



Invariant Solutions and Conservation Laws of the Variable-Coefficient Heisenberg Ferromagnetic Spin Chain Equation

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The variable-coefficient Heisenberg ferromagnetic spin chain (vcHFSC) equation is investigated using the Lie group method. The infinitesimal generators and Lie point symmetries are reported. Four types of similarity reductions are acquired by virtue of the optimal system of one-dimensional subalgebras. Several invariant solutions are derived, including trigonometric and hyperbolic function solutions. Furthermore, conservation laws for the vcHFSC equation are obtained with the help of Lagrangian and non-linear self-adjointness.

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INTRODUCTION

The investigation of physical phenomenon modeled by non-linear partial differential equations (NLPDEs) and searching for their underlying dynamics remain the hot issue of research for applied and theoretical sciences. A lot of attention has been concentrated on looking for the explicit solutions of NLPDEs, for they can provide accurate information with which to understand some interesting physical phenomena. A great many powerful methods have been proposed to construct the explicit solutions of NLPDEs, such as the inverse scattering method [1], the Lie group method [2–5], the Hirota bilinear method [6, 7], the extended tanh method [8–10], the homoclinic test method [11–13], the F-expansion technique [14], and so on [15–18]. Among these methods, the Lie group method is a powerful and prolific method for the study of NLPDEs. On the one hand, based on the Lie group method, we can obtain new exact solutions directly or from the known ones or via similarity reductions; on the other hand, the conservation laws can be constructed via the corresponding Lie point symmetries. Recently, invariant solutions of a class of constant and variable coefficient NLPDEs have been obtained by virtue of this method, such as Keller-Segel models [19], generalized fifth-order non-linear integrable equation [20], KdV equation [21], and Davey-Stewartson equation [22].

So far, many effective methods have been extended to construct exact solutions of different types of differential equations. For example, the generalized Bernoulli sub-ODE and the generalized tanh methods have been applied to establish optical soliton solutions of the Chen-Lee-Liu equation [23]. The Lie group method has been used to find the exact solutions of the time fractional Abrahams–Tsuneto reaction diffusion system [24] and the non-linear transmission line equation [25].

In this work, we will focus on the (2+1)-dimensional variable-coefficient Heisenberg ferromagnetic spin chain (vcHFSC) equation

$$iq_t + \delta_1(t)q_{xx} + \delta_2(t)q_{yy} + \delta_3(t)q_{xy} + \delta_4(t)|q|^2q = 0, \quad (1)$$

where $\delta_1(t)$, $\delta_2(t)$, $\delta_3(t)$, and $\delta_4(t)$ are arbitrary functions with respect to t . The interaction properties and stability of the bright and dark solitons are presented in [26]. Non-autonomous complex wave and analytic solutions are obtained in [27]. When $\delta_i(t)$ ($i = 1, \dots, 4$) are arbitrary constants, Equation (1) can be reduced to the following (2+1)-dimensional Heisenberg ferromagnetic spin chain (HFSC) equation:

$$iq_t + \delta_1q_{xx} + \delta_2q_{yy} + \delta_3q_{xy} + \delta_4|q|^2q = 0. \quad (2)$$

Latha and Vasanthi [28] obtained multisoliton solutions by employing Darboux transformation and analyzed the interaction properties of Equation (2). Anitha et al. [29] derived the dynamical equations of motion by employing long wavelength approximation and discussed the complete non-linear excitation with the aid of sine-cosine function method. Periodic solutions were obtained by Triki and Wazwaz [30], and they also discussed conditions for the existence and uniqueness of wave solutions. Tang et al. [31] reported the explicit power series solutions and bright and dark soliton solutions of Equation (2), and they also obtained some other exact solutions via the sub-ODE method.

However, the Lie symmetries, invariant solutions, and conservation laws of the (2+1)-dimensional vcHFSC equation (1) have not been studied. In the current work, we study the vcHFSC equation (1) via the Lie group method and obtain new invariant solutions, including the trigonometric and hyperbolic function solutions. Moreover, based on non-linear self-adjointness, conservation laws for vcHFSC equation (1) are constructed.

The main structure of this paper is as follows. In section Lie Symmetry Analysis and Optimal System, based on the Lie symmetry analysis, we construct the Lie point symmetries and the optimal system of one-dimensional subalgebras for Equation (1). In section Symmetry Reductions and Invariant Solutions, four types of similarity reductions and some invariant solutions are studied by virtue of the optimal system. In section Non-linear Self-Adjointness and Conservation Laws, conservation laws for Equation (1) are obtained with the help of Lagrangian and non-linear self-adjointness. Section Results and Discussion provides the results and discussion. Finally, the conclusion is given in section Conclusion.

LIE SYMMETRY ANALYSIS AND OPTIMAL SYSTEM

In this section, our aim is to obtain the Lie point symmetries and the optimal system of the vcHFSC equation (1) by employing the Lie group method.

The vcHFSC equation (1) can be changed to the following system

$$\begin{cases} F_1 = u_t + \delta_1(t)v_{xx} + \delta_2(t)v_{yy} + \delta_3(t)v_{xy} \\ \quad + \delta_4(t)(u^2v + v^3) = 0, \\ F_2 = -v_t + \delta_1(t)u_{xx} + \delta_2(t)u_{yy} + \delta_3(t)u_{xy} \\ \quad + \delta_4(t)(u^3 + uv^2) = 0, \end{cases} \quad (3)$$

by using the transformation

$$q(x, y, t) = u(x, y, t) + iv(x, y, t), \quad (4)$$

where $u(x, y, t)$ and $v(x, y, t)$ are real and smooth functions.

Suppose that the associated vector field of system (3) is as follows:

$$\begin{aligned} V = & \xi^1(x, y, t, u, v) \frac{\partial}{\partial x} + \xi^2(x, y, t, u, v) \frac{\partial}{\partial y} + \xi^3(x, y, t, u, v) \frac{\partial}{\partial t} \\ & + \eta^1(x, y, t, u, v) \frac{\partial}{\partial u} + \eta^2(x, y, t, u, v) \frac{\partial}{\partial v}, \end{aligned} \quad (5)$$

where $\xi^1(x, y, t, u, v)$, $\xi^2(x, y, t, u, v)$, $\xi^3(x, y, t, u, v)$, $\eta^1(x, y, t, u, v)$ and $\eta^2(x, y, t, u, v)$ are unknown functions that need to be determined.

If vector field (5) generates a symmetry of system (3), then V must satisfy symmetry condition

$$pr^{(2)}V(\Delta_1) \Big|_{\Delta_1} = 0, pr^{(2)}V(\Delta_2) \Big|_{\Delta_2} = 0, \quad (6)$$

where

$$\begin{cases} \Delta_1 = u_t + \delta_1(t)v_{xx} + \delta_2(t)v_{yy} + \delta_3(t)v_{xy} + \delta_4(t)(u^2v + v^3), \\ \Delta_2 = -v_t + \delta_1(t)u_{xx} + \delta_2(t)u_{yy} + \delta_3(t)u_{xy} + \delta_4(t)(u^3 + uv^2). \end{cases}$$

