\gamma} - 8\overline{\delta\bar\delta^4\gamma\bar\gamma} - 2\overline{\gamma\overline{\bar\delta\bar\delta^4\gamma}} + 2\overline{\bar\delta\bar\delta\gamma}\overline{\bar\delta^3\gamma} + \bar\delta\left(\overline{\bar\delta^2\gamma}\right)^2 \\
& - 36\overline{(\bar\gamma^2\delta\bar\delta\overline{(\delta\bar\gamma)})} - 48\bar\delta\overline{(\bar\gamma^2\delta\overline{(\bar\delta\gamma)})} - 16\overline{(\gamma\bar\delta\overline{(\bar\gamma\bar\delta^2\gamma)})} \\
& - 8\bar\delta\overline{(\delta\bar\gamma\overline{(\delta\bar\gamma^2)})} + 8\bar\delta^2\overline{(\bar\delta^2\gamma\delta\bar\gamma)}.
\end{aligned}
$$

By using these functions, we obtained the next theorem, 2, which enables us to find the maximum multiplicity in which the integral is like $\int \gamma(t)\,\overline{\delta(t)}dt$; *bar "—" shows that integral* $\overline{\delta(t)} = \int_0^t \delta(t)\,dt$ *is definite.*

Theorem 2.2. *The solution* $z = 0$ *of* (6) *has a multiplicity* k, *wherever* $2 \leq k \leq 10$ *iff* $\eta_n = 0$ *for* $2 \leq n \leq k-1$ *and* $\eta_n \neq 0$ *where*

$$
\begin{aligned}
\eta_2 &= \int_0^\beta \delta \\
\eta_3 &= \int_0^\beta \gamma \\
\eta_4 &= \int_0^\beta \gamma\bar\delta \\
\eta_5 &= \int_0^\beta \gamma\bar\delta^2
\end{aligned}
$$

$\eta_6 = \int_0^\beta \gamma\overline{\delta}^3 - \frac{1}{2}\overline{\gamma}^2\delta$
$\eta_7 = \int_0^\beta \gamma\overline{\delta}^4 + 2\gamma\overline{\delta}^2\overline{\gamma}$
$\eta_8 = \int_0^\beta \gamma\overline{\delta}^5 + 3\gamma\overline{\delta}^3\overline{\gamma} + \gamma\overline{\delta}^2\overline{\overline{\delta}\gamma} - \frac{1}{2}\overline{\gamma}^3\delta$
$\eta_9 = \int_0^\beta \gamma\overline{\delta}^6 - 5\gamma\overline{\delta}^4\overline{\gamma} - 2\overline{\delta}^3\overline{\delta\overline{\gamma}} + 20\overline{\delta\overline{\gamma}^2} + 2\overline{\delta\overline{\gamma}}\delta\overline{\gamma}^2$
and
$\eta_{10} = \int_0^\beta \gamma\overline{\delta}^7 - \frac{1235}{6}\gamma\overline{\gamma}\overline{\delta}^5 - \frac{970}{3}\gamma\overline{\gamma}^2\overline{\delta}^3 - 237\delta\overline{\delta}^2\overline{\gamma}^3 - 24\gamma\overline{\gamma}^2\delta\overline{\delta}^2 - 70\overline{\gamma}\overline{\delta}^3\gamma^2 - 21\overline{\gamma}^4\delta - 74\gamma\overline{\gamma}^3\overline{\delta} + \frac{5}{2}\overline{\gamma}^2\delta\overline{\delta}^4 + 32\overline{\delta}^4\gamma\overline{\delta\overline{\gamma}} - 16\delta\overline{\delta}^4\overline{\gamma} - 15\overline{\delta}^5\gamma^2 - 36\delta\overline{\delta}\overline{\gamma}^2\overline{\delta\overline{\gamma}} - 8\delta\overline{\delta}^4\gamma\overline{\gamma}.$

In theorem (2.1) some functions $a_2, a_3, ..., a_{10}$ are given that are obtained from Equation (7) and are helpful in calculating the periodic solutions. As future work, one can calculate a maximum multiplicity >10 by firstly generalizing theorem 2.1 and 2.2. This should be calculated by substituting the value of $i > 10$ into Equation (7).

3. CONDITIONS FOR THE CENTER AND METHOD OF PERTURBATION

In this section, we describe some conditions for the center. From theorem 2.2, we find the maximum value μ for different classes of equations. We have to stop calculating multiplicity η_k. We need some conditions that assure that there is no need to proceed further with η_k. For this, we require some conditions that are sufficient for $z = 0$ as a center. The conditions are given in theorem 3 and corollary 4.

Theorem 3.1. *Consider that there are continuous functions f, g defined on $I = \sigma([0,\alpha])$ and differentiable function σ with $\sigma(\alpha) = \sigma(0)$ such that*

$$\gamma(t) = f(\sigma(t))\dot{\sigma}$$

$$\delta(t) = g(\sigma(t))\dot{\sigma},$$

then the origin is a center for (6).

Corollary 3.2. *Consider that if any δ or γ is identically 0 and the other has mean value zero. The origin is a center.*

For more detail, see [17, 19, 20]. After determining the maximum multiplicity μ, we now have to make a series of perturbations of the coefficients, every one of which results in one periodic solution coming out of origin.

For this, suppose the equation of the form given below:

$$\dot{z} = \gamma(t)z^3 + \delta(t)z^2 \tag{8}$$

having multiplicity $\mu = j$ (suppose). Let U be in the region near 0 in the complex plane containing no periodic solution except $z = 0$. From theorem (2.4) in Alwash and Llyod [22], the initial point that is contained in U remained fixed concerning a total number of periodic solutions. With the condition that perturbations of the coefficients considered are small enough, our goal is to get $\eta_2 = \eta_3 = ... = \eta_{j-2} = 0$ but $\eta_{j-1} \neq 0$ by perturbing and making suitable choices of γ and δ, if possible. Obviously the most effective solutions in U and ψ are zero solutions while we get periodic solution $\psi(t)$, where $\psi(0) \in U$ as a non-trivial solution. By considering the underlying fact that the complex solutions always appear in conjugate pairs, we can say that ψ is real. Now, let U_1 and V_1 be the neighborhood of zero and ψ, respectively, such that $V_1 \cup U_1 \subset U$ and $V_1 \cap U_1 = \gamma$. The periodic solutions around V_1 and U_1 are preserved when we make a small perturbation in the coefficients. By applying the same procedure as above, our choice is to perturb the coefficients such that $\eta_k = 0$ for $k = 2, 3, ..., j-3$, but $\eta_j \neq 0$. So that we get $\mu = j - 2$. By applying that procedure, we get two non-trivial real periodic solutions and the zero solution is of multiplicity $j - 2$. Continuously, in this way, we end up with Equation (8) with $\mu = 2$ and $j - 2$ being distinct non-trivial (other than zero) real periodic solutions.

