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On the Boundary of Incidence Energy and Its Extremum Structure of Tricycle Graphs

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With the wide application of graph theory in circuit layout, signal flow chart and power system, more and more attention has been paid to the network topology analysis method of graph theory. In this paper, we construct a graph transformation which can reflect the monotonicity of coefficients and reduce the number of graphs. A sharp lower bound for incidence energy in the tricyclic graphs is given and all the extremal structures are characterized. The most interesting things that we find two different classes tricyclic graphs have the same signless Laplacian characteristic polynomials and one of the extremal graphs beyond all expectations.

Keywords: incidence energy, extremal graph, tricyclic graph, Laplacian matrix, signless Laplacian coefficients

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1. INTRODUCTION

Graph theory is a branch of discrete mathematics, Its research object is abstracted from the actual problem. For example, the geometric structure of an electrical network can be represented as a corresponding line graph. In the graph, the properties of circuit elements are ignored, the length and bending of edges are not important, but the connection between nodes and branches is highlighted. Each element in the network is replaced by a line segment, which is called a branch, and the endpoint of each element or the point connected by several elements is represented by an origin, which is called a node. The set of points and lines is called a network graph and is represented by G. Let $G = (V, E)$ be a simple connected graph with n vertices, m edges [1]. Let P_n, C_n and S_n be the path, the cycle and the star with n vertices, respectively [1]. Let $N_G(v) = \{u|uv \in E(G)\}$, denote by $d_G(v) = |N_G(v)|$ the degree of the vertex v of G. We know that $L(G) = D(G) - A(G)$ is the *Laplacian matrix* of G, and $A(G)$ is $(0, 1)$ adjacency matrix, $D(G)$ is degree diagonal matrix. Corresponding to the Laplacian matrix, $Q(G) = D(G)+A(G)$ is called the *signless Laplacian matrix* of a graph [2]. The Laplacian characteristic polynomials and signless Laplacian characteristic are defined as the following

$$L(G; \lambda) = det(\lambda I - L(G)) = \sum_{i=0}^{n}(-1)^i c_i(G)\lambda^{n-i},$$

$$Q(G; \lambda) = det(\lambda I - Q(G)) = \sum_{i=0}^{n}(-1)^i \varphi_i(G)\lambda^{n-i}.$$

For G, H, if $c_i(G) \leq c_i(H)$,$i = 1, 2, \ldots, n$, we call that $G \preceq' H$. If $\varphi_i(G) \leq \varphi_i(H)$, $i = 1, 2, \ldots, n$, we call that $G \preceq H$ [3, 4].

Denote by $\mathscr{G}_{n,m}$ the set of simple connected graphs of order n and size m. If $m = n-1+c$, G denotes a *c-cyclic graph*. If $c = 0,1,2$, and 3, G represents a tree, unicyclic graph, bicyclic graph and tricyclic graph, respectively [1]. Recently, with further research on the power system network, the study of the structure and properties of the partial ordering sets $(\mathscr{G}_{n,m}, \preceq')$ and $(\mathscr{G}_{n,m}, \preceq)$ have attracted much attention. For $m = n-1$, Mohar [5] proved that there is unique maximal element and unique minimal element in $(\mathscr{G}_{n,n-1}, \preceq')$. Since $L(G;\lambda) = Q(G;\lambda)$ for bipartite graph, then $(\mathscr{G}_{n,n-1}, \preceq)$ has the same structure and properties as $(\mathscr{G}_{n,n-1}, \preceq')$. For $m = n$, Stevanović and Ilić [6] showed that there is also unique maximal element and unique minimal element in $(\mathscr{G}_{n,n}, \preceq')$. But for $(\mathscr{G}_{n,n}, \preceq)$, Li et al. [7] given the extremal elements in $(\mathscr{G}_{n,n}, \preceq)$. He and Shan [8] obtained the unique minimal element in $(\mathscr{G}_{n,n+1}, \preceq')$, and in Zhang and Zhang [3], two minimal elements in $(\mathscr{G}_{n,n+1}, \preceq)$ were determined by Zhang and Zhang. For simplicity, denote the class of connected tricyclic graphs order n, i.e., $\mathscr{G}_{n,n+2}$ by $\mathscr{T}_n$ [9]. Pai et al. [10] characterized the unique minimal element in $(\mathscr{T}_n, \preceq')$. Based on these works, we focus on the structure and properties of the partial ordering sets $(\mathscr{T}_n, \preceq)$.

2. PRELIMINARIES

In this section, we introduce some graphic transformations and lemmas, which will be used to prove our main results.

If a connected graph has only one cycle whose length is odd, the graph is odd unicyclic. If the components of a spanning subgraph of G are trees or odd unicyclic graphs, the subgraph is called a *TU-subgraph* of G [3]. Let H be a TU-subgraph of G, which contains c odd unicyclic graphs and s trees $T_1,\ldots,T_s$ of orders $n_1,\ldots,n_s$, respectively. So the weight of H $\omega(H) = 4^c\prod_{i=1}^{s} n_i$. If there contains no tree in H, so $\omega(H) = 4^c$. If H is empty graph,there is no H, so $\omega(H) = 0$. We can express the signless Laplacian coefficients $\varphi_i(G)$ by the weight of TU-subgraphs of G [11].