The infinitesimals ξ^1 , ξ^2 , ξ^3 , η^1 , and η^2 must satisfy the following invariant conditions

$$\begin{cases} \eta_t^1 + \xi^3 \delta_1'(t)v_{xx} + \delta_1(t)\eta_{xx}^2 + \xi^3 \delta_2'(t)v_{yy} + \delta_2(t)\eta_{yy}^2 \\ \quad + \xi^3 \delta_3'(t)v_{xy} \\ \quad + \delta_3(t)\eta_{xy}^2 + \xi^3 \delta_4'(t)(u^2v + v^3) + \delta_4(t)(2u\eta^1v \\ \quad + u^2\eta^2 + 3v^2\eta^2) = 0, \\ -\eta_t^2 + \xi^3 \delta_1'(t)u_{xx} + \delta_1(t)\eta_{xx}^1 + \xi^3 \delta_2'(t)u_{yy} \\ \quad + \delta_2(t)\eta_{yy}^1 + \xi^3 \delta_3'(t)u_{xy} \\ \quad + \delta_3(t)\eta_{xy}^1 + \xi^3 \delta_4'(t)(u^3 + uv^2) + \delta_4(t)(3u^2\eta^1 \\ \quad + \eta^1v^2 + 2uv\eta^2) = 0, \end{cases} \quad (7)$$

where

$$\begin{aligned} \eta_t^1 &= D_t(\eta^1 - \xi^1u_x - \xi^2u_y - \xi^3u_t) + \xi^1u_{xt} + \xi^2u_{yt} \\ &\quad + \xi^3u_{tt}, \\ \eta_{xx}^1 &= D_{xx}(\eta^1 - \xi^1u_x - \xi^2u_y - \xi^3u_t) + \xi^1u_{xxx} \\ &\quad + \xi^2u_{xxy} + \xi^3u_{xxt}, \\ \eta_{xy}^1 &= D_{xy}(\eta^1 - \xi^1u_x - \xi^2u_y - \xi^3u_t) + \xi^1u_{xxy} \\ &\quad + \xi^2u_{xyy} + \xi^3u_{xyt}, \\ \eta_{yy}^1 &= D_{yy}(\eta^1 - \xi^1u_x - \xi^2u_y - \xi^3u_t) + \xi^1u_{xyy} \end{aligned}$$

$$\begin{aligned}
 & +\xi^2 u_{yyy} + \xi^3 u_{yyt}, \\
 \eta_t^2 & = D_t(\eta^2 - \xi^1 v_x - \xi^2 v_y - \xi^3 v_t) + \xi^1 v_{xt} \\
 & + \xi^2 v_{yt} + \xi^3 v_{tt}, \\
 \eta_{xx}^2 & = D_{xx}(\eta^2 - \xi^1 v_x - \xi^2 v_y - \xi^3 v_t) + \xi^1 v_{xxx} \\
 & + \xi^2 v_{xxy} + \xi^3 v_{xxt}, \\
 \eta_{xy}^2 & = D_{xy}(\eta^2 - \xi^1 v_x - \xi^2 v_y - \xi^3 v_t) \\
 & + \xi^1 v_{xxy} + \xi^2 v_{xyy} + \xi^3 v_{xyt}, \\
 \eta_{yy}^2 & = D_{yy}(\eta^2 - \xi^1 v_x - \xi^2 v_y \\
 & - \xi^3 v_t) + \xi^1 v_{xyy} + \xi^2 v_{yyy} + \xi^3 v_{yyt}.
 \end{aligned}$$

Solving Equation (7), one can obtain

$$\begin{aligned}
 \xi^1 & = c_1 x + c_2, \xi^2 = c_1 y + c_3, \xi^3 = \frac{2c_1 \int \delta_1(t) dt}{\delta_1(t)} \\
 & + \frac{c_4}{\delta_1(t)}, \eta^1 = c_1 u, \eta^2 = c_1 v, \quad (8)
 \end{aligned}$$

where $c_1, c_2, c_3,$ and c_4 are arbitrary constants, and the coefficient functions $\delta_1(t), \delta_2(t), \delta_3(t),$ and $\delta_4(t)$ are determined by

$$\begin{aligned}
 \xi^3 \delta_{2t} + \xi_t^3 \delta_2 - 2\delta_2 c_1 & = 0, \\
 \xi^3 \delta_{3t} + \xi_t^3 \delta_3 - 2\delta_3 c_1 & = 0, \\
 \xi^3 \delta_{4t} + \xi_t^3 \delta_4 + 2c_1 \delta_4 & = 0. \quad (9)
 \end{aligned}$$

The Lie algebra of infinitesimal symmetries of system (3) is generated by the four vector fields:

$$\begin{aligned}
 \tilde{\mathfrak{J}}_1 & = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{2 \int \delta_1(t) dt}{\delta_1(t)} \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}, \\
 \tilde{\mathfrak{J}}_2 & = \frac{\partial}{\partial x}, \tilde{\mathfrak{J}}_3 = \frac{\partial}{\partial y}, \tilde{\mathfrak{J}}_4 = \frac{1}{\delta_1(t)} \frac{\partial}{\partial t}. \quad (10)
 \end{aligned}$$

The one-parameter groups g_i generated by the $\tilde{\mathfrak{J}}_i$ are given as follows:

$$\begin{aligned}
 g_1 & : (x, y, t, u, v) \rightarrow \left(x e^\varepsilon, y e^\varepsilon, t + \varepsilon \frac{2 \int \delta_1(t) dt}{\delta_1(t)}, u e^\varepsilon, v e^\varepsilon \right), \\
 g_2 & : (x, y, t, u, v) \rightarrow (x + \varepsilon, y, t, u, v), \\
 g_3 & : (x, y, t, u, v) \rightarrow (x, y + \varepsilon, t, u, v), \\
 g_4 & : (x, y, t, u, v) \rightarrow \left(x, y, t + \frac{\varepsilon}{\delta_1(t)}, u, v \right). \quad (11)
 \end{aligned}$$

If $\{u = U(x, y, t), v = V(x, y, t)\}$ is a solution of system (3), by employing symmetry groups g_i ($i = 1, 2, 3, 4$), we can obtain the following new solutions

$$\begin{aligned}
 (u^{(1)}, v^{(1)}) & \rightarrow \left(e^\varepsilon U \left(x e^{-\varepsilon}, y e^{-\varepsilon}, t - \varepsilon \frac{2 \int \delta_1(t) dt}{\delta_1(t)} \right), \right. \\
 & \left. e^\varepsilon V \left(x e^{-\varepsilon}, y e^{-\varepsilon}, t - \varepsilon \frac{2 \int \delta_1(t) dt}{\delta_1(t)} \right) \right), \\
 (u^{(2)}, v^{(2)}) & \rightarrow (U(x - \varepsilon, y, t), V(x - \varepsilon, y, t)),
 \end{aligned}$$

TABLE 1 | Commutator table of the vector fields of system (3).

$[\tilde{\mathfrak{J}}_i, \tilde{\mathfrak{J}}_j]$	$\tilde{\mathfrak{J}}_1$	$\tilde{\mathfrak{J}}_2$	$\tilde{\mathfrak{J}}_3$	$\tilde{\mathfrak{J}}_4$
$\tilde{\mathfrak{J}}_1$	0	$-\tilde{\mathfrak{J}}_2$	$-\tilde{\mathfrak{J}}_3$	$-2\tilde{\mathfrak{J}}_4$
$\tilde{\mathfrak{J}}_2$	$\tilde{\mathfrak{J}}_2$	0	0	0
$\tilde{\mathfrak{J}}_3$	$\tilde{\mathfrak{J}}_3$	0	0	0
$\tilde{\mathfrak{J}}_4$	$2\tilde{\mathfrak{J}}_4$	0	0	0

TABLE 2 | Adjoint table of the vector fields of system (3).

Ad	$\tilde{\mathfrak{J}}_1$	$\tilde{\mathfrak{J}}_2$	$\tilde{\mathfrak{J}}_3$	$\tilde{\mathfrak{J}}_4$
$\tilde{\mathfrak{J}}_1$	$\tilde{\mathfrak{J}}_1$	$\tilde{\mathfrak{J}}_2 e^\varepsilon$	$\tilde{\mathfrak{J}}_3 e^\varepsilon$	$\tilde{\mathfrak{J}}_4 e^{2\varepsilon}$
$\tilde{\mathfrak{J}}_2$	$\tilde{\mathfrak{J}}_1 - \varepsilon \tilde{\mathfrak{J}}_2$	$\tilde{\mathfrak{J}}_2$	$\tilde{\mathfrak{J}}_3$	$\tilde{\mathfrak{J}}_4$
$\tilde{\mathfrak{J}}_3$	$\tilde{\mathfrak{J}}_1 - \varepsilon \tilde{\mathfrak{J}}_3$	$\tilde{\mathfrak{J}}_2$	$\tilde{\mathfrak{J}}_3$	$\tilde{\mathfrak{J}}_4$
$\tilde{\mathfrak{J}}_4$	$\tilde{\mathfrak{J}}_1 - 2\varepsilon \tilde{\mathfrak{J}}_4$	$\tilde{\mathfrak{J}}_2$	$\tilde{\mathfrak{J}}_3$	$\tilde{\mathfrak{J}}_4$

$$\begin{aligned}
 (u^{(3)}, v^{(3)}) & \rightarrow (U(x, y - \varepsilon, t), V(x, y - \varepsilon, t)), \\
 (u^{(4)}, v^{(4)}) & \rightarrow \left(U \left(x, y, t - \frac{\varepsilon}{\delta_1(t)} \right), V \left(x, y, t - \frac{\varepsilon}{\delta_1(t)} \right) \right). \quad (12)
 \end{aligned}$$