4. POLYNOMIAL COEFFICIENTS FOR SOME CLASSES

Let $C_{i,j}$ indicate the class of the shape (6) in which the degree of γ is i and δ is j and these are polynomial in "t" only. We consider some classes $C_{7,3}, C_{7,5}, C_{7,6}, C_{3,8}, C_{4,3}, C_{9,1}$ and will evaluate the maximum multiplicity; for more classes, see [19, 20, 28]. These are described below in the form of theorems as:

Theorem 4.1. *Let $C_{7,3}$ be class of equation of the form*

$$\dot{z} = \gamma(t)z^3 + \delta(t)z^2. \tag{9}$$

with

$$\gamma(t) = a + b(2t) + c(2t)^2 + d(2t)^3 + e(2t)^4 + h(2t)^7,$$

$$\delta(t) = i + l(2t)^3.$$

where the degree of $\gamma(t)$ is 7 and $\delta(t)$ is 3. Then, $\mu_{\max}(C_{7,3}) \geq 10$.

Proof. By using theorem 2.2, we calculate

$$\eta_2 = i + 2l, \tag{10}$$

$$\eta_3 = a + b + \frac{4}{3}c + 2d + \frac{16}{5}e + 16h. \tag{11}$$

Thus, multiplicity of $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$, and multiplicity $\mu = 3$ if $\eta_2 = 0$ but $\eta_3 \neq 0$. If $\eta_2 = \eta_3 = 0$, then from (10) and (11) , we take

$$i = -2l, \tag{12}$$

and

$$a = -b - c\frac{4}{3} - 2d - \frac{16}{5}e - 16h. \tag{13}$$

Now, by using (12) and (13) , $\gamma(t)$ and $\delta(t)$ take the form of:

$$\gamma(t) = b(2t-1) + c[(2t)^2 - \frac{4}{3}] + d[(2t)^3 - 2] + e[(2t)^4 - \frac{16}{5}] + h[(2t)^7 - 16],$$

$$\delta(t) = l[(2t)^3 - 2].$$

and η_4 is a constant multiple of "l" given as:

$$\eta_4 = -\frac{l(-3920h - 224e + 90c + 105b)}{1575}.$$

If $\eta_4 = 0$ then either $l = 0$ or

$$h = -\frac{224}{3920}e + \frac{90}{3920}c + \frac{105}{3920}b. \tag{14}$$

If $l = 0$, then $\delta(t) = 0$ and $\eta_3 = 0$ shows the mean value of $\gamma(t)$ is zero. So by corollary 3.2, the origin is a center. Suppose $l \neq 0$. If (14) holds, then η_5 is calculated as:

$$\eta_5 = -\frac{8l^2(264e - 35c + 165b)}{675675}.$$

If $\eta_5 = 0$, then either $l = 0$ or $264e - 35c + 165b = 0$. But $l \neq 0\ \left(\textit{taken above}\right)$, so we substitute

$$e = \frac{35}{264}c - \frac{165}{264}b. \tag{15}$$

and calculate η_6 as:

$$\eta_6 = \frac{cl(200984c + 6416388l^2 + 534699b)}{81723972330}.$$

If $\eta_6 = 0$, then either $c = 0$ or

$$c = -\frac{6416388}{200984}l^2 - \frac{534699}{200984}b, \tag{16}$$

because we already take $l \neq 0$. If $c = 0$ then by using (15), $\gamma(t)$, and $\delta(t)$ take the following form:

$$\gamma(t) = [8t^3 - 2][b\left(t^4 - t\right) + d],$$

$$\delta(t) = l[8t^3 - 2].$$

Let $\sigma\ (t) = 2t^4 - 2t$; then $\dot{\sigma}\ (t) = 8t^3 - 2$. Also, $\sigma\ (0) = \sigma\ (1)$. So $\gamma(t)$ and $\delta(t)$ are as follows:

$$\gamma(t) = [b\left(t^4 - t\right) + d]\dot{\sigma},$$

$$\delta(t) = l\dot{\sigma}.$$

By using theorem 3.1 the origin is a center having $f\ (\sigma) = [b\left(t^4 - t\right) + d]$ and $g\ (\sigma) = l$. Thus, suppose $c \neq 0$. By using (16), we have η_7 as follows:

$$\eta_7 = \frac{491l^2(12l^2 + b)(2060488754705b + 22506362768324l^2 - 757960558278d)}{1336288792941284910}.$$

Recall that $l \neq 0$ (considered above). If $\eta_7 = 0$ then either $b = -12l^2$ or

$$b = -\frac{22506362768324}{2060488754705}l^2 - \frac{757960558278}{2060488754705}d. \tag{17}$$

If (17) holds, then we find

$$\eta_8 = \frac{491l(307857d - 901484l^2)\rho}{17032366139772045417310567951383403732758840000 0}.$$

Here

$$\begin{aligned}\rho = {} & 173156925093579519341141346 81d^2 - \\ & 56338816236088458375833229507 44dl^2 \\ & -50148902845361498768071379821226736l^4.\end{aligned} \tag{18}$$

Now if $\eta_8 = 0$ then either

$$d = -\frac{901484}{307857}l^2, \tag{19}$$

or because $l \neq 0, \rho \neq 0$. If $l \neq 0$, $307857d - 901484l^2 \neq 0$ but (18) holds, then we have $d = p_i l^2$. For $i = 1, 2$, with $p_1 = 340.28404100$, $p_2 = -377.123069100$, and in each case η_9 is a multiple of l^7, and $l \neq 0$ (taken above). If (19) holds, then we compute η_9 as:

$$\eta_9 = \frac{32l^5(37026759569911l - 1736569072760400)}{999670687490475}.$$

If $\varkappa_9 = 0$ then, as $l \neq 0$ considered above gives that $l^5 \neq 0$, we takes value of l as:

$$l = \frac{1736569072760400}{37026759569911}. \tag{20}$$

If (20) holds, then we calculate $\varkappa_{10}$ as:

$$\varkappa_{10} = -\frac{\begin{matrix}5981909850866458913426128067910330261654972 1\\ 83986994762723753707656668378300214360133 8\\ 80839216718016650781737345237712655039420363\\ 082187505657708544000000000000000000\end{matrix}}{\begin{matrix}4973878997017328732440906864985754494327312\\ 9235904875486944604484243602120619367493715\\ 9764415673497474229420134524147096811967 58\\ 182064297581227921\end{matrix}}.$$

Here, $\varkappa_{10}$ is equal to a constant number that is non-zero. Thus, we conclude that the multiplicity of class $C_{7,3}$ is 10, i.e., $\mu_{\max}\left(C_{7,3}\right) \geq 10$.