Lemma 2.1. *[12] Let $Q(G;\lambda) = det(\lambda I - Q(G)) = \sum_{i=0}^{n}(-1)^i\varphi_i(G)\lambda^{n-i}$ be the characteristic polynomial of the signless Laplacian matrix of a graph G of order n. Then $\varphi_i(G) = \sum_{H_i}\omega(H_i)$, $i = 1,\ldots,n$, where the summation runs over all TU-subgraph H_i of G with i edges.*

Definition 1. *[8] Let G be a simple connected graph with n vertices and uv be a non-pendent edge, which is not contained in any cycles of G. Let $G_{uv} = G - \{vx|x \in N_G(v)\setminus\{u\}\} + \{ux|x \in N_G(v)\setminus\{u\}\}$. We say that G_{uv} is an α-transformation of G.*

Lemma 2.2. *[3] Let G be a connected graph of order $n \geq 4$, and G_{uv} be obtained from G by α-transformation. Then $G_{uv} \preceq G$, i.e., $\varphi_i(G_{uv}) \leq \varphi_i(G)$, $i = 0,1,\ldots,n$, with equality if and only if either $i \in \{0,1,n\}$ when G is non-bipartite, or $i \in \{0,1,n-1,n\}$ for otherwise.*

The proof of the following lemma can be found in many places in the literature (see, such as [13]).

Lemma 2.3. *[14] $L(G;\lambda) = Q(G;\lambda)$ if and only if the graph G is bipartite.*

Lemma 2.4. *[15] Let $f(\lambda)$ and $g(\lambda)$ be two real polynomials arranged according to decreasing exponents. If their coefficients are alternate about positive and negative, then the coefficients of $f(\lambda)g(\lambda)$ also are alternate about positive and negative.*

Let G be a connected graph with at least one cycle, the base of G is represented by $\widehat{G}$, which is the minimal connected subgraph containing all the cycles of G [16]. So $\widehat{G}$ is the unique subgraph of G, which contains no pendant vertex. G can be obtained from $\widehat{G}$ by planting trees to some vertices of $\widehat{G}$ [17]. Hoffman and Smith [18] define an *internal path* of G as a walk $u_0u_1\ldots u_s (s \geq 1)$,and the vertices $u_0, u_1, \ldots, u_{s-1}$ are distinct, $d(u_0) > 2, d(u_s) > 2$, and $d(u_i) = 2$, whenever $0 < i < s$. An internal path is closed, if $u_0 = u_s$.

Definition 2. *[19] Let $G = (V,E)$ be a connected graph and the base of G is $\widehat{G}$. Let u, v, w be three consecutive vertices in an internal path of length at least 4 of $\widehat{G}$, which satisfy $N_G(u)\cap N_G(v) = \emptyset$, $N_G(w)\cap N_G(v) = \emptyset$ and $N_G(u)\cap N_G(w) = \{v\}$. We can delete all edges vz for $z \in N_G(v)\setminus\{u,w\}$, wz for $z \in N_G(w)$ and add all edges uz for $z \in (N_G(v)\cup N_G(w))\setminus\{u,v\}$ from G and get the graph $G'(u,v,w)$. G to $G'(u,v,w)$ is called a β-transformation of G.*

Lemma 2.5. *Let $G = (V,E)$ be a connected graph and the base of G is $\widehat{G}$. Let u, v, w be three consecutive vertices in an internal path of length at least 4 of $\widehat{G}$, and $G'(u,v,w)$ be a graph obtained from G by β-transformation [19]. So $G'(u,v,w) \preceq G$, that is, $\varphi_i(G'(u,v,w)) \leq \varphi_i(G)$ for $i \in \{0,1,2,\ldots,n\}$, with equality if and only if $i \in \{0,1\}$ when G is non-bipartite, and $i \in \{0,1,n\}$ when G is bipartite.*

Proof: $\varphi_0(G'(u,v,w)) = \varphi_0(G) = 1$ and $\varphi_1(G'(u,v,w)) = \varphi_1(G) = 2|E|$. Moreover, $\varphi_n(G'(u,v,w)) = \varphi_n(G) = 0$ for bipartite graph. Now assume that $2 \leq i \leq n$. Let $\mathscr{H}$ and $\mathscr{H}$ be the set of all TU-subgraphs of $G'(u,v,w)$ and G with i edges, respectively. For an arbitrary TU-subgraph $H' \in \mathscr{H}$, denote by the R' connected component of H' containing u [3]. Let $f:\mathscr{H} \to \mathscr{H}$ with $H' \to H = f(H')$, where $V(H) = V(H')$ and

$$E(H) = E(H') - \{ux|x \in N_{R'}(u)\cap N_G(v)\} - \{ux|x \in N_{R'}(u)\cap N_G(w)\setminus\{v\}\} + \{vx|x \in N_{R'}(u)\cap N_G(v)\} + \{wx|x \in N_{R'}(u)\cap N_G(w)\setminus\{v\}\}.$$

Then f is injective from $\mathscr{H} \to \mathscr{H}$.

Case 1. u,v,w belongs the component S'. So $f(S')$ is a component of H, which in the same order as S'. Then $\omega(H) = \omega(H')$.

Case 2. u,v,w belong to at least two components of H'.

Case 2.1. u is not in an odd unicyclic component of H'. Then u is contained in a tree component of H'. Assume that there exist x_1+1 vertices in the connected component which contains u in $H-uv$ [3], x_2+1 vertices in the connected component which contains w in $H-wv$ and x_3+1 vertices in the connected component which contains v in $H-uv-vw$, where $x_1,x_2,x_3 \geq 0$. Let N indicate the weight of the components of H', which contain no u,v,w.