In order to construct the optimal system for system (3), we apply the formula

$$Ad(\exp(\varepsilon \tilde{\mathfrak{J}}_i)) \tilde{\mathfrak{J}}_j = \tilde{\mathfrak{J}}_j - \varepsilon [\tilde{\mathfrak{J}}_i, \tilde{\mathfrak{J}}_j] + \frac{\varepsilon^2}{2} [\tilde{\mathfrak{J}}_i, [\tilde{\mathfrak{J}}_i, \tilde{\mathfrak{J}}_j]] - \dots, \quad (13)$$

where $[\tilde{\mathfrak{J}}_i, \tilde{\mathfrak{J}}_j] = \tilde{\mathfrak{J}}_i \tilde{\mathfrak{J}}_j - \tilde{\mathfrak{J}}_j \tilde{\mathfrak{J}}_i$ and ε is a parameter. The commutator table and the adjoint table of system (3) have been constructed and are presented in Tables 1, 2, respectively.

Based on Tables 1, 2, system (3) has the following optimal system [3, 32]

$$\text{(i) } \tilde{\mathfrak{J}}_1; \text{ (ii) } \tilde{\mathfrak{J}}_2 + \alpha \tilde{\mathfrak{J}}_3 + \beta \tilde{\mathfrak{J}}_4; \text{ (iii) } \tilde{\mathfrak{J}}_3 + \chi \tilde{\mathfrak{J}}_4; \text{ (iv) } \tilde{\mathfrak{J}}_4, \quad (14)$$

where $\alpha, \beta,$ and χ are arbitrary constants.

SYMMETRY REDUCTIONS AND INVARIANT SOLUTIONS

Based on the optimal system (14), our major goal is to deal with the similarity reductions and invariant solutions for system (3).

Subalgebra $\tilde{\mathfrak{J}}_1$

The characteristic equations of subalgebra $\tilde{\mathfrak{J}}_1$ can be written as

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{\frac{2}{\delta_1(t)} \int \delta_1(t) dt} = \frac{du}{u} = \frac{dv}{v}. \quad (15)$$

Solving these equations yields the four similarity variables

$$\begin{aligned}
 r & = x \left(\int \delta_1(t) dt \right)^{-\frac{1}{2}}, s = y \left(\int \delta_1(t) dt \right)^{-\frac{1}{2}}, \\
 u & = F(r, s) \cdot \left(\int \delta_1(t) dt \right)^{\frac{1}{2}}, v = H(r, s) \cdot \left(\int \delta_1(t) dt \right)^{\frac{1}{2}}, \quad (16)
 \end{aligned}$$

and solving the constrained conditions (9), we get

$$\begin{aligned} \delta_2(t) &= k_1\delta_1(t), \delta_3(t) = k_2\delta_1(t), \\ \delta_4(t) &= k_3\delta_1(t) \left(\int \delta_1(t) dt \right)^{-2}, \end{aligned} \tag{17}$$

where $k_1, k_2,$ and k_3 are arbitrary constants. These variables reduce system (3) to the following (1+1)-dimensional PDEs

$$\begin{cases} F - rF_r - sF_s + 2H_{rr} + 2k_1H_{ss} + 2k_2H_{rs} \\ + 2k_3(F^2H + H^3) = 0, \\ -H + rH_r + sH_s + 2F_{rr} + 2k_1F_{ss} + 2k_2F_{rs} + 2k_3(F^3 \\ + FH^2) = 0. \end{cases} \tag{18}$$

Subalgebra $\tilde{\mathcal{J}}_1$ does not give any group-invariant solutions.

Subalgebra $\tilde{\mathcal{J}}_2 + \alpha\tilde{\mathcal{J}}_3 + \beta\tilde{\mathcal{J}}_4$

The similarity variables of this generator are

$$\begin{aligned} r &= \alpha x - y, s = \beta x - \int \delta_1(t) dt, \\ u &= F(r, s), v = H(r, s), \end{aligned} \tag{19}$$

and solving the constrained conditions (9), we get

$$\delta_2(t) = k_1\delta_1(t), \delta_3(t) = k_2\delta_1(t), \delta_4(t) = k_3\delta_1(t), \tag{20}$$

where $k_i (i = 1, 2, 3, 4)$ are arbitrary constants. Substituting Equations (19) and (20) into (3), we have

$$\begin{cases} F_s - (\alpha^2 + k_1 - \alpha k_2)H_{rr} - \beta^2H_{ss} - (2\alpha\beta - \beta k_2)H_{rs} \\ - k_3(F^2H + H^3) = 0, \\ H_s + (\alpha^2 + k_1 - \alpha k_2)F_{rr} + \beta^2F_{ss} + (2\alpha\beta - \beta k_2)F_{rs} \\ + k_3(F^3 + FH^2) = 0. \end{cases} \tag{21}$$

For solving Equation (21), we use the transformation $\zeta = r - \kappa s, F = f(\zeta), H = h(\zeta)$, where κ is an arbitrary constant, and then (21) can be reduced to the following ODEs

$$\begin{cases} -\kappa f' + (2\alpha\beta\kappa - \beta k_2\kappa - \beta^2\kappa^2 - \alpha^2 - k_1 + \alpha k_2)h'' \\ - k_3(f^2h + h^3) = 0, \\ -\kappa h' - (2\alpha\beta\lambda - \beta k_2\lambda - \beta^2\lambda^2 - \alpha^2 - k_1 + \alpha k_2)f'' \\ + k_3(f^3 + fh^2) = 0. \end{cases} \tag{22}$$

Solving Equation (22) yields

$$\begin{cases} f = -B_1 \\ + A_1 \tan \left(r - \frac{4\alpha\beta - 2\beta k_2 + 1 - \sqrt{4\beta^2(k_2^2 - 4k_1) + 4\beta(2\alpha - k_2) + 1}}{4\beta^2} s \right), \\ h = A_1 \\ + B_1 \tan \left(r - \frac{4\alpha\beta - 2\beta k_2 + 1 - \sqrt{4\beta^2(k_2^2 - 4k_1) + 4\beta(2\alpha - k_2) + 1}}{4\beta^2} s \right), \end{cases} \tag{23}$$

and

$$\begin{cases} f = -B_1 \\ + A_1 \cot \left(r - \frac{4\alpha\beta - 2\beta k_2 + 1 - \sqrt{4\beta^2(k_2^2 - 4k_1) + 4\beta(2\alpha - k_2) + 1}}{4\beta^2} s \right), \\ h = A_1 \\ + B_1 \cot \left(r - \frac{4\alpha\beta - 2\beta k_2 + 1 - \sqrt{4\beta^2(k_2^2 - 4k_1) + 4\beta(2\alpha - k_2) + 1}}{4\beta^2} s \right), \end{cases} \tag{24}$$

where $k_3 = -\frac{4\alpha\beta - 2\beta k_2 + 1 - \sqrt{4\beta^2(k_2^2 - 4k_1) + 4\beta(2\alpha - k_2) + 1}}{4\beta^2(A_1^2 + B_1^2)}$ and A_1, B_1 are free parameters.