Theorem 4.2. *For equation*

$$\dot{z} = \gamma(t)z^3 + \delta(t)z^2 \tag{21}$$

With

$$\left.\begin{aligned}\gamma(t) = &-\frac{1802968}{307857}\left(\frac{1736569072760400}{37026759569911}+\epsilon_1\right)^2-\frac{163228503184}{41209750941}\epsilon_2+\frac{656215}{121027}\epsilon_3\\ &-\frac{1080}{539}\epsilon_4-\frac{16}{7}\epsilon_5-16\epsilon_6+\epsilon_7+2(-12(\frac{1736569072760400}{37026759569911}+\epsilon_1)^2-\\ &(\frac{757960558278}{2060488754705}\epsilon_2+\epsilon_3)t+4(\frac{8065930672107}{8241955018820}\epsilon_2-\frac{534699}{200984}\epsilon_3+\epsilon_4)t^2\\ &+8(\frac{1736569072760400}{37026759569911}+\epsilon_1)^2+\epsilon_2)t^3+16(\frac{4742797224165}{13187128030112}\epsilon_2-\frac{224575}{229696}\epsilon_3\\ &+\frac{35}{264}\epsilon_4+\frac{15}{2}(\frac{1736569072760400}{37026759569911}+\epsilon_1)^2+\epsilon_5)t^4+128(-\frac{30780466431}{3878567067680}\epsilon_2+\\ &\frac{1699711}{78785728}\epsilon_3+\frac{199}{12936}\epsilon_4-(\frac{1736569072760400}{37026759569911}+\epsilon_1)^2-\frac{2}{35}\epsilon_5+\epsilon_6)t^7.\end{aligned}\right\} \quad (22)$$

$$\delta(t) = -\frac{3473138145520800}{37026759569911}-2\epsilon_1+\epsilon_8 +8(\frac{1736569072760400}{37026759569911}+\epsilon_1)t^3. \quad (23)$$

Choose ϵ_j for $1 \leq j \leq 8$ to be non-zero and small as compared to ϵ_{j-1}. Then (21) has eight distinct non-trivial real periodic solutions.

Proof. If we substitute $\epsilon_p = 0, \forall\, p = 1, 2, ..., 8$, and coefficients are as given in Equations (22) and (23). So, the multiplicity of the origin $\varkappa$ is 10. Now, choose $\epsilon_1 \neq 0$ and $\epsilon_p = 0$ for $2 \leq p \leq 8$; then it can be easily seen that $\varkappa_9$ is a constant multiple of ϵ_1, but $\varkappa_2 = \varkappa_3 = ... = \varkappa_7 = \varkappa_8 = 0$. So, the multiplicity reduces by one and $\varkappa = 9$. For that reason, one periodic solution bifurcates out of the origin. Now, set $\epsilon_1 \neq 0$, $\epsilon_2 \neq 0$ and $\epsilon_p = 0$ for $3 \leq p \leq 8$; then we have $\varkappa_p = 0$ for $p = 2, 3, ..., 7$. But $\varkappa_8$ results in a form of ϵ_2 with some constant multiple. So, $\varkappa = 8$. Now, set $\epsilon_1 \neq 0$, $\epsilon_2 \neq 0, \epsilon_3 \neq 0$ and $\epsilon_p = 0$ for $4 \leq p \leq 8$; then we have $\varkappa_p = 0$ for $p = 2, 3, ..., 6$. But $\varkappa_7$ results in a form of ϵ_3 with some constant multiple. If ϵ_2 is sufficient small, then there are two non-trivial real periodic solutions. Further, moving in the present way, we have eight real periodic non-trivial solutions.

Corollary 4.1. For an equation

$$\dot{z} = \gamma(t)z^3 + \delta(t)z^2 + \gamma + \upsilon. \quad (24)$$

if $\gamma(t)$ and $\delta(t)$ are as given in theorem 4.1, Equation (24) has ten real periodic solutions if γ and υ are small enough.

Proof. If $\gamma = 0$ and $\upsilon = 0$, $\mu = 2$ then (24) has eight real periodic solutions. If $\gamma \neq 0$ but is small enough, then $\mu = 1$ and by using the same arguments as in the above theorem, there are nine distinct periodic solutions other than 0; $z = 0$ is another solution. Thus, we have ten real periodic solutions.

Theorem 4.3. *For class $C_{7,5}$ consider $\delta(t) = i + n(2t+1)^5$ and*

$$\gamma(t) = a + b(2t+1) + c(2t+1)^2 + f(2t+1)^5 + h(2t+1)^7.$$

Then $\mu_{\max}(C_{7,5}) \geq 8$.

Proof. By using theorem 2.2, we have

$$\eta_2 = \frac{182}{3}n + i,$$

$$\eta_3 = 410h + \frac{182}{3}f + \frac{13}{3}c + 2b + a.$$

Thus, the multiplicity of $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$. And multiplicity $\mu = 3$ if $\eta_2 = 0$ but $\eta_3 \neq 0$. If $\eta_2 = \eta_3 = 0$, then we calculate η_4 as:

$$\eta_4 = -\frac{n(-114470h + 1079c + 372b)}{189}.$$

If $\eta_4 = 0$ then either $n = 0$ or

$$h = \frac{1079}{114470}c + \frac{372}{114470}b. \quad (25)$$

If $n = 0$, then $\eta_3 = 0$ shows $i = 0$; hence, $\delta(t) = 0$ and $\eta_2 = 0$ implies that the mean value of $\gamma(t)$ is zero. By corollary 3.2, the origin is a center. Suppose $n \neq 0$. By using (25), we compute η_5 as:

$$\eta_5 = -\frac{8n^2(886519816c + 499588553b)}{2109395925}.$$

If $\eta_5 = 0$ then

$$c = -\frac{499588553}{886519816}b. \quad (26)$$

because we already take $n \neq 0$. If (26) holds, then η_6 is:

$$\eta_6 = \frac{2bn(-379667599239958655624n^2 + 5813410092109719b)}{815558371657762539807}.$$

If $\eta_6 = 0$ then either $b = 0$ or

$$b = \frac{379667599239958655624}{5813410092109719}n^2, \quad (27)$$

because $n \neq 0$. If $b = 0$, then by using (26), (25), $\gamma(t)$ and $\delta(t)$ take the form:

$$\gamma(t) = f[(2t+1)^5 - \frac{182}{3}],$$

$$\delta(t) = n[(2t+1)^5 - \frac{182}{3}].$$

Let $\sigma(t) = \frac{1}{2}(2t+1)^6 - 182t$; then $\dot{\sigma}(t) = 3(2t+1)^5 - 182$. Also, $\sigma(0) = \sigma(1)$. So, we can write it as:

$$\gamma(t) = \frac{1}{3} f\dot{\sigma},$$
$$\delta(t) = \frac{1}{3} n\dot{\sigma}.$$