(i) If $uv \in E(H')anduw \notin E(H')$, then

$$\begin{aligned}\omega(H') &= (x_1+x_2+x_3+2)\cdot 1\cdot N,\\ \omega(H) &= (x_1+x_3+2)(x_2+1)\cdot 1\cdot N,\\ \omega(H)-\omega(H') &= x_2(x_1+x_3+1)N \geq 0.\end{aligned}$$

(ii) If $uv \notin E(H')anduw \in E(H')$, then

$$\begin{aligned}\omega(H') &= (x_1+x_2+x_3+2)\cdot 1\cdot N,\\ \omega(H) &= (x_2+x_3+2)(x_1+1)\cdot N,\\ \omega(H)-\omega(H') &= x_1(x_2+x_3+1)N \geq 0.\end{aligned}$$

(iii) If $uv \notin E(H')anduw \notin E(H')$, then

$$\begin{aligned}\omega(H') &= (x_1+x_2+x_3+1)\cdot 1\cdot 1\cdot N,\\ \omega(H) &= (x_1+1)(x_2+1)(x_3+1)\cdot N,\\ \omega(H)-\omega(H') &= (x_1x_2x_3+x_1x_2+x_1x_3+x_2x_3-1)N \geq 0.\end{aligned}$$

Case 2.2. u is in an odd unicyclic component S' of H'. Let C' be a subgraph of S', which corresponds to an odd cycle C in G.

(i) If $uv \notin E(H'), uw \notin E(H')$, and $C = C'$, let S be the component containing C in H. So there are the same components in H' and H, except for S', $\{v\}$, $\{w\}$ in H', which correspond to the component S containing u, two components S_1 containing v and S_2 containing w of order at least 1, respectively, in H. If $uv \notin E(H'), uw \notin E(H')$, and $C \neq C'$. So there are the same components in H' and H, except for S', $\{v\}$, $\{w\}$ in H', which correspond to two tree components S_1 containing u, w of order at least 4 since u, v, w are three consecutive vertices in an internal path of length at least 4 of $\widehat{G}$, and S_2 containing v of order at least 1, in H. So

$$\begin{aligned}\omega(H') &= 4\cdot 1\cdot 1\cdot N,\\ \omega(H) &\geq 4\cdot 1\cdot 1\cdot N,\\ \omega(H)-\omega(H') &\geq 0.\end{aligned}$$

(ii) If $uv \notin E(H'), uw \in E(H')$ or $uv \in E(H'), uw \notin E(H')$, and $C = C'$, So there are the same components in H' and H, except for S', $\{v\}$ or $\{w\}$ in H', which correspond to an odd unicyclic component S containing C and a tree component S_1 containing v, w of order at least 2. So

$$\begin{aligned}\omega(H') &= 4\cdot 1\cdot N,\\ \omega(H) &\geq 4\cdot 2\cdot N,\\ \omega(H)-\omega(H') &\geq 4N > 0.\end{aligned}$$

If $uv \notin E(H'), uw \in E(H')$ or $uv \in E(H'), uw \notin E(H')$, and $C \neq C'$, So there are the same components in H' and H, except for S', $\{v\}$ or $\{w\}$ in H', which correspond to a tree component S containing u, v, w of order at least 5. So

$$\begin{aligned}\omega(H') &= 4\cdot 1\cdot N,\\ \omega(H) &\geq 4\cdot N,\\ \omega(H)-\omega(H') &\geq 0.\end{aligned}$$

Then by Lemma 2.1, we have $\varphi_i(G'(u,v,w)) = \sum_{H'_i\in\mathscr{G}}\omega(H'_i) \leq \sum_{H_i\in\mathscr{G}}\omega(H_i) = \varphi_i(G)$

Hence the results hold. □

Similarly, we can prove the following result.

Lemma 2.6. [19] *Let $G = (V, E)$ be a connected graph with base $\widehat{G}$. Let u, v, w be three consecutive vertices in an internal path $P = u_1u_2\ldots u_k$ with $k = 4$ of $\widehat{G}$ and $u_1u_k \notin E(\widehat{G})$. Let $G'(u, v, w)$ be a graph obtained from G by β-transformation, then $G'(u, v, w) \preceq G$, that is, $\varphi_i(G'(u, v, w)) \leq \varphi_i(G)$ for $i \in \{0, 1, 2, \ldots, n\}$, with equality if and only if $i \in \{0, 1\}$ when G is non-bipartite, and $i \in \{0, 1, n\}$ when G is bipartite.*

By Li et al. [20], There are the following four types of bases in tricyclic graphs(as shown in **Figures 1–4**): $G^3_j(j = 1, \ldots, 7)$, $G^4_j(j = 1, \ldots, 4)$, $G^6_j(j = 1, \ldots, 3)$ and G^7_1. Let

$$\mathscr{T}^3_n = \{G|\widehat{G} \cong G^3_j, j \in \{1, \ldots, 7\}\}; \quad \mathscr{T}^4_n = \{G|\widehat{G} \cong G^4_j, j \in \{1, \ldots, 4\}\};$$
$$\mathscr{T}^6_n = \{G|\widehat{G} \cong G^6_j, j \in \{1, \ldots, 3\}\}; \quad \mathscr{T}^7_n = \{G|\widehat{G} \cong G^7_1\}.$$

Then $\mathscr{T}_n = \mathscr{T}^3_n \cup \mathscr{T}^4_n \cup \mathscr{T}^6_n \cup \mathscr{T}^7_n$.

Let $T^3_1(n-7, 0, 0, 0, 0, 0, 0)$, $T^4_1(n-6, 0, 0, 0, 0, 0)$, $T^6_1(n-5, 0, 0, 0, 0)$ and $T^7_1(n-4, 0, 0, 0)$ be the graphs as shown in **Figure 5**.