Based on Equations (19), (23), and (24), we obtain the following trigonometric function solutions for system (3)

$$\begin{cases} u = -B_1 \\ + A_1 \tan \left(\alpha x - y - \frac{4\alpha\beta - 2\beta k_2 + 1 - \sqrt{4\beta^2(k_2^2 - 4k_1) + 4\beta(2\alpha - k_2) + 1}}{4\beta^2} (\beta x - \int \delta_1(t) dt) \right), \\ v = A_1 \\ + B_1 \tan \left(\alpha x - y - \frac{4\alpha\beta - 2\beta k_2 + 1 - \sqrt{4\beta^2(k_2^2 - 4k_1) + 4\beta(2\alpha - k_2) + 1}}{4\beta^2} (\beta x - \int \delta_1(t) dt) \right), \end{cases} \tag{25}$$

and

$$\begin{cases} u = -B_1 \\ + A_1 \cot \left(\alpha x - y - \frac{4\alpha\beta - 2\beta k_2 + 1 - \sqrt{4\beta^2(k_2^2 - 4k_1) + 4\beta(2\alpha - k_2) + 1}}{4\beta^2} (\beta x - \int \delta_1(t) dt) \right), \\ v = A_1 \\ + B_1 \cot \left(\alpha x - y - \frac{4\alpha\beta - 2\beta k_2 + 1 - \sqrt{4\beta^2(k_2^2 - 4k_1) + 4\beta(2\alpha - k_2) + 1}}{4\beta^2} (\beta x - \int \delta_1(t) dt) \right), \end{cases} \tag{26}$$

where $k_3 = -\frac{4\alpha\beta - 2\beta k_2 + 1 - \sqrt{4\beta^2(k_2^2 - 4k_1) + 4\beta(2\alpha - k_2) + 1}}{4\beta^2(A_1^2 + B_1^2)}$ and A_1, B_1 are free parameters.

Subalgebra $\tilde{\mathcal{J}}_3 + \chi\tilde{\mathcal{J}}_4$

The similarity variables of this generator are

$$\begin{aligned} r &= x, s = \chi y - \int \delta_1(t) dt, \\ u &= F(r, s), v = H(r, s), \end{aligned} \tag{27}$$

and solving the constrained conditions (9), we get

$$\delta_2(t) = k_1\delta_1(t), \delta_3(t) = k_2\delta_1(t), \delta_4(t) = k_3\delta_1(t), \tag{28}$$

where $k_i (i = 1, 2, 3, 4)$ are arbitrary constants. System (3) can then be transformed to

$$\begin{cases} F_s - H_{rr} - \chi^2 k_1 H_{ss} - \chi k_2 H_{rs} - k_3(F^2H + H^3) = 0, \\ H_s + F_{rr} + \chi^2 k_1 F_{ss} + \chi k_2 F_{rs} + k_3(F^3 + FH^2) = 0. \end{cases} \tag{29}$$

For solving Equation (29), we use the transformation $\zeta = r - \kappa s, F = f(\zeta), H = h(\zeta)$, where κ is an arbitrary constant; Equation (29) can then be written as

$$\begin{cases} -\kappa f' + (\chi k_2\kappa - \chi^2\kappa^2 k_1 - 1)h'' - k_3(f^2h + h^3) = 0, \\ -\kappa h' - (\chi k_2\kappa - \chi^2\kappa^2 k_1 - 1)f'' + k_3(f^3 + fh^2) = 0. \end{cases} \tag{30}$$

To obtain the solutions of Equation (30), we shall apply the $\left(\frac{G'}{G}\right)$ method, as described in [33].

Let us consider the solutions of (30), as

$$f = \sum_{i=0}^n A_i \left(\frac{G'}{G}\right)^i, h = \sum_{i=0}^m B_i \left(\frac{G'}{G}\right)^i. \tag{31}$$

By balancing the highest order derivative term and non-linear term in (30), we obtain $m = n = 1$, and $G = G(\zeta)$ satisfies second-order ODE

$$G'' + \lambda G' + \mu G = 0.$$

Solving Equation (30), we obtain

$$\begin{aligned} \mu &= \frac{\lambda^2(A_1^2 + B_1^2) + 4B_0(B_0 - \lambda B_1)}{4A_1^2}, \\ A_0 &= \frac{\lambda(A_1^2 + B_1^2) - 2B_0B_1}{2A_1}, \kappa = \frac{k_3(A_1^2 + B_1^2)(2B_0 - \lambda B_1)}{2A_1}, \\ k_1 &= \frac{-2A_1((A_1^2 + B_1^2)(\lambda\chi B_1 k_2 k_3 - 2\chi A_1 B_0 k_2 k_3 + 2A_1 k_3) + 2A_1)}{\chi^2 k_3^2 (\lambda A_1^2 B_1 + \lambda B_1^3 - 2A_1^2 B_0 - 2B_0 B_1^2)}, \end{aligned} \tag{32}$$

where $\lambda, \chi, d_1, B_0, B_1, k_2$, and k_3 are arbitrary constants.

Substituting (32) into (30), we obtain two types of solutions of (30), as follows:

When $\lambda^2 - 4\mu > 0$,

$$\begin{cases} f = \frac{\lambda B_1 - 2B_0}{2i} \times \left(\frac{C_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right) + C_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right)}{C_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right)} \right) \\ - \frac{\lambda(\lambda B_1 - 2B_0)}{2i\sqrt{\lambda^2 - 4\mu}} - \frac{\lambda B_0 - 2\mu B_1}{i\sqrt{\lambda^2 - 4\mu}}, \\ h = \frac{B_1}{2}\sqrt{\lambda^2 - 4\mu} \times \left(\frac{C_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right) + C_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right)}{C_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right)} \right) \\ - \frac{\lambda B_1}{2} + B_0, \end{cases} \tag{33}$$

where

$$\begin{aligned} k_1 &= \frac{2k_3(\lambda B_0 B_1 - \mu B_1^2 - B_0^2) + \lambda^2 - 4\mu + 2i\chi k_2 k_3(\lambda B_0 B_1 - \mu B_1^2 - B_0^2)\sqrt{\lambda^2 - 4\mu}}{4\chi^2 k_3^2 (\lambda B_0 B_1 - \mu B_1^2 - B_0^2)^2}, \\ \zeta &= r - \frac{2k_3(\lambda B_0 B_1 - \mu B_1^2 - B_0^2)}{i\sqrt{\lambda^2 - 4\mu}} s, \lambda, \mu, \chi, B_0, B_1, \\ &C_1, C_2, k_2, \text{ and } k_3 \text{ are arbitrary constants.} \end{aligned}$$

When $\lambda^2 - 4\mu < 0$,

$$\begin{cases} f = \frac{\lambda B_1 - 2B_0}{2} \times \left(\frac{C_1 \cosh\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right) - C_2 \sinh\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right)}{C_1 \sinh\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right)} \right) \\ - \frac{\lambda(\lambda B_1 - 2B_0)}{2\sqrt{4\mu - \lambda^2}} - \frac{\lambda B_0 - 2\mu B_1}{\sqrt{4\mu - \lambda^2}}, \\ h = \frac{B_1}{2}\sqrt{4\mu - \lambda^2} \times \left(\frac{C_1 \cosh\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right) - C_2 \sinh\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right)}{C_1 \sinh\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right)} \right) \\ - \frac{\lambda B_1}{2} + B_0, \end{cases} \tag{34}$$

where

$$\begin{aligned} k_1 &= \frac{2k_3(\lambda B_0 B_1 - \mu B_1^2 - B_0^2) + \lambda^2 - 4\mu + 2\chi k_2 k_3(\lambda B_0 B_1 - \mu B_1^2 - B_0^2)\sqrt{4\mu - \lambda^2}}{4\chi^2 k_3^2 (\lambda B_0 B_1 - \mu B_1^2 - B_0^2)^2}, \\ \zeta &= r - \frac{2k_3(\mu B_1^2 + B_0^2 - \lambda B_0 B_1)}{\sqrt{\lambda^2 - 4\mu}} s, \lambda, \mu, \chi, B_0, B_1, C_1, C_2, k_2, \\ &\text{and } k_3 \text{ are arbitrary constants.} \end{aligned}$$