Then, by theorem 3.1, the origin is a center with $f(\sigma) = \frac{1}{3}f$ and $g(\sigma) = \frac{1}{3}n$. So we take $b \neq 0$. If (27) holds, then η_7 is as follows:

$$\eta_7 = -\frac{586877587954432n^4(-9888560913986218905316\,8639013752599n^2 + 5425568321674937983254\,3161535450f)}{392738038968759387059090863949933323736325}.$$

Now, if $\eta_7 = 0$, recalling that $n \neq 0$ then

$$f = \frac{98885609139862189053168639013752599}{54255683216749379832543161535450} n^2. \tag{28}$$

With holding (28), we have

$$\eta_8 = \frac{2300397726692597332894199628622545916981609410130707279936768}{793823482030814697509595313095537997395261781753311944606875} n^7.$$

Which is a constant multiple of n^7 and is non-zero because $n \neq 0$ (taken above). Thus we conclude that the multiplicity of class $C_{7,5}$ is 8, i.e., $\mu_{\max}(C_{7,5}) \geq 8$.

Theorem 4.4. *For equation*

$$\dot{z} = \gamma(t)z^3 + \delta(t)z^2 + \sigma_1 z + \sigma_2. \tag{29}$$

Let

$$\delta(t) = -\frac{182}{3}n + \epsilon_6 + n(2t+1)^5,$$

$$\gamma(t) = -v_1 + \epsilon_5 + b(2t+1) + c(2t+1)^2 + f(2t+1)^5 + h(2t+1)^7.$$

Proof. With

$$\begin{aligned}
v_1 &= -410h - \frac{182}{3}f - \frac{13}{3}c - 2b,\\
b &= \frac{379667599239958655624}{5813410092109719}n^2 + \epsilon_2,\\
c &= -\frac{194701348602454824917 47}{529020318381984429}n^2 - \frac{499588553}{886519816}\epsilon_2 + \epsilon_3,\\
f &= \frac{98885609139862189053168639013752599}{54255683216749379832543161535450}n^2 + \epsilon_1,
\end{aligned}$$

and

$$h = -\frac{576907546430940497}{4283565331028214}n^2 - \frac{281257}{136387664}\epsilon_2 + \frac{1079}{114470}\epsilon_3 + \epsilon_4.$$

If $\epsilon_j\ (1 \leq j \leq 6)$, σ_1 and σ_2 are chosen to be non-zero and also

$$|\sigma_2| \ll |\sigma_1| \ll |\epsilon_6| \ll |\epsilon_5| \ll \ldots \ll |\epsilon_1|.$$

Then (29) has eight distinct real periodic solutions other than zero.

Theorem 4.5. *Let* $\delta(t) = j + p(t-1)^6$ *and*

$$\gamma(t) = a + c(t-1)^2 + d(t-1)^3 + g(t-1)^6 + h(t-1)^7.$$

For class $C_{7,6}$ *of the form* (9), *then* $\mu_{\max}(C_{7,6}) \geq 8$.

Proof. By using theorem 2.2, we have

$$\eta_2 = \frac{1}{7}p + j, \tag{30}$$

$$\eta_3 = -\frac{1}{8}h + \frac{1}{7}g - \frac{1}{4}d + \frac{1}{3}c + a. \tag{31}$$

Thus, the multiplicity of $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$, and multiplicity $\mu = 3$ if $\eta_2 = 0$ but $\eta_3 \neq 0$. If $\eta_2 = \eta_3 = 0$, by using values of a & j,, $\delta(t)$ and $\gamma(t)$ are as follows:

$$\gamma(t) = c[(t-1)^2 - \frac{1}{3}] + d[(t-1)^3 + \frac{1}{4}] + g[(t-1)^6 - \frac{1}{7}]$$
$$+h[(t-1)^7 - \frac{1}{8}],$$

$$\delta(t) = p[(t-1)^6 - \frac{1}{7}].$$

Also, we calculate η_4 as:

$$\eta_4 = \frac{p(792c - 486d + 77h)}{221760}.$$

If $\eta_4 = 0$ then either $p = 0$ or

$$h = \frac{486}{77}d - \frac{792}{77}c. \tag{32}$$

If $p = 0$, then $\delta(t) = 0$ and $\eta_3 = 0$ implies that mean value $\gamma(t) = 0$. From corollary 3.2, the origin is a center, so consider $p \neq 0$. If (32) holds, then we have η_5 as:

$$\eta_5 = -\frac{p^2(484c - 51d)}{54428220}.$$

If $\eta_5 = 0$, then as $p \neq 0$ (considered above) implies

$$d = \frac{484}{51}c. \tag{33}$$

And by using (33), we calculate η_6 as:

$$\eta_6 = \frac{cp(51382814976p^2 + 520996995995c)}{115437830545837800}.$$

If $\eta_6 = 0$, then either $c = 0$ or

$$c = -\frac{51382814976}{520996995995}p^2, \tag{34}$$

because $p \neq 0$. If $c = 0$ then $\gamma(t)$ and $\delta(t)$ are:

$$\gamma(t) = g[(t-1)^6 - \frac{1}{7}],$$
$$\delta(t) = p[(t-1)^6 - \frac{1}{7}].$$

Let $\sigma(t) = (t-1)^7 - t$, then $\dot{\sigma}(t) = 7(t-1)^6 - 1$. Also, $\sigma(0) = \sigma(1)$. So it takes new form as:

$$\gamma(t) = \frac{g}{7}\dot{\sigma},$$
$$\delta(t) = \frac{p}{7}\dot{\sigma}.$$

By theorem 3.1, having $f(\sigma) = \frac{g}{7}$ and $g(\sigma) = \frac{p}{7}$, the origin is a center, so take $c \neq 0$. Using (34), we have η_7 as:

$$\eta_7 = -\frac{41979424p^4(41353646531078094777799p^2 + 7481442954213656422 40g)}{5365753244092271448722629098254731 25}.$$

If $\eta_7 = 0$, recalling that $p \neq 0$ (η_5), then

$$g = -\frac{41353646531078094777799}{748144295421365642240}p^2. \tag{35}$$

If (35) holds, then we find η_8 as:

$$\eta_8 = \frac{679565224952546531953909700744690288107705 0577}{3176729706559774306700768187469584051223368153410160 0000}p^7.$$

which is a constant multiple of p^5, and p is also non-zero (as shown above). Thus, we conclude that the multiplicity of class $C_{7,6}$ is 8, i.e., $\mu_{\max}(C_{7,6}) \geq 8$.

Theorem 4.6. *Let $C_{9,1}$ be a class of equation of the form* (9)*, with*

$$\gamma(t) = c + dt + et^2 + ft^3 + kt^8 + lt^9.$$

$$\delta(t) = m + nt.$$

We then see that $\mu_{\max}(C_{9,1}) \geq 10$.