Lemma 2.7. [10]

(i) *If $G \in \mathscr{T}^3_n$, then for every $i = 0, 1, \ldots, n$, $c_i(G) \geq c_i(T^3_1(n-7, 0, 0, 0, 0, 0, 0))$, with equality if and only if $i \in \{0, 1, n\}$.*
(ii) *If $G \in \mathscr{T}^4_n$, then for every $i = 0, 1, \ldots, n$, $c_i(G) \geq c_i(T^4_1(n-6, 0, 0, 0, 0, 0))$, with equality if and only if $i \in \{0, 1, n\}$.*
(iii) *If $G \in \mathscr{T}^6_n$, then for every $i = 0, 1, \ldots, n$, $c_i(G) \geq c_i(T^6_1(n-5, 0, 0, 0, 0))$, with equality if and only if $i \in \{0, 1, n\}$.*
(iv) *If $G \in \mathscr{T}^7_n$, then for every $i = 0, 1, \ldots, n$, $c_i(G) \geq c_i(T^7_1(n-4, 0, 0, 0))$, with equality if and only if $i \in \{0, 1, n\}$.*

For $i = 3, 4, 6, 7$, let $\mathscr{T}^{i,e}_n$ (resp., $\mathscr{T}^{i,o}_n$) be the set of bipartite tricyclic graphs (resp., non-bipartite tricyclic graphs) in $\mathscr{T}^i_n$, then $\mathscr{T}^i_n = \mathscr{T}^{i,e}_n \cup \mathscr{T}^{i,o}_n$. From lemmas 2.3 and 2.7, we get

Corollary 2.8. [10]

(i) *If $G \in \mathscr{T}^{3,e}_n$, then for every $i = 0, 1, \ldots, n$, $\varphi_i(G) \geq \varphi_i(T^3_1(n-7, 0, 0, 0, 0, 0, 0))$, with equality if and only if $i \in \{0, 1, n\}$.*
(ii) *If $G \in \mathscr{T}^{4,e}_n$, then for every $i = 0, 1, \ldots, n$, $\varphi_i(G) \geq \varphi_i(T^4_1(n-6, 0, 0, 0, 0, 0))$, with equality if and only if $i \in \{0, 1, n\}$.*
(iii) *If $G \in \mathscr{T}^{6,e}_n$, then for every $i = 0, 1, \ldots, n$, $\varphi_i(G) \geq \varphi_i(T^6_1(n-5, 0, 0, 0, 0))$, with equality if and only if $i \in \{0, 1, n\}$.*
(iv) *If $G \in \mathscr{T}^{7,e}_n$, then for every $i = 0, 1, \ldots, n$, $\varphi_i(G) \geq \varphi_i(T^7_1(n-4, 0, 0, 0))$, with equality if and only if $i \in \{0, 1, n\}$.*

Theorem 2.9. [10] *Let G be a connected tricyclic graph on n vertices and i be an integer, $0 \leq i \leq n$. Then $c_i(G) \geq c_i(T^7_1(n-4, 0, 0, 0))$.*

Repeated by lemmas 2.2, 2.5, and 2.6, we get the following conclusion

Theorem 2.10. *Let G be a graph in $\mathscr{T}^{3,o}_n \cup \mathscr{T}^{4,o}_n \cup \mathscr{T}^{6,o}_n \cup \mathscr{T}^{7,o}_n$. So there is a tricyclic graph G' with order n, such that $G' \preceq G$,. The base of G' is one of graphs in $\{T^3_i|j = 1, 2, \ldots, 9\} \cup \{T^4_i|j = 1, 2, \ldots, 20\} \cup \{T^6_i|j = 1, 2, \ldots, 24\} \cup \{T^7_i|j = 1, 2, \ldots, 7\}$(these base graphs are as shown in **Figures 7–9**.*

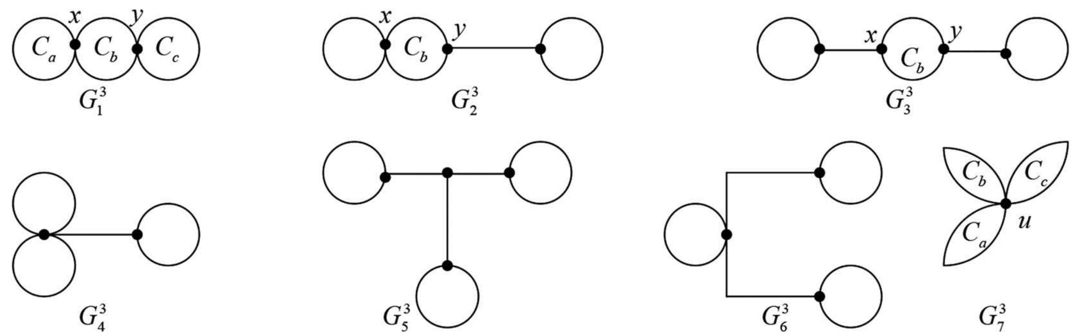

FIGURE 1 | The graphs $G_i^3 (i = 1, 2, \ldots, 7)$.

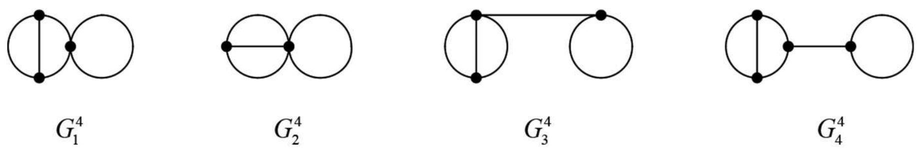

FIGURE 2 | The graphs $G_i^4 (i = 1, 2, \ldots, 4)$.

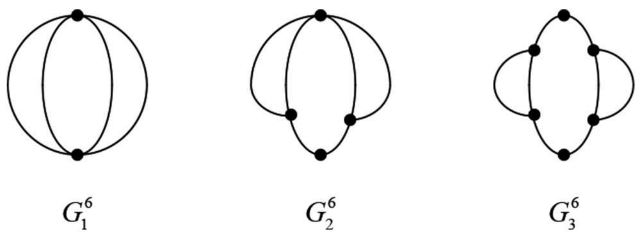

FIGURE 3 | The graphs $G_i^6 (i = 1, 2, 3)$.