Taking into account Equations (27), (33), and (34), we obtain the hyperbolic function solutions for system (3):

$$\begin{cases} u = \frac{\lambda B_1 - 2B_0}{2i} \times \left(\frac{C_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right) + C_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right)}{C_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right)} \right) \\ - \frac{\lambda(\lambda B_1 - 2B_0)}{2i\sqrt{\lambda^2 - 4\mu}} - \frac{\lambda B_0 - 2\mu B_1}{i\sqrt{\lambda^2 - 4\mu}}, \\ v = \frac{B_1}{2}\sqrt{\lambda^2 - 4\mu} \times \left(\frac{C_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right) + C_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right)}{C_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right)} \right) \\ - \frac{\lambda B_1}{2} + B_0, \end{cases} \tag{35}$$

where $\lambda^2 - 4\mu > 0$,

$$\begin{aligned} k_1 &= \frac{2k_3(\lambda B_0 B_1 - \mu B_1^2 - B_0^2) + \lambda^2 - 4\mu + 2i\chi k_2 k_3(\lambda B_0 B_1 - \mu B_1^2 - B_0^2)\sqrt{\lambda^2 - 4\mu}}{4\chi^2 k_3^2 (\lambda B_0 B_1 - \mu B_1^2 - B_0^2)^2}, \\ \zeta &= x - \frac{2k_3(\lambda B_0 B_1 - \mu B_1^2 - B_0^2)}{i\sqrt{\lambda^2 - 4\mu}} \left(\chi y - \int \delta_1(t) dt \right), \\ &\lambda, \mu, \chi, B_0, B_1, C_1, C_2, k_2, \\ &\text{and } k_3 \text{ are arbitrary constants.} \end{aligned}$$

$$\begin{cases} u = \frac{\lambda B_1 - 2B_0}{2} \times \left(\frac{C_1 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right) - C_2 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right)}{C_1 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right) + C_2 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right)} \right) \\ - \frac{\lambda(\lambda B_1 - 2B_0)}{2\sqrt{4\mu - \lambda^2}} - \frac{\lambda B_0 - 2\mu B_1}{\sqrt{4\mu - \lambda^2}}, \\ v = \frac{B_1}{2}\sqrt{4\mu - \lambda^2} \times \left(\frac{C_1 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right) - C_2 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right)}{C_1 \sin\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right) + C_2 \cos\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\zeta\right)} \right) \\ - \frac{\lambda B_1}{2} + B_0, \end{cases} \tag{36}$$

where $\lambda^2 - 4\mu < 0$,

$$\begin{aligned} k_1 &= \frac{2k_3(\lambda B_0 B_1 - \mu B_1^2 - B_0^2) + \lambda^2 - 4\mu + 2\chi k_2 k_3(\lambda B_0 B_1 - \mu B_1^2 - B_0^2)\sqrt{4\mu - \lambda^2}}{4\chi^2 k_3^2 (\lambda B_0 B_1 - \mu B_1^2 - B_0^2)^2}, \\ \zeta &= x - \frac{2k_3(\mu B_1^2 + B_0^2 - \lambda B_0 B_1)}{\sqrt{\lambda^2 - 4\mu}} \left(\chi y - \int \delta_1(t) dt \right), \\ &\lambda, \mu, \chi, B_0, B_1, C_1, C_2, k_2, \text{ and } k_3 \text{ are arbitrary constants.} \end{aligned}$$

Subalgebra $\tilde{\mathcal{J}}_4 = \frac{1}{\delta_1(t)} \frac{\partial}{\partial t}$

The similarity variables of this generator are

$$\begin{aligned} r &= x, s = y, \\ u &= F(r, s), v = H(r, s), \end{aligned} \tag{37}$$

and solving the constrained conditions (9), we get

$$\delta_2(t) = k_1\delta_1(t), \delta_3(t) = k_2\delta_1(t), \delta_4(t) = k_3\delta_1(t), \quad (38)$$

where k_i ($i = 1, 2, 3$) are arbitrary constants. Thus, system (3) can be transformed to

$$\begin{cases} H_{rr} + k_1H_{ss} + k_2H_{rs} + k_3(F^2H + H^3) = 0, \\ F_{rr} + k_1F_{ss} + k_2F_{rs} + k_3(F^3 + FH^2) = 0. \end{cases} \quad (39)$$

For solving Equation (39), we use the transformation $\zeta = r - \kappa s$, $F = f(\zeta)$, $H = h(\zeta)$, where λ is an arbitrary constant, and then (39) can be reduced to the following ODEs

$$\begin{cases} (1 + \kappa^2k_1 - \kappa k_2)h'' + k_3(f^2h + h^3) = 0, \\ (1 + \kappa^2k_1 - \kappa k_2)f'' + k_3(f^3 + fh^2) = 0. \end{cases} \quad (40)$$

Solving Equation (40) yields

$$\begin{cases} f = C_1 \sin\left(r - \frac{k_2 + \sqrt{4k_1k_3(C_1^2 + C_2^2) + k_2^2 - 4k_1}}{2k_1}s\right) \\ \quad - C_2 \cos\left(r - \frac{k_2 + \sqrt{4k_1k_3(C_1^2 + C_2^2) + k_2^2 - 4k_1}}{2k_1}s\right), \\ h = C_2 \sin\left(r - \frac{k_2 + \sqrt{4k_1k_3(C_1^2 + C_2^2) + k_2^2 - 4k_1}}{2k_1}s\right) \\ \quad + C_1 \cos\left(r - \frac{k_2 + \sqrt{4k_1k_3(C_1^2 + C_2^2) + k_2^2 - 4k_1}}{2k_1}s\right), \end{cases} \quad (41)$$

where $C_1, C_2, k_1, k_2,$ and k_3 are arbitrary constants.

On combining Equations (37) and (41), we obtain the periodic function solutions for system (3):

$$\begin{cases} u = C_1 \sin\left(x - \frac{k_2 + \sqrt{4k_1k_3(C_1^2 + C_2^2) + k_2^2 - 4k_1}}{2k_1}y\right) \\ \quad - C_2 \cos\left(x - \frac{k_2 + \sqrt{4k_1k_3(C_1^2 + C_2^2) + k_2^2 - 4k_1}}{2k_1}y\right), \\ v = C_2 \sin\left(x - \frac{k_2 + \sqrt{4k_1k_3(C_1^2 + C_2^2) + k_2^2 - 4k_1}}{2k_1}y\right) \\ \quad + C_1 \cos\left(x - \frac{k_2 + \sqrt{4k_1k_3(C_1^2 + C_2^2) + k_2^2 - 4k_1}}{2k_1}y\right), \end{cases} \quad (42)$$

where $C_1, C_2, k_1, k_2,$ and k_3 are arbitrary constants.

NON-LINEAR SELF-ADJOINTNESS AND CONSERVATION LAWS

Conservation laws have been extensively researched due to their important physical significance. Many effective approaches have been proposed to construct conservation laws for NPDEs, such as Noether's theorem [34], the multiplier approach [35], and so on [36, 37]. Ibragimov [38, 39] proposed a new conservation theorem that does not require the existence of a Lagrangian and is based on the concept of an adjoint equation for NLPDEs. In this section, we will construct non-linear self-adjointness and conservation laws for vchFSC equation (1).

Non-linear Self-Adjointness

Based on the method of constructing Lagrangians [38], we have the following formal Lagrangian \mathcal{L} in the symmetric form

$$\begin{aligned} \mathcal{L} = \bar{u} & \left[u_t + \delta_1(t)v_{xx} + \delta_2(t)v_{yy} + \frac{1}{2}\delta_3(t)v_{xy} \right. \\ & \left. + \frac{1}{2}\delta_3(t)v_{yx} + \delta_4(t)(u^2v + v^3) \right] \\ & + \bar{v} \left[-v_t + \delta_1(t)u_{xx} + \delta_2(t)u_{yy} + \frac{1}{2}\delta_3(t)u_{xy} \right. \\ & \left. + \frac{1}{2}\delta_3(t)u_{yx} + \delta_4(t)(u^3 + uv^2) \right], \end{aligned} \quad (43)$$

where \bar{u} and \bar{v} are two new dependent variables.