Proof. Using theorem 2.2, we take

$$\eta_2 = m + \frac{1}{2}n,$$
$$\eta_3 = c + \frac{1}{2}d + \frac{1}{3}e + \frac{1}{4}f + \frac{1}{9}k + \frac{1}{10}l.$$

Thus, the multiplicity of $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$, and the multiplicity $\mu = 3$ if $\eta_2 = 0$ but $\eta_3 \neq 0$. If $\eta_2 = \eta_3 = 0$, then by using values of "k" and "a," $\gamma(t)$ and $\delta(t)$ are as follows:

$$\gamma(t) = d(t - \frac{1}{2}) + e\left(t^2 - \frac{1}{3}\right) + f\left(t^3 - \frac{1}{4}\right) + k\left(t^8 - \frac{1}{9}\right) + l\left(t^9 - \frac{1}{10}\right), \tag{36}$$

$$\delta(t) = n\left(t - \frac{1}{2}\right). \tag{37}$$

Also, we compute η_4, given below as:

$$\eta_4 = \frac{n(108l + 112k + 99f + 66e)}{23760}.$$

If $\eta_4 = 0$ then either $n = 0$ or

$$l = -\frac{112}{108}k - \frac{99}{108}f - \frac{66}{108}e. \tag{38}$$

If $n = 0$, then $\delta(t) = 0$ and $\eta_3 = 0$ gives that the mean value of $\gamma(t)$ is zero. Thus, the origin is a center from corollary 3.2, so consider $n \neq 0$. Now, if (38) holds, then η_5 is as below:

$$\eta_5 = -\frac{n^2(56k + 297f + 198e)}{7207200}.$$

If $\eta_5 = 0$, then as we already take $n \neq 0$ it implies

$$k = -\frac{297}{56}f - \frac{198}{56}e. \tag{39}$$

and by using (39), η_6 is:

$$\eta_6 = -\frac{n(2e + 3f)(32e - 57n^2 + 29f)}{836559360}.$$

If $\eta_6 = 0$, then as we already consider $n \neq 0$ either $f = -\frac{2}{3}e$ or

$$e = \frac{57}{32}n^2 - \frac{29}{32}f. \tag{40}$$

If $f = -\frac{2}{3}e$ then (36) and (37) are of the following form:

$$\gamma(t) = \left(t - \frac{1}{2}\right)\left[d + e\left(-\frac{2}{3}t^2 + \frac{2}{3}t - \frac{1}{3}\right)\right],$$

$$\delta(t) = n\left(t - \frac{1}{2}\right).$$

Let $\sigma(t) = t^2 - t$; then $\dot{\sigma}(t) = 2t - 1$. Also, $\sigma(0) = \sigma(1)$. So we can write

$$\gamma(t) = \frac{1}{2}\left[d + c\left(-\frac{2}{3}t^2 + \frac{2}{3}t\right)\right]\dot{\sigma},$$

$$\delta(t) = \frac{n}{2}\dot{\sigma}.$$

Thus, from theorem 3.1, the origin is a center with

$$f(\sigma) = \frac{1}{2}\left[d + c\left(-\frac{2}{3}t^2 + \frac{2}{3}t\right)\right],$$

and $g(\sigma) = \frac{n}{2}$, so we take $f \neq -\frac{2}{3}e$. Holding (40), we compute η_7 as:

$$\eta_7 = \frac{19n^2(3n^2 + f)(-348307f + 1697231n^2 + 1445136d)}{254514115215360}.$$

If $\eta_7 = 0$, recalling that $n \neq 0$, then either $f = -3n^2$ or

$$f = \frac{1697231}{348307}n^2 + \frac{1445136}{348307}d. \tag{41}$$

If $f = -3n^2$, then

$$\gamma(t) = \frac{1}{2}\left[d + n^2\left(-3t^2 + 3t\right)\right]\dot{\sigma},$$

$$\delta(t) = \frac{n}{2}\dot{\sigma}.$$

From theorem (3.1), the origin is a center with $f(\sigma) = \frac{1}{2}\left[d + n^2\left(-3t^2 + 3t\right)\right]$ and $g(\sigma) = \frac{n}{2}$, so consider $f \neq -3n^2$. Using (41), we calculate η_8 as:

$$\eta_8 = \frac{23n(1122d + 2129n^2)\zeta}{10294742845940798944790227648 51200}.$$

where $\zeta = 273879615326996052d^2 + 171373534155545550 8dn^2 - 132695961322089231627n^4$. Now, if $\eta_8 = 0$ then either $\zeta = 0$ or

$$d = -\frac{2129}{1122}n^2. \tag{42}$$

because $n \neq 0$. If Equation (42) $\neq 0$, $n \neq 0$, but $\zeta = 0$, then $b = r_i n^2$ for $i = 1, 2$, where $r_1 = 38.208145460$, $r_2 = -50.722660920$. If (42) holds, but $\zeta \neq 0, n \neq 0$, then we compute η_9

$$\eta_9 = -\frac{n^5(5168n + 449059)}{2749402656}.$$

If $\eta_9 = 0$, then we substitute $n = -\frac{449059}{5168}$ and calculate η_{10} as:

$$\eta_{10} = -\frac{27752605638865243090865101496287308874092968151486717758 8679}{164738202978811962713659536291595 7451747785441280}$$

That is a non-zero constant number. Thus, we conclude that the multiplicity of class $C_{9,1}$ is 10, i.e., $\mu_{\max}\left(C_{9,1}\right) \geq 10$.

Theorem 4.7. *Choose n, k, l with $nl \neq 0$. In the equation*

$$\dot{z} = \gamma(t)z^3 + \delta(t)z^2 + \sigma_1 z + \sigma_2. \tag{43}$$

Let

$$\delta(t) = \frac{449059}{10336} - \frac{1}{2}\epsilon_1 + \epsilon_8 + (-\frac{449059}{5168} + \epsilon_1)t,$$

and

$$\gamma(t) = u_1 + dt + et^2 + ft^3 + kt^8 + lt^9.$$

With

$$\begin{aligned}
u_1 &= \frac{223}{1122}(-\frac{449059}{5168} + \epsilon_1)^2 - \frac{287723}{4179684}\epsilon_2 + \frac{419}{4032}\epsilon_3 \\
&\quad -\frac{31}{126}\epsilon_4 - \frac{1}{135}\epsilon_5 - \frac{1}{10}\epsilon_6 + \epsilon_7, \\
d &= -\frac{2129}{1122}(-\frac{449059}{5168} + \epsilon_1)^2 + \epsilon_2, \\
f &= -3(-\frac{449059}{5168} + \epsilon_1)^2 + \frac{1445136}{348307}\epsilon_2 + \epsilon_3, \\
e &= \frac{9}{2}(-\frac{449059}{5168} + \epsilon_1)^2 - \frac{2619309}{696614}\epsilon_2 - \frac{29}{32}\epsilon_3 + \epsilon_4, \\
k &= -\frac{24270543}{2786456}\epsilon_2 - \frac{1881}{896}\epsilon_3 - \frac{99}{28}\epsilon_4 + \epsilon_5,
\end{aligned}$$

and

$$l = \frac{31461815}{4179684}\epsilon_2 + \frac{1045}{576}\epsilon_3 + \frac{55}{18}\epsilon_4 - \frac{28}{27}\epsilon_5 + \epsilon_6.$$