3. THE SIGNLESS LAPLACIAN COEFFICIENTS OF GRAPHS IN T_N

Now we consider the minimal element in the partial ordering set $(\mathscr{T}_n, \preceq)$.

For $i = 1, 2, \ldots, 9$, let $T_i^3(s_1, s_2, \ldots, s_{|T_i^3|})$ be the graph obtained from T_i^3 (as shown in **Figure 6**) by attaching s_j pendent edges at $u_j (j = 1, 2, \ldots, |T_i^3|)$, where $n = s_1 + s_2 + \cdots + s_{|T_i^3|} + |T_i^3|$.

Lemma 3.1. *For* $j = 1, 2, \ldots, 9$, $T_j^3(s_1 + s_2 + \cdots + s_{|T_j^3|}, 0, \ldots, 0) \preceq T_j^3(s_1, s_2, \ldots, s_{|T_j^3|})$, *that is,* $\phi_i(T_j^3(s_1 + s_2 + \cdots + s_{|T_j^3|}, 0, \ldots, 0)) \leq \phi_i(T_j^3(s_1, s_2, \ldots, s_{|T_j^3|}))$, $i = 0, 1, \ldots, n$. *The equality holds if and only if* $s_2 = \cdots = s_{|T_j^3|} = 0$.

Proof: For convenience, let $G = T_j^3(s_1, s_2, \ldots, s_{|T_j^3|})$ and $G' = T_j^3(s_1 + s_2 + \cdots + s_{|T_j^3|}, 0, \ldots, 0)$ for $j = 1, 2, \ldots, 9$. Note that $\phi_0(G) = 1 = \phi_0(G')$, $\phi_1(G) = 2(n+2) = \phi_1(G')$. For $2 \leq i \leq n$,

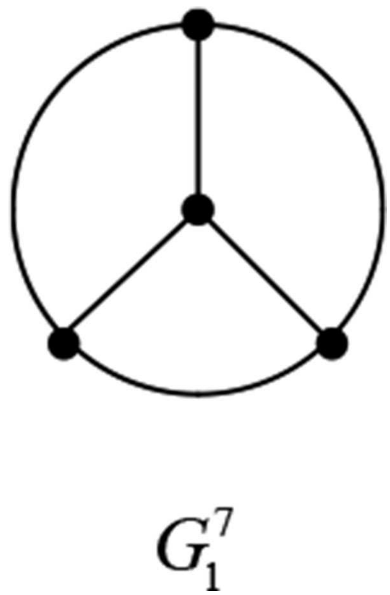

FIGURE 4 | The graph G_1^7.

let $\mathscr{H}'$ and $\mathscr{H}$ be the set of all TU-subgraphs of G' and G with exactly i edges, respectively [3]. Let

$$\mathscr{H}'^{(1)} = \{H' \in \mathscr{H}' | H' \textit{ contains no odd cycle}\},$$
$$\mathscr{H}'^{(2)} = \{H' \in \mathscr{H}' | H' \textit{ contains an odd cycle}\},$$
$$\mathscr{H}'^{(3)} = \{H' \in \mathscr{H}' | H' \textit{ contains two odd cycles}\}.$$

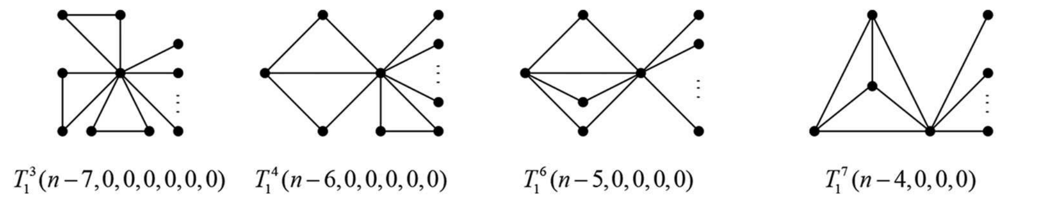

FIGURE 5 | The extremal graphs.

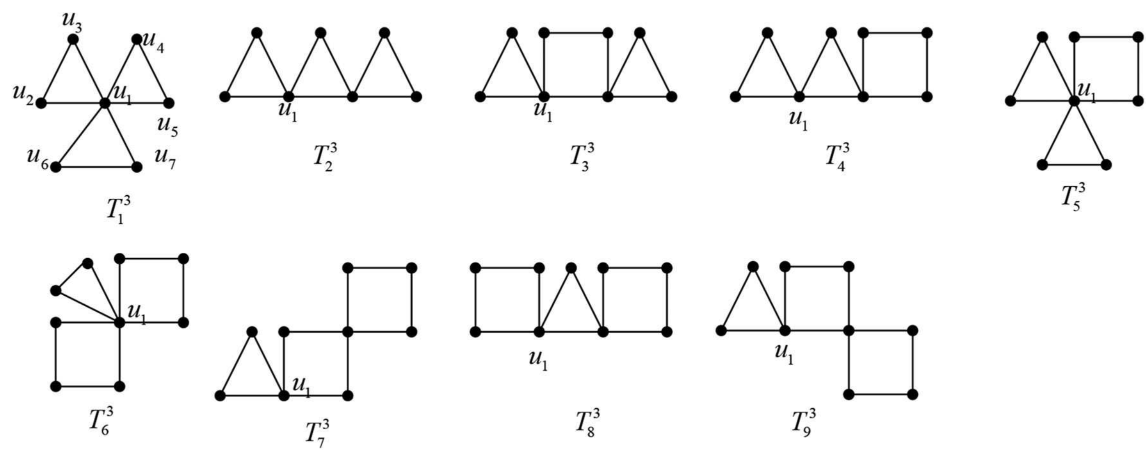

FIGURE 6 | The graphs $T_i^3 (i = 1, 2, \dots, 9)$.