The adjoint system of system (3) is

$$\begin{cases} F_1^* = \frac{\delta \mathcal{L}}{\delta \bar{u}} = 0, \\ F_2^* = \frac{\delta \mathcal{L}}{\delta \bar{v}} = 0, \end{cases} \quad (44)$$

where

$$\frac{\delta \mathcal{L}}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} - D_t \frac{\partial \mathcal{L}}{\partial u_t} + D_x D_x \frac{\partial \mathcal{L}}{\partial u_{xx}} + D_x D_y \frac{\partial \mathcal{L}}{\partial u_{xy}} + D_y D_y \frac{\partial \mathcal{L}}{\partial u_{yy}}, \quad (45)$$

$$\frac{\delta \mathcal{L}}{\delta v} = \frac{\partial \mathcal{L}}{\partial v} - D_t \frac{\partial \mathcal{L}}{\partial v_t} + D_x D_x \frac{\partial \mathcal{L}}{\partial v_{xx}} + D_x D_y \frac{\partial \mathcal{L}}{\partial v_{xy}} + D_y D_y \frac{\partial \mathcal{L}}{\partial v_{yy}}, \quad (46)$$

with $D_x, D_y,$ and D_t the total differentiations with respect to $x, y,$ and t .

For illustration, D_x can be expressed as

$$\begin{aligned} D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + u_{xx} \frac{\partial}{\partial u_x} + v_{xx} \frac{\partial}{\partial v_x} + u_{xt} \frac{\partial}{\partial u_t} \\ + v_{xt} \frac{\partial}{\partial v_t} + \dots \end{aligned}$$

Substituting (43), (45), and (46) into (44), the adjoint system for system (3) is

$$\begin{cases} F_1^* = -\bar{u}_t + \delta_1(t)\bar{v}_{xx} + \delta_2(t)\bar{v}_{yy} + \delta_3(t)\bar{v}_{xy} \\ \quad + 2\delta_4(t)\bar{u}uv + \delta_4(t)\bar{v}(3u^2 + v^2), \\ F_2^* = \bar{v}_t + \delta_1(t)\bar{u}_{xx} + \delta_2(t)\bar{u}_{yy} + \delta_3(t)\bar{u}_{xy} \\ \quad + 2\delta_4(t)\bar{v}uv + \delta_4(t)\bar{u}(u^2 + 3v^2). \end{cases} \quad (47)$$

The system (3) is non-linear self-adjoint when adjoint system (47) satisfy the following conditions

$$\begin{cases} F_1^* \Big|_{\bar{u} = \phi(x,y,t,u,v), \bar{v} = \psi(x,y,t,u,v)} = \lambda_{11}F_1 + \lambda_{12}F_2, \\ F_2^* \Big|_{\bar{u} = \phi(x,y,t,u,v), \bar{v} = \psi(x,y,t,u,v)} = \lambda_{21}F_1 + \lambda_{22}F_2, \end{cases} \quad (48)$$

where $\phi(x, y, t, u, v) \neq 0$ or $\psi(x, y, t, u, v) \neq 0$, and λ_{ij} ($i, j = 1, 2$) are undetermined coefficients.

Substituting the expressions of F_i ($i=1,2$) and F_i^* ($i=1,2$) into (48), we obtain the following equations

$$\begin{aligned}
 &(\lambda_{12} - \psi_u)(\delta_1(t)u_{xx} - \delta_2(t)u_{yy} - \delta_3(t)u_{xy}) \\
 & - (\lambda_{11} - \psi_v)(\delta_1(t)v_{xx} + \delta_2(t)v_{yy} + \delta_3(t)v_{xy}) \\
 & - (\lambda_{11} + \phi_u)u_t + (\lambda_{12} - \phi_v)v_t + \psi_{uv}(2\delta_1(t)u_x v_x \\
 & + 2\delta_2(t)u_y v_y + \delta_3(t)u_x v_y + \delta_3(t)u_y v_x) \\
 & + \psi_{uu}(\delta_1(t)u_x^2 + \delta_2(t)u_y^2 + \delta_3(t)u_x u_y) + \psi_{vv}(\delta_1(t)v_x^2 \\
 & + \delta_2(t)v_y^2 + \delta_3(t)v_x v_y) \\
 & + (2\delta_1(t)\psi_{xu} + \delta_3(t)\psi_{yu})u_x + (2\delta_2(t)\psi_{yu} + \delta_3(t)\psi_{xu})u_y \\
 & + (2\delta_1(t)\psi_{xv} + \delta_3(t)\psi_{yv})v_x \\
 & + (2\delta_2(t)\psi_{yv} + \delta_3(t)\psi_{xv})v_y + \delta_1(t)\psi_{xx} + \delta_2(t)\psi_{yy} \\
 & + \delta_3(t)\psi_{xy} - \lambda_{11}\delta_4(t)(u^2 v + v^3) \\
 & - \lambda_{12}\delta_4(t)(uv^2 + u^3) + 2\delta_4(t)\phi uv + 3\delta_4(t)\psi u^2 \\
 & + \delta_4(t)\psi v^2 - \phi_t = 0, \\
 & - (\lambda_{22} - \phi_u)(\delta_1(t)u_{xx} + \delta_2(t)u_{yy} + \delta_3(t)u_{xy}) \\
 & - (\lambda_{21} - \phi_v)(\delta_1(t)v_{xx} + \delta_2(t)v_{yy} + \delta_3(t)v_{xy}) \\
 & - (\lambda_{21} - \psi_u)u_t + (\lambda_{22} + \psi_v)v_t + \phi_{uv}(2\delta_1(t)u_x v_x \\
 & + 2\delta_2(t)u_y v_y + \delta_3(t)u_x v_y + \delta_3(t)u_y v_x) \\
 & + \phi_{uu}(\delta_1(t)u_x^2 + \delta_2(t)u_y^2 + \delta_3(t)u_x u_y) + \phi_{vv}(\delta_1(t)v_x^2 \\
 & + \delta_2(t)v_y^2 + \delta_3(t)v_x v_y) \\
 & + (2\delta_1(t)\phi_{xu} + \delta_3(t)\phi_{yu})u_x \\
 & + (2\delta_2(t)\phi_{yu} + \delta_3(t)\phi_{xu})u_y + (2\delta_1(t)\phi_{xv} + \delta_3(t)\phi_{yv})v_x \\
 & + (2\delta_2(t)\phi_{yv} + \delta_3(t)\phi_{xv})v_y + \delta_1(t)\phi_{xx} + \delta_2(t)\phi_{yy} \\
 & + \delta_3(t)\phi_{xy} - \lambda_{21}\delta_4(t)(u^2 v + v^3) \\
 & - \lambda_{22}\delta_4(t)(uv^2 + u^3) + 2\delta_4(t)\psi uv + 3\delta_4(t)\phi v^2 \\
 & + \delta_4(t)\phi u^2 + \psi_t = 0.
 \end{aligned} \tag{49}$$

Solving the above systems, we have

$$\phi = -Cu, \psi = Cv, \lambda_{12} = \lambda_{21} = 0, \lambda_{11} = C, \lambda_{22} = -C. \tag{51}$$

Theorem 4.1. System (3) is non-linearly self-adjoint.