If ϵ_k $\left(1 \leq k \leq 8\right)$, σ_1 and σ_2 are chosen to be non-zero such that

$$|\sigma_2| \ll |\sigma_1| \ll |\epsilon_8| \ll |\epsilon_7| \ll ... \ll |\epsilon_1|.$$

then (43) *has ten real distinct non-trivial periodic solutions.*

Proof. The proof is similar to that in theorem in (4.2), so it is omitted.

It is pertinent to mention that $\mu_{\max}\left(C_{4,3}\right) \geq 5$ given in Yasmin and Ashraf [20] but we succeeded in increasing its multiplicity from 5 to 8, i.e., $\mu_{\max}\left(C_{4,3}\right) \geq 8$ by using variable (t) instead of (2t-1).

Theorem 4.8. *Let*

$$\gamma(t) = e + ft + gt^2 + ht^3 + it^4,$$

$$\delta(t) = a + dt^3.$$

for the class $C_{4,3}$ of form (9). *Then we prove that $\mu_{\max}\left(C_{4,3}\right) \geq 8$.*

Proof. From theorem 2.2, we calculate solutions as:

$$\eta_2 = a + \frac{d}{4}.$$

$$\eta_3 = e + \frac{f}{2} + \frac{g}{3} + \frac{h}{4} + \frac{i}{5}.$$

Thus, the multiplicity of $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$, and multiplicity $\mu = 3$ if $\eta_2 = 0$ but $\eta_3 \neq 0$. If $\eta_2 = \eta_3 = 0$, then

$$a = -\frac{d}{4}. \tag{44}$$

$$e = -\frac{f}{2} - \frac{g}{3} - \frac{h}{4} - \frac{i}{5}. \tag{45}$$

By using (44) and (45), $\gamma(t)$ and $\delta(t)$ are as given below:

$$\gamma(t) = f\left(t - \frac{1}{2}\right) + g\left(t^2 - \frac{1}{3}\right) + h\left(t^3 - \frac{1}{4}\right) + i\left(t^4 - \frac{1}{5}\right), \tag{46}$$

$$\delta(t) = d\left(t^3 - \frac{1}{4}\right). \tag{47}$$

Also we calculate η_4 as:

$$\eta_4 = -\frac{d(-28i + 45g + 105f)}{25200}.$$

If $\eta_4 = 0$ then either $d = 0$ or

$$i = \frac{(45g + 105f)}{28}. \tag{48}$$

If $d = 0$, then $\delta(t) = 0$, and also $\eta_3 = 0$ gives that $\delta(t)$ has mean value 0. From corollary 3.2, the origin is a center. Consider $d \neq 0$. By using (48), we compute η_5 as:

$$\eta_5 = -\frac{d^2(199g + 1617f)}{60540480}.$$

If $\eta_5 = 0$ as $d \neq 0$, then

$$f = -\frac{199}{1617}g. \tag{49}$$

By using (49), we compute η_6 as:

$$\eta_6 = \frac{gd(78600753d^2 + 13597688g)}{2050291017815040}.$$

If $\eta_6 = 0$, then either $g = 0$ or

$$g = -\frac{78600753}{13597688}d^2, \tag{50}$$

because we already take $d \neq 0$. If $g = 0$, then (46) and (47) can be written as:

$$\gamma(t) = h\left(t^3 - \frac{1}{4}\right),$$

$$\delta(t) = d\left(t^3 - \frac{1}{4}\right).$$

Consider $\sigma(t) = t^4 - t$; then $\dot{\sigma}(t) = 4t^3 - 1$. Also $\sigma(0) = \sigma(1)$, so it can written as:

$$\gamma(t) = \frac{h}{4}\dot{\sigma}(t),$$

$$\delta(t) = \frac{d}{4}\dot{\sigma}(t).$$

By theorem 3.1, the origin is a center with $f(\sigma) = \frac{1}{4}h$ and $g(\sigma) = \frac{1}{4}d$. So take $g \neq 0$. If (50) holds, then η_7 is:

$$\eta_7 = -\frac{491d^4(-704698056497d^2 + 61560932862h)}{102299877970079928320}.$$

If $\eta_7 = 0$, recalling that $d \neq 0$, then we take

$$h = \frac{704698056497}{61560932862}d^2. \tag{51}$$

If (51) holds, then we compute η_8 as:

$$\eta_8 = \frac{466979144516058112634649177}{233844781864904008056474186956 80}d^7.$$

where η_8 is a constant multiple of d^7 and $d \neq 0$ (*taken above*). Thus, we conclude that the multiplicity of class $C_{4,3}$ is 8, i.e., $\mu_{\max}(C_{4,3}) \geq 8$.

Theorem 4.9. *Let $C_{3,8}$ be a class of equation of the form* (9) *with*

$$\gamma(t) = bt + dt^3,$$

$$\delta(t) = ft + ht^3 + jt^5 + lt^7 + mt^8.$$

Then $\mu_{\max}(C_{3,8}) \geq 8$ when $j = 1$.

Proof. First we suppose that $j \neq 1$. From theorem 2.2, we see:

$$\eta_2 = \frac{1}{2}f + \frac{1}{4}h + \frac{1}{6}j + \frac{1}{8}l + \frac{1}{9}m. \tag{52}$$

$$\eta_3 = \frac{1}{2}b + \frac{1}{4}d. \tag{53}$$

Thus the multiplicity of $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$, and multiplicity $\mu = 3$ if $\eta_2 = 0$ but $\eta_3 \neq 0$. If $\eta_2 = \eta_3 = 0$, then from (52) and (53) we take:

$$f = -\frac{1}{2}h - \frac{1}{3}j - \frac{1}{4}l - \frac{2}{9}m, \tag{54}$$

and

$$b = -\frac{1}{2}d. \tag{55}$$

Now by using (54) and (55), we calculate η_4 as:

$$\eta_4 = -\frac{d(1400m + 1287l + 858j)}{1235520}.$$

If $\eta_4 = 0$ then either $d = 0$ or

$$m = -\frac{1287}{1400}l - \frac{858}{1400}j. \tag{56}$$

If $d = 0$, then from (55), $\gamma(t) = 0$ and $\eta_3 = 0$ gives that the mean value of $\delta(t)$ is zero. So by corollary 3.2, the origin is a center. Thus, suppose $d \neq 0$. If (56) holds, then η_5 is:

$$\eta_5 = -\frac{d(2j + 3l)(30744j + 59850h + 8041l)}{683726400000}.$$

If $\eta_5 = 0$ then either

$$2j + 3l = 0, \tag{57}$$

or

$$h = -\frac{30744}{59850}j - \frac{8041}{59850}l, \tag{58}$$

because we already take $d \neq 0$. If (57) holds, then $\gamma(t)$ *and* $\delta(t)$ are of the form:

$$\gamma(t) = d\left(t^3 - \frac{t}{2}\right),$$

$$\delta(t) = \left(t^3 - \frac{t}{2}\right)\left[h + j\left(\frac{\frac{3}{5600}t^2 + \frac{88}{525}t}{\frac{3}{700}t^5 + \frac{2}{3}t^4 + \frac{3}{1400}t^3 + \frac{4}{3}t^2 - \frac{3}{2800}t + \frac{2}{5}}\right)\right].$$

Let $\sigma(t) = \frac{t^4}{4} - \frac{t^2}{4}$, then $\dot{\sigma}(t) = t^3 - \frac{t}{2}$. Also $\sigma(0) = \sigma(1)$. $\gamma(t)$ and $\delta(t)$ are thus written as:

$$\gamma(t) = d\dot{\sigma}(t),$$

$$\delta(t) = \left[h + j\left(\frac{\frac{3}{5600}t^2 + \frac{88}{525}t}{\frac{3}{700}t^5 + \frac{2}{3}t^4 + \frac{3}{1400}t^3 + \frac{4}{3}t^2 - d32800t + \frac{2}{5}}\right)\right]\dot{\sigma}(t).$$

So by theorem 3.1, the origin is a center with $f(\sigma) = d$ and

$$g(\sigma) = \left[h + j\left(\frac{\frac{3}{5600}t^2 + \frac{88}{525}t}{\frac{3}{700}t^5 + \frac{2}{3}t^4 + \frac{3}{1400}t^3 + \frac{4}{3}t^2 - \frac{3}{2800}t + \frac{2}{5}}\right)\right].$$

Therefore, we suppose that $3l+2j \neq 0$. Holding (58), we compute η_6, which is a constant multiple of ξ, as:

$$\eta_6 = -d(3l + 2j)\xi.$$

where

$$\xi = 1532329720524j^2 + 1273229285332jl + 1690381942750100d + 25542706569l^2.$$

Now $\eta_6 = 0$ only if $\xi = 0$, because we have already discussed the possibility of $d = 0, (3l + 2j) = 0$; in each case, the origin is a center. For $\xi = 0$, we substitute

$$d = -\frac{153232639720524}{169038161950100}j^2 - \frac{127322239285332}{169038942750100}jl - \frac{25542700956569}{169038161942750100}l^2.$$

and obtain

$$\eta_7 = d\left(3l + 2j\right)\left(\text{homogeneous cubic in } j \text{ and } l\right).$$

and

$$\eta_8 = -d\left(3l + 2j\right)\left(\text{homogeneous quartic in } j \text{ and } l\right).$$

We cannot draw any conclusion looking at η_7 and η_8. Thus, for simplification, we substitute $j = 1$. Then, η_7 and η_8 becomes

$$\eta_7 = \begin{cases} d\left(3l+2\right)\left(-\frac{20052663449157741455869750}{437}\right. \\ +\frac{52546142442075209795770 9100}{4807}l - \\ \frac{26054076800585569433880148475}{302841}l^2 \\ \left. +\frac{2486363753968216260443 00}{11}l^3\right). \end{cases}$$

and

$$\eta_8 = \begin{cases} -d\left(3l+2\right)(-60491257464742335697522 \\ 710660183586465326 27l + \\ 70498585156938778217550785087929520 7088l^2 \\ -367735177988424214344 \\ 949040302512071231 17l^3 + 72607900452360540 \\ 071435525781134 38l^4). \end{cases}$$

For multiplicity >6, we have to prove that the cubic in η_7 and quartic in η_8 have no common zero. For this, we suppose that both have a common zero. Then we get a linear relation in j and l, $j + kl = 0(say)$, and for this value of j, η_8 is a constant multiple of $dl^5 \neq 0$. Thus, we conclude that the multiplicity of class $C_{3,8}$ is 8, i.e., $\mu_{\max}\left(C_{3,8}\right) \geq 8$ with $j = 1$.

Theorem 4.10. *Suppose $l = \alpha$ be a real root of the equation*

$$\begin{cases} -\frac{20052663449157741455869750}{437} \\ +\frac{525461424420752097957709100}{4807}l - \\ \frac{26054076800585569433880148475}{302841}l^2 \\ +\frac{248636375396821626044300}{11}l^3 = 0. \end{cases}$$

Choose

$$\begin{aligned} l &= \alpha + \epsilon_1, \\ d &= -\frac{5197145561}{573321672577500}j^2 - \frac{38865152407}{5159895053197500}j(\alpha + \epsilon_1) - \\ &\quad \frac{1110552215503}{734948530185870000}(\alpha + \epsilon_1)^2 + \epsilon_2, \end{aligned}$$

$$h = -\frac{244}{475}j - \frac{8041}{59850}l + \epsilon_3,$$
$$m = -\frac{1287}{1400}\alpha - \frac{1287}{1400}\epsilon_1 - \frac{429}{700}j + \epsilon_4,$$
$$f = \frac{397}{6650}j + \frac{8041}{119700} - \frac{1}{2}\epsilon_3 - \frac{8}{175}\alpha - \frac{8}{175}\epsilon_1 - \frac{2}{9}\epsilon_4 + \epsilon_6,$$
$$b = \frac{5195561}{11461500}j^2 + \frac{3886407}{1035000}j(\alpha+\epsilon_1) + \frac{11103}{14600}(\alpha+\epsilon_1)^2 - \frac{1}{2}\epsilon_2 + \epsilon_5,$$

Such that $|\epsilon_6| \ll |\epsilon_5| \ll |\epsilon_4| \ll |\epsilon_3| \ll |\epsilon_2| \ll |\epsilon_1|$. *Then equation* $\dot{z} = \gamma(t)z^3 + \delta(t)z^2$, *has six real periodic non-trivial solutions. Where*

$$\gamma(t) = bt + dt^3.$$

$$\delta(t) = ft + ht^3 + jt^5 + lt^7 + mt^8.$$

with $j = 1$.

Proof. If we put ϵ_j for j as; $1 \le j \le 6$, instead of $1 \le j \le 8$ then the proof is similar to that for theorem (4.2), so it is omitted.

5. EXAMPLES

The following examples demonstrate the applicability of our main results.