Similarly for $\mathscr{H}'^{(1)}, \mathscr{H}'^{(2)}$, and $\mathscr{H}'^{(3)}$. We only prove the case for $j = 1$, the others can be proved similarly.

Let $f : \mathscr{H}' \to \mathscr{H}$ with $H' \to H = f(H')$, where $V(H) = V(H')$ and

$$\begin{aligned} E(H') = E(H) &- \{u_1x | x \in N_{R'}(u_1) \cap N_G(u_2) \setminus \{u_3\}\} \\ &- \{u_1x | x \in N_{R'}(u_1) \cap N_G(u_3) \setminus \{u_2\}\} \\ &- \{u_1x | x \in N_{R'}(u_1) \cap N_G(u_4) \setminus \{u_5\}\} \\ &- \{u_1x | x \in N_{R'}(u_1) \cap N_G(u_5) \setminus \{u_4\}\} \\ &- \{u_1x | x \in N_{R'}(u_1) \cap N_G(u_6) \setminus \{u_7\}\} \\ &- \{u_1x | x \in N_{R'}(u_1) \cap N_G(u_7) \setminus \{u_6\}\} \\ &+ \{u_2x | x \in N_{R'}(u_1) \cap N_G(u_2) \setminus \{u_3\}\} \\ &+ \{u_3x | x \in N_{R'}(u_1) \cap N_G(u_3) \setminus \{u_2\}\} \\ &+ \{u_4x | x \in N_{R'}(u_1) \cap N_G(u_4) \setminus \{u_5\}\} \\ &+ \{u_5x | x \in N_{R'}(u_1) \cap N_G(u_5) \setminus \{u_4\}\} \\ &+ \{u_6x | x \in N_{R'}(u_1) \cap N_G(u_6) \setminus \{u_7\}\} \\ &+ \{u_7x | x \in N_{R'}(u_1) \cap N_G(u_7) \setminus \{u_6\}\} \end{aligned}$$

for R' being a component of H' containing u_1. Obviously, f is injective and $f(\mathscr{H}'^{(k)}) \subseteq \mathscr{H}^{(k)}$ for $j = 1, 2, 3$. From the procedure of proof in Theorem 3.1 [10], we have

$$\sum_{H' \in \mathscr{H}'^{(1)}} \omega(H') < \sum_{H \in \mathscr{H}^{(1)}} \omega(H).$$

Note that $\mathscr{H}'^{(3)} = \emptyset$ for $j = 1$. For $H' \in \mathscr{H}'^{(2)}$, without loss of generality, we assume that R' contains $C_3 = u_1u_2u_3u_1$ as a subgraph. Let R be the component of H corresponding to R', obviously, R also contains $C_3 = u_1u_2u_3u_1$. It is obvious that H', H have the same number of components and the product of the order of components which contain no $u_i (i = 1, 2, \dots, 7)$ of H' is the same as H. The order of the tree components of H', which include at least one of $u_i (i = 4, \dots, 7)$ are no more than the corresponding ones of H, then $\omega(f(H')) \geq \omega(H')$. Hence

$$\begin{aligned} \phi_i(G) &= \sum_{H \in \mathscr{H}^{(1)}} \omega(H) + \sum_{H \in \mathscr{H}^{(2)}} \omega(H) + \sum_{H \in \mathscr{H}^{(3)}} \omega(H) \\ &\geq \sum_{H' \in \mathscr{H}'^{(1)}} \omega(H') + \sum_{H' \in \mathscr{H}'^{(2)}} \omega(H') + \sum_{H' \in \mathscr{H}'^{(3)}} \omega(H') = \phi_i(G'). \end{aligned}$$

The equality holds if and only if $s_2 = \cdots = s_7 = 0$. □

For $i = 1, 2, \dots, 20$, let $T_i^4(s_1, s_2, \dots, s_{|T_i^4|})$ be the graph obtained from T_i^4 (as shown in **Figure 7**) by attaching s_j pendent edges at $u_j (j = 1, 2, \dots, |T_i^4|)$, where $n = s_1 + s_2 + \cdots + s_{|T_i^4|} + |T_i^4|$. Similar to the proof of Lemma 3.1, we have

Lemma 3.2. *For $j = 1, 2, \dots, 20$, $T_j^4(s_1 + s_2 + \cdots + s_{|T_j^4|}, 0, \dots, 0) \preceq T_j^4(s_1, s_2, \dots, s_{|T_j^4|})$, that is, $\phi_i(T_j^4(s_1 + s_2 + \cdots + s_{|T_j^4|}, 0, \dots, 0)) \leq \phi_i(T_j^4(s_1, s_2, \dots, s_{|T_j^4|})), i = 0, 1, \dots, n$. The equality holds if and only if $s_2 = \cdots = s_{|T_j^4|} = 0$.*

For $i = 1, 2, \dots, 7$, let $T_i^7(s_1, s_2, \dots, s_{|T_i^7|})$ be the graph obtained from T_i^7 (as shown in **Figure 8**) by attaching s_j pendent edges at $u_j (j = 1, 2, \dots, |T_i^7|)$, where $n = s_1 + s_2 + \cdots + s_{|T_i^7|} + |T_i^7|$. Similar to the proof of Lemma 3.1, we have

Lemma 3.3. *For* $j = 1, 2, \ldots, 7$, $T_j^7(s_1+s_2+\cdots+s_{|T_j^7|}, 0, \ldots, 0) \preceq T_j^7(s_1, s_2, \ldots, s_{|T_j^7|})$, *that is,* $\phi_i(T_j^7(s_1+s_2+\cdots+s_{|T_j^7|}, 0, \ldots, 0)) \leq \phi_i(T_j^7(s_1, s_2, \ldots, s_{|T_j^7|}))$, $i = 0, 1, \ldots, n$. *The equality holds if and only if* $s_2 = \cdots = s_{|T_j^7|} = 0$.