The formal Lagrangian corresponding to (43) reads as,

$$\begin{aligned}
 L = & -Cu[u_t + \delta_1(t)v_{xx} + \delta_2(t)v_{yy} + \frac{1}{2}\delta_3(t)v_{xy} \\
 & + \frac{1}{2}\delta_3(t)v_{yx} + \delta_4(t)(u^2 v + v^3)] \\
 & + Cv[-v_t + \delta_1(t)u_{xx} + \delta_2(t)u_{yy} + \frac{1}{2}\delta_3(t)u_{xy} \\
 & + \frac{1}{2}\delta_3(t)u_{yx} + \delta_4(t)(u^3 + uv^2)].
 \end{aligned} \tag{52}$$

Conservation Laws

In this section, we will construct the conservation laws for system (3) by Ibragimov's theorem. Next, we briefly review the notations used in the following sections. Let $x = (x^1, x^2, \dots, x^n)$ be n independent variables, $u = (u^1, u^2, \dots, u^m)$ be m dependent variables,

$$X = \xi_i(x, u, u_{(1)}, \dots) \frac{\partial}{\partial x^i} + \eta_s(x, u, u_{(1)}, \dots) \frac{\partial}{\partial u^s}, \tag{53}$$

be a symmetry of m equations

$$F_s(x, u, u_{(1)}, \dots, u_{(N)}) = 0, s = 1, 2, \dots, m. \tag{54}$$

and the corresponding adjoint equation

$$\begin{aligned}
 F_s^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(N)}, v_{(N)}) \\
 = \frac{\delta(v^s F_i)}{\delta u^s} = 0, s = 1, 2, \dots, m.
 \end{aligned} \tag{55}$$

Theorem 4.2. Any Lie point, Lie-Bäcklund and non-local symmetry X , as given in (53), of Equation (54) provides a conservation law for the system (54) and its adjoint system (55). The conserved vector is defined by

$$\begin{aligned}
 T^i = & \xi_i \mathcal{L} + W^s \left[\frac{\partial \mathcal{L}}{\partial u_i^s} - D_{x^j} \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^s} \right) + D_{x^j} D_{x^k} \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^s} \right) - \dots \right] \\
 & + D_{x^j} (W^s) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^s} - D_{x^k} \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^s} \right) + D_{x^k} D_{x^r} \left(\frac{\partial \mathcal{L}}{\partial u_{ijk r}^s} \right) - \dots \right] \\
 & + D_{x^j} D_{x^k} (W^s) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^s} - D_{x^r} \left(\frac{\partial \mathcal{L}}{\partial u_{ijk r}^s} \right) + \dots \right] + \dots,
 \end{aligned} \tag{56}$$

where $W^s = \eta_s - \xi_i u_i^s$ is the Lie characteristic function and $\mathcal{L} = \sum_{i=1}^m v^i F_i$ is the formal Lagrangian.

Based on the formula in Theorem 4.2, we next construct conserved vectors for system (3) by employing the formal Lagrangian (43) and the symmetry operator (10). For system (3), Equation (56) becomes the following form

$$\begin{aligned}
 T^x = & \xi \mathcal{L} - W^1 \left[D_x \left(\frac{\partial \mathcal{L}}{\partial u_{xx}} \right) + D_y \left(\frac{\partial \mathcal{L}}{\partial u_{xy}} \right) \right] \\
 & + D_x(W^1) \left(\frac{\partial \mathcal{L}}{\partial u_{xx}} \right) + D_y(W^1) \left(\frac{\partial \mathcal{L}}{\partial u_{xy}} \right) \\
 & - W^2 \left[D_x \left(\frac{\partial \mathcal{L}}{\partial v_{xx}} \right) + D_y \left(\frac{\partial \mathcal{L}}{\partial v_{xy}} \right) \right] + D_x(W^2) \left(\frac{\partial \mathcal{L}}{\partial v_{xx}} \right) \\
 & + D_y(W^2) \left(\frac{\partial \mathcal{L}}{\partial v_{xy}} \right)
 \end{aligned} \tag{57}$$

$$\begin{aligned}
 = & \xi \mathcal{L} - W^1 C(\delta_1(t)v_x + \frac{1}{2}\delta_3(t)v_y) \\
 & + D_x(W^1) (C\delta_1(t)v) + D_y(W^1) (\frac{1}{2}C\delta_3(t)v) \\
 & + W^2 C(\delta_1(t)u_x + \frac{1}{2}\delta_3(t)u_y) - D_x(W^2) (C\delta_1(t)u) \\
 & - D_y(W^2) (\frac{1}{2}C\delta_3(t)u),
 \end{aligned}$$

$$\begin{aligned}
 T^y = & \eta \mathcal{L} - W^1 \left[D_x \left(\frac{\partial \mathcal{L}}{\partial u_{yx}} \right) + D_y \left(\frac{\partial \mathcal{L}}{\partial u_{yy}} \right) \right] \\
 & + D_x(W^1) \left(\frac{\partial \mathcal{L}}{\partial u_{yx}} \right) + D_y(W^1) \left(\frac{\partial \mathcal{L}}{\partial u_{yy}} \right) \\
 & - W^2 \left[D_x \left(\frac{\partial \mathcal{L}}{\partial v_{yx}} \right) + D_y \left(\frac{\partial \mathcal{L}}{\partial v_{yy}} \right) \right] \\
 & + D_x(W^2) \left(\frac{\partial \mathcal{L}}{\partial v_{yx}} \right) + D_y(W^2) \left(\frac{\partial \mathcal{L}}{\partial v_{yy}} \right) \\
 = & \eta \mathcal{L} - W^1 C \left[\frac{1}{2}\delta_3(t)v_x + \delta_2(t)v_y \right] + D_x(W^1) (\frac{1}{2}C\delta_3(t)v) \\
 & + D_y(W^1) (C\delta_2(t)v) \\
 & + W^2 C \left[\frac{1}{2}\delta_3(t)u_x + \delta_2(t)u_y \right] - D_x(W^2) (\frac{1}{2}C\delta_3(t)u) \\
 & - D_y(W^2) (C\delta_2(t)u),
 \end{aligned} \tag{58}$$

$$\begin{aligned}
 T^t = & \tau \mathcal{L} + W^1 \left(\frac{\partial \mathcal{L}}{\partial u_t} \right) + W^2 \left(\frac{\partial \mathcal{L}}{\partial v_t} \right) = \tau \\
 & \mathcal{L} - W^1 (Cu) - W^2 (Cv),
 \end{aligned} \tag{59}$$

with

$$\begin{aligned}
 W^1 = & \Phi - \xi u_x - \eta u_y - \tau u_t, \\
 W^2 = & \Omega - \xi v_x - \eta v_y - \tau v_t.
 \end{aligned}$$

Case 1 $\mathfrak{J}_1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{2 \int \delta_1(t) dt}{\delta_1(t)} \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}$

The Lie characteristic functions for this operator are

$$W^1 = u - xu_x - yu_y - \frac{2 \int \delta_1(t) dt}{\delta_1(t)} u_t, \tag{60}$$

$$W^2 = v - xv_x - yv_y - \frac{2 \int \delta_1(t) dt}{\delta_1(t)} v_t. \tag{61}$$

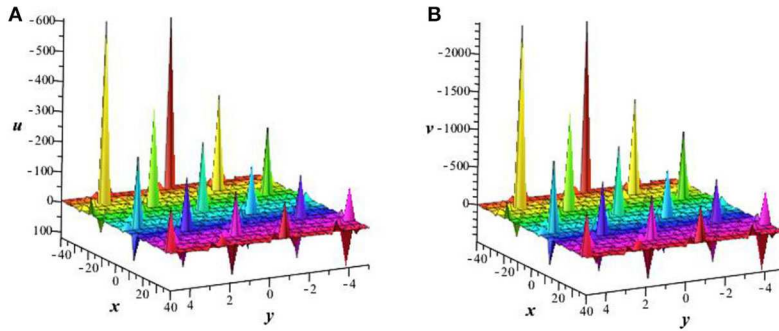


FIGURE 1 | Plot of invariant solution (25) with $\delta_1(t) = \sin t$, $A_1 = 1$, $B_1 = 4$, $\alpha = \beta = k_1 = 1$, $k_2 = 3$ at $t = 0$. **(A)** Perspective view of the solution u . **(B)** Perspective view of the solution v .

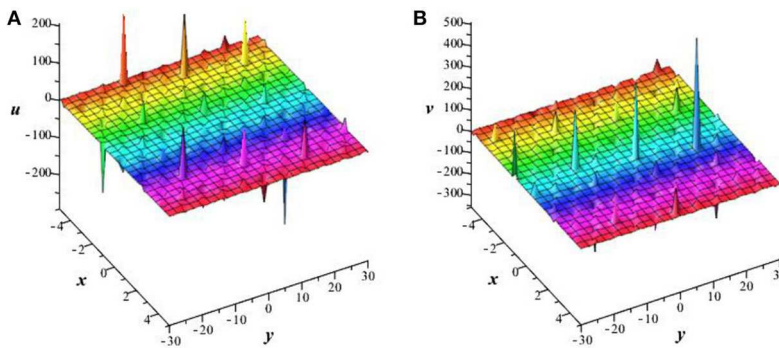


FIGURE 2 | Plot of invariant solution (36) with $\delta_1(t) = 1$, $C_1 = 2$, $C_2 = 1$, $\lambda = \mu = \chi = 1$, $B_0 = B_1 = k_2 = k_3 = 1$ at $t = 5$. **(A)** Perspective view of the solution u . **(B)** Perspective view of the solution v .