Example: Consider the differential equation:

$$\frac{dz}{dt} = \left(e^t\right) z^3 + (\cos t)\, z^2. \tag{59}$$

Here $\gamma(t), \delta(t)$ are transcendental functions, but we use the power series representations by neglecting the terms 't^n' for $n > 4$. Like $\gamma(t) = e^t = a + bt + ct^2 + dt^3 + et^4$, $\delta(t) = \cos t = j - kt^2 + lt^4$, with $a = b = j = 1$, $c = k = \frac{1}{2!}$, $d = \frac{1}{3!}$, $e = \frac{1}{4!}$, and then calculate the periodic solutions.

SOLUTION: We substitute $k = 0$, and by using theorem 2.2, we calculate:

$$\eta_2 = j + \frac{1}{120}l, \tag{60}$$

$$\eta_3 = a + \frac{1}{2}b + \frac{1}{6}c + \frac{1}{24}d + \frac{1}{120}e. \tag{61}$$

Thus, the multiplicity of $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$, and multiplicity $\mu = 3$ if $\eta_2 = 0$ but $\eta_3 \neq 0$. If $\eta_2 = \eta_3 = 0$, then we take $j = -\frac{1}{120}l$, and $a = -\frac{1}{2}b - \frac{1}{6}c - \frac{1}{24}d - \frac{1}{120}e$. By using these values, η_4 is calculated as:

$$\eta_4 = -\frac{l(14d + 105c + 360b)}{1814400}.$$

If $\eta_4 = 0$ then either $l = 0$ or

$$d = -\frac{105}{14}c - \frac{360}{14}b. \tag{62}$$

If $l = 0$, then $\delta(t) = 0$ and $\eta_3 = 0$ shows that the mean value of $\gamma(t)$ is zero. So by corollary 3.2, the origin is a center. Suppose $l \neq 0$. If (62) holds, we have

$$\eta_5 = -\frac{l^2(28c + 325b)}{6054048000}.$$

If $\eta_5 = 0$, then $l \neq 0$ $\left(\textit{taken above}\right)$, and we substitute $c = -\frac{325}{28}b$, and calculate η_6 as:

$$\eta_6 = \frac{bl(-21078407l^2 + 10315069600b)}{1261504744980480000}.$$

If $\eta_6 = 0$, then either $b = 0$ or

$$b = \frac{21078407}{10315069600}l^2, \tag{63}$$

because $l \neq 0$. If $b = 0$ then $d = c = 0$, by using these values, $\gamma(t)$ and $\delta(t)$ takes the following form:

$$\gamma(t) = e\left(t^4 - \frac{1}{5}\right),$$

$$\delta(t) = l\left(t^4 - \frac{1}{5}\right).$$

Let $\sigma(t) = t^5 - t$, then $\dot{\sigma}(t) = 5t^4 - 1$. Also, $\sigma(0) = \sigma(1)$. Therefore, we can write $\gamma(t) = \frac{1}{5}e\dot{\sigma}$ and $\delta(t) = \frac{1}{5}l\dot{\sigma}$. From theorem 3.1, The origin is a center having $f(\sigma) = \frac{1}{5}e$ and $g(\sigma) = \frac{1}{5}l$. Thus, suppose that $b \neq 0$. By using (63), we have η_7 as follows:

$$\eta_7 = -\frac{3011201l^4(4904530106070545l^2 + 18047929302961536e)}{11937513330495722763883212963840000000}.$$

Recalling that $l \neq 0$ (considered above), if $\eta_7 = 0$ then

$$e = -\frac{4904530106070545}{18047929302961536}l^2. \tag{64}$$

If (64) holds, then we find

$$\eta_8 = \frac{373952841430967319297259961 89267}{860144982077104954669374286063444008999321600000000}l^7.$$

which is constant multiple of l^7, and $l \neq 0$. Thus we conclude that (59) have eight periodic solutions.

Example: Consider the differential equation:

$$\frac{dz}{dt} = \gamma(t)\, z^3 + \delta(t)\, z^2. \tag{65}$$

With $\gamma(t)$ =equation of circle $= ax^2 + by^2 + cx + dy + f$, $\delta(t)$ =quadratic equation$= gx^2 - h$, we calculate the periodic solutions.

SOLUTION: By using theorem 2.2, we calculate:

$$\eta_2 = \frac{g}{3} - h,$$

$$\eta_3 = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{2}c + \frac{1}{2}d + f.$$

Thus, the multiplicity of $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$, and multiplicity $\mu = 3$ if $\eta_2 = 0$ but $\eta_3 \neq 0$. If $\eta_2 = \eta_3 = 0$, then we put $h = \frac{g}{3}$, and $f = -\frac{1}{3}a - \frac{1}{3}b - \frac{1}{2}c - \frac{1}{2}d$. By using these values, we get

$$\eta_4 = -\frac{1}{360}cg.$$

If $\eta_4 = 0$ then $c = g = 0$. If $g = 0$, then $\delta(t) = 0$ and $\eta_3 = 0$ shows the mean value of $\gamma(t)$ is zero. So by corollary 3.2, the origin is a center. Suppose that $g \neq 0$. Now by using the value of $c = 0$, we calculate $\eta_5 = 0$. Thus we conclude that (65) have four periodic solutions.

6. CONCLUSION AND DISCUSSION

In this article, periodic solutions are calculated. The solutions satisfying $\mathfrak{z}(\beta) = \mathfrak{z}(0)$, are called periodic orbits of the Equation (1). The periodic orbits that are isolated in the set of all periodic orbits are usually called the limit cycle. Periodic solutions are found for algebraic coefficients for various classes by using bifurcation analysis. We examined classes $C_{3,8}, C_{4,3}, C_{7,3}, C_{7,5}, C_{7,6}, C_{9,1}$. We could only get a maximum multiplicity of 10 by using the classical formulas existing in the literature. We succeeded in developing the formula η_{10} by which classes $C_{7,3}$, and $C_{9,1}$ have maximum multiplicity 10, which is the highest known until this time. We also improved some already calculated results of Yasmin and Ashraf [20] for class $C_{4,3}$, where $\mu_{\max}$ is improved from 5 to 8. A systematic procedure has been established in defining coefficients of higher-order polynomial functions. A perturbation method has been properly established for making the maximal number of limit cycles in section 3, which was used numerically to calculate all the classes mentioned in the article. Some examples are also presented to show the applicability of the method. Since the journey toward solving Hilbert's 16th problem is still far at an end, searching for more limit cycles and raising the general lower bound form could be an effective choice for approaching the problem.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

SA and AN: conceptualization. SA, AN, and NY: writing original draft. DB and KN: methodology, review, and editing. AG and DB: formal analysis. SA, AG, and KN: software. All authors contributed to the article and approved the submitted version.

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