For $i = 1, 2, \ldots, 24$, let $T_i^6(s_1, s_2, \ldots, s_{|T_i^6|})$ be the graph obtained from T_i^6 (as shown in **Figure 9**) by attaching s_j pendent edges at $u_j (j = 1, 2, \ldots, |T_i^6|)$, where $n = s_1+s_2+\cdots+s_{|T_i^6|}+|T_i^6|$. Similar to the proof of Lemma 3.1, we have

Lemma 3.4. *For* $j = 1, 2, \ldots, 24$, $T_j^6(s_1 + s_2 + \cdots + s_{|T_j^6|}, 0, \ldots, 0) \preceq T_j^6(s_1, s_2, \ldots, s_{|T_j^6|})$, *that is,* $\phi_i(T_j^6(s_1+s_2+\cdots+s_{|T_j^6|}, 0, \ldots, 0)) \leq \phi_i(T_j^6(s_1, s_2, \ldots, s_{|T_j^6|}))$, $i = 0, 1, \ldots, n$. *The equality holds if and only if* $s_2 = \cdots = s_{|T_j^6|} = 0$.

Lemma 3.5. *For* $n \geq |T_j^3| (j = 1, 2, \ldots, 9)$,

(i) $T_2^3(n-7, 0, 0, 0, 0, 0, 0) \succeq T_1^3(n-7, 0, 0, 0, 0, 0, 0)$.

(ii) $T_j^3(n-8, 0, 0, 0, 0, 0, 0, 0) \succeq T_5^3(n-8, 0, 0, 0, 0, 0, 0, 0, 0)$ *for* $j = 3, 4$.

(iii) $T_j^3(n-9, 0, 0, 0, 0, 0, 0, 0, 0) \succeq T_6^3(n-9, 0, 0, 0, 0, 0, 0, 0, 0)$ *for* $j = 7, 8, 9$.

Proof: (i) We have

$$\begin{aligned}
&Q(T_2^3(n-7, 0, \ldots, 0)) - Q(T_1^3(n-7, 0, 0, 0, 0, 0, 0)) \\
= \ &(x-1)^{n-8}[2(n-5)x^6 - (18n-90)x^5 + (62n-302)x^4 \\
&-(102n-462)x^3 \\
&+(80n-296)x^2 - (24n-24)x + 32].
\end{aligned}$$

Further by Lemma 2.3, $T_2^3(n-7, 0, 0, 0, 0, 0, 0) \succeq T_1^3(n-7, 0, 0, 0, 0, 0, 0)$.

$$\begin{aligned}
&Q(T_3^3(n-8, 0, 0, 0, 0, 0, 0, 0, 0)) - Q(T_5^3(n-8, 0, 0, 0, 0, 0, 0, 0, 0)) \\
= \ &(x-1)^{n-9}[(2n-12)x^7 - (22n-132)x^6 + (96n-568)x^5 \\
&-(212n-1208)x^4 \\
&+(250n-1308)x^3 - (150n-628)x^2 + (36n-32)x - 48], \\
&Q(T_4^3(n-8, 0, 0, 0, 0, 0, 0, 0, 0)) - Q(T_5^3(n-8, 0, 0, 0, 0, 0, 0, 0, 0)) \\
= \ &(x-1)^{n-9}[(2n-11)x^7 - (23n-125)x^6 + (104n-546)x^5 \\
&-(233n-1128)x^4 \\
&+(266n-1047)x^3 - (140n-219)x^2 + (24n+200)x - 68], \\
&Q(T_7^3(n-9, 0, 0, 0, 0, 0, 0, 0, 0)) - Q(T_6^3(n-9, 0, 0, 0, 0, 0, 0, 0, 0)) \\
= \ &(x-1)^{n-10}[(2n-14)x^8 - (26n-182)x^7 + (138n-958)x^6 \\
&-(382n-2594)x^5 \\
&+(580n-3748)x^4 - (456n-5608)x^3 + (144n-464)x^2 - 192x], \\
&Q(T_8^3(n-9, 0, 0, 0, 0, 0, 0, 0, 0)) - Q(T_6^3(n-9, 0, 0, 0, 0, 0, 0, 0, 0)) \\
= \ &(x-1)^{n-10}[(2n-14)x^8 - (26n-182)x^7 + (138n-948)x^6 \\
&-(364n-2512)x^5 \\
&+(520n-3520)x^4 - (368n-2400)x^3 + (96n-576)x^2], \\
&Q(T_9^3(n-9, 0, 0, 0, 0, 0, 0, 0, 0)) - Q(T_6^3(n-9, 0, 0, 0, 0, 0, 0, 0, 0)) \\
= \ &(x-1)^{n-10}[(2n-14)x^8 - (26n-182)x^7 + (136n-944)x^6 \\
&-(364n-2468)x^5 \\
&+(522n-3350)x^4 - (378n-2102)x^3 + (108n-300)x^2 - 144x].
\end{aligned}$$

So (ii) and (iii) hold. □

Lemma 3.6. *For* $n \geq |T_j^4| (j = 1, \ldots, 20)$,

(i) $T_j^4(n-|T_j^4|, 0, \ldots, 0) \succeq T_1^4(n-6, 0, \ldots, 0)$ *for* $j = 2, 5, 6, 10, 15, 16, 17$

(ii) $T_j^4(n-|T_j^4|, 0, \ldots, 0) \succeq T_4^4(n-7, 0, \ldots, 0)$ *for* $j = 3, 7, 8, 9, 18, 19, 20$.