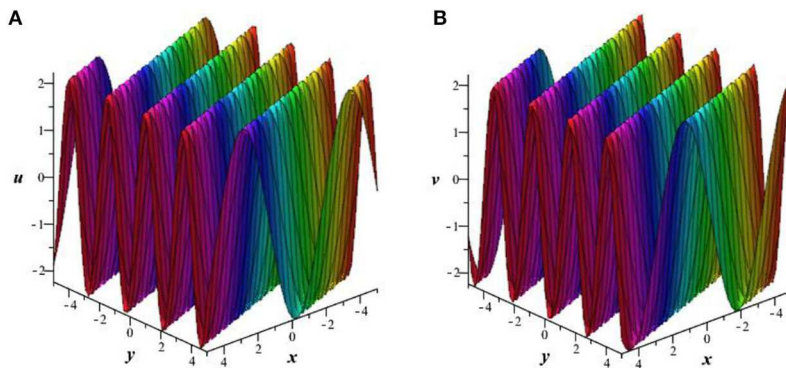


FIGURE 3 | Plot of invariant solution (42) with $C_1 = 1$, $C_2 = 2$, $k_1 = k_2 = k_3 = 1$ at $t = 0$. **(A)** Perspective view of the solution u . **(B)** Perspective view of the solution v .

The corresponding conservation laws are

$$\begin{aligned}
 T^x = & -\frac{1}{2}C [2k_1\delta_1(t)(uv_{yy} - u_{yy}v) + k_2\delta_1(t) \\
 & (uv_{xy} - u_xv_y + u_yv_x - u_{xy}v) + 2(uu_t + vv_t)]x \\
 & -\frac{1}{2}C [k_2\delta_1(t)(u_{yy}v - uv_{yy}) + 2\delta_1(t) \\
 & (u_{xy}v - u_yv_x - uv_{xy} + u_xv_y)]y \\
 & -\frac{1}{2}C \int \delta_1(t)dt [2k_2(u_{ty}v - uv_{ty} - u_tv_y + u_yv_t) \\
 & + 4(u_{tx}v - uv_{tx} - u_tv_x + u_xv_t)] \\
 & -\frac{1}{2}C [k_2\delta_1(t)(uv_y - u_yv) + 2\delta_1(t)(uv_x - u_xv)],
 \end{aligned} \tag{62}$$

$$\begin{aligned}
 T^y = & \frac{1}{2}C [2k_1\delta_1(t)(uv_{xy} + u_xv_y - u_yv_x - u_{xy}v) + k_2\delta_1(t) \\
 & (uv_{xx} - u_{xx}v)]x \\
 & +\frac{1}{2}C [2\delta_1(t)(u_{xx}v - uv_{xx}) + k_2\delta_1(t) \\
 & (u_yv_x + u_{xy}v - uv_{xy} - u_xv_y) - 2(uu_t + vv_t)]y
 \end{aligned} \tag{63}$$

$$\begin{aligned}
 & +\frac{1}{2}C \int \delta_1(t)dt [2k_2(uv_{tx} - u_{tx}v + u_tv_x - u_xv_t) \\
 & + 4k_1(uv_{ty} - u_{ty}v + u_tv_y - u_yv_t)] \\
 & +\frac{1}{2}C [k_2\delta_1(t)(u_xv - uv_x) + 2k_1\delta_1(t)(u_yv - uv_y)], \\
 T^t = & C [(uu_x + vv_x)x + (uu_y + vv_y)y - (u^2 + v^2)] \\
 & -C \int \delta_1(t)dt [2k_1(uv_{yy} - u_{yy}v) \\
 & + 2k_2(uv_{xy} - u_{xy}v) + 2(uv_{xx} - u_{xx}v)].
 \end{aligned} \tag{64}$$

Case 2 $\mathfrak{J}_2 = \frac{\partial}{\partial x}$

The Lie characteristic functions for this operator are

$$W^1 = -u_x, W^2 = -v_x. \quad (65)$$

The corresponding conservation laws are

$$T^x = -\frac{1}{2}C [2\delta_2(t)(uv_{yy} - u_{yy}v) + \delta_3(t)(uv_{xy} - u_{xy}v - u_xv_y + u_yv_x) + 2(uu_t + vv_t)], \quad (66)$$

$$T^y = \frac{1}{2}C [2\delta_2(t)(uv_{xy} - u_{xy}v + u_xv_y - u_yv_x) + \delta_3(t)(uv_{xx} - u_{xx}v)], \quad (67)$$

$$T^t = C(uu_x + vv_x). \quad (68)$$

Case 3 $\mathfrak{J}_3 = \frac{\partial}{\partial y}$

The Lie characteristic functions for this operator are

$$W^1 = -u_y, W^2 = -v_y. \quad (69)$$

The corresponding conservation laws are

$$T^x = \frac{1}{2}C [2\delta_1(t)(uv_{xy} - u_{xy}v - u_xv_y + u_yv_x) + \delta_3(t)(uv_{yy} - u_{yy}v)], \quad (70)$$

$$T^y = \frac{1}{2}C [2\delta_1(t)(u_{xx}v - uv_{xx}) - \delta_3(t)(uv_{xy} - u_{xy}v + u_xv_y - u_yv_x) - 2(uu_t + vv_t)], \quad (71)$$

$$T^t = C(uu_y + vv_y). \quad (72)$$

Case 4 $\mathfrak{J}_4 = \frac{1}{\delta_1(t)} \frac{\partial}{\partial t}$

The Lie characteristic functions for this operator are

$$W^1 = -\frac{1}{\delta_1(t)}u_t, W^2 = -\frac{1}{\delta_1(t)}v_t. \quad (73)$$

The corresponding conservation laws are,

$$T^x = \frac{1}{2}C [k_2(uv_{ty} - u_{ty}v + u_tv_y - u_yv_t) + 2(uv_{tx} - u_{tx}v + u_tv_x - u_xv_t)], \quad (74)$$

$$T^y = \frac{1}{2}C [k_2(uv_{tx} - u_{tx}v + u_tv_x - u_xv_t) + 2k_1(uv_{ty} - u_{ty}v + u_tv_y - u_yv_t)], \quad (75)$$

$$T^t = C [k_1(u_{yy}v - uv_{yy}) + k_2(u_{xy}v - uv_{xy}) + (u_{xx}v - uv_{xx})]. \quad (76)$$

RESULTS AND DISCUSSION

The Lie group method has been successfully used to establish the invariant solutions for the vHFSC equation. Some results for the vHFSC equation have been published in the literature.

Huang et al. [26] used the Hirota bilinear method and found the bright and dark solitons to Equation (1). Peng [27] reported some new non-autonomous complex wave and analytic solutions to Equation (1) with the aid of the (G'/G) method. In this article, we constructed the trigonometric and hyperbolic function solutions to the studied equation. Compared with the solutions obtained in references [26, 27], our results are new. To better understand the characteristics of the obtained solutions, the 3D graphical illustrations are plotted in **Figures 1–3**.

With the Lagrangian, we find that the vHFSC equation is non-linearly self-adjoint. Furthermore, a new conservation theorem has been used to construct conservation laws for the vHFSC equation. Based on the four infinitesimal generators, we obtained four conserved vectors. It worth noting that the conservation laws obtained in this article have been verified by Maple software.

CONCLUSION

In this research, the infinitesimal generators and Lie point symmetries of the vHFSC equation have been investigated using the Lie group method. Based on the optimal system of one-dimensional subalgebras, four types of similarity reductions are presented. Taking similarity reductions into account, the invariant solutions are provided, including trigonometric and hyperbolic function solutions. Furthermore, conservation laws for the vHFSC equation are derived by non-linear self-adjointness and a new conservation theorem.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article; further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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Conflict of Interest: The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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