(iii) $T_j^4(n-|T_j^4|, 0, \ldots, 0) \succeq T_{14}^4(n-7, 0, \ldots, 0)$ *for* $j = 11, 12, 13$.

Proof:

$$\begin{aligned}
&Q(T_2^4(n-6, 0, \ldots, 0)) - Q(T_1^4(n-6, 0, \ldots, 0)) \\
= \ &(x-1)^{n-7}[(n-4)x^5 - (8n-32)x^4 + (23n-88)x^3 \\
&-(28n-92)x^2 + (12n-16)x - 16], \\
&Q(T_3^4(n-7, 0, \ldots, 0)) - Q(T_4^4(n-7, 0, \ldots, 0)) \\
= \ &(x-1)^{n-8}[(n-5)x^6 - (10n-50)x^5 + (38n-188)x^4 \\
&-(68n-382)x^3 + \\
&(56n-256)x^2 - (16n-64)x], \\
&Q(T_{11}^4(n-8, 0, \ldots, 0)) - Q(T_{14}^4(n-7, 0, \ldots, 0)) \\
= \ &(x-1)^{n-9}[(2n-10)x^7 - (24n-120)x^6 + (110n-539)x^5 \\
&-(241n-1107)x^4 + \\
&(255n-971)x^3 - (111n-161)x^2 + (9n+144)x - 12],
\end{aligned}$$

By the results of **Appendix**, the results hold. □

Lemma 3.7. *For* $n \geq |T_j^7| (j = 1, \ldots, 7)$, $T_j^7(n-|T_j^7|, 0, \ldots, 0) \succeq T_1^7(n-4, 0, 0, 0)$.

Proof: We have

$$\begin{aligned}
&Q(T_2^7(n-5, 0, \ldots, 0)) - Q(T_1^7(n-4, 0, \ldots, 0)) \\
= \ &(x-1)^{n-6}[(n-3)x^4 - (8n-28)x^3 + (18n-68)x^2 \\
&-(12n-48)x], \\
&Q(T_3^7(n-6, 0, \ldots, 0)) - Q(T_1^7(n-4, 0, \ldots, 0)) \\
= \ &(x-1)^{n-7}[(2n-7)x^5 - (19n-73)x^4 + (55n-213)x^3 \\
&-(57n-199)x^2 + (19n-40)x - 12], \\
&Q(T_4^7(n-7, 0, \ldots, 0)) - Q(T_1^7(n-4, 0, \ldots, 0)) \\
= \ &(x-1)^{n-8}[(3n-12)x^6 - (33n-140)x^5 + (126n-536)x^4 \\
&-(210n-828)x^3 + (151n-432)x^2 - (37n+48)x + 60], \\
&Q(T_5^7(n-7, 0, \ldots, 0)) - Q(T_1^7(n-4, 0, \ldots, 0)) \\
= \ &(x-1)^{n-8}[(3n-12)x^6 - (33n-140)x^5 + (126n-540)x^4 \\
&-(210n-858)x^3 + (152n-514)x^2 - (39n+36)x + 39], \\
&Q(T_6^7(n-8, 0, \ldots, 0)) - Q(T_1^7(n-4, 0, \ldots, 0)) \\
= \ &(x-1)^{n-9}[(4n-18)x^7 - (50n-238)x^6 + (233n-1139)x^5 \\
&-(521n-2541)x^4 + (584n-2713)x^3 \\
&-(301n-1171)x^2 + (50n-40) - 32], \\
&Q(T_7^7(n-9, 0, \ldots, 0)) - Q(T_1^7(n-4, 0, \ldots, 0)) \\
= \ &(x-1)^{n-10}[(5n-25)x^8 - (70n-368)x^7 + (385n-2086)x^6 \\
&-(1085n-5938)x^5 + (2684n-9039)x^4 - (1415n-7004)x^3 \\
&+(572n-21122)x^2 - (76n+128)x + 80].
\end{aligned}$$

So the results hold. □

Theorem 3.8. *For $G \in \mathscr{T}_n^3 \cup \mathscr{T}_n^4 \cup \mathscr{T}_n^7$, $G \succeq T_1^7(n-4,0,0,0)$. The equality holds if and only if $G \cong T_1^7(n-4,0,0,0)$.*

Proof: If $G \in \mathscr{T}_n^{3,e} \cup \mathscr{T}_n^{4,e} \cup \mathscr{T}_n^{7,e}$, by Theorem 2.9, the results hold.
If $G \in \mathscr{T}_n^{3,o} \cup \mathscr{T}_n^{4,o} \cup \mathscr{T}_n^{7,o}$, by direct calculation, we have

$$\begin{aligned}
&Q(T_1^3(n-7,0,\dots,0)) - Q(T_1^7(n-4,0,\dots,0))\\
=\ &(x-1)^{n-8}[3x^6-(3n+16)x^5+(16n+54)x^4\\
&-(30n+156)x^3+(24n+259)x^2-(7n+204)x+60]\\
&Q(T_5^3(n-8,0,\dots,0)) - Q(T_1^7(n-4,0,\dots,0))\\
=\ &(x-1)^{n-9}[nx^7-(14n-16)x^6+(66n-84)x^5\\
&-(144n-108)x^4+(157n+100)x^3-(82n+316)x^2\\
&+(16n+224)x-48],\\
&Q(T_6^3(n-9,0,\dots,0)) - Q(T_1^7(n-4,0,\dots,0))\\
=\ &(x-1)^{n-10}[(2n-4)x^8-(28n-70)x^7+(148n-391)x^6\\
&-(393n-994)x^5+(570n-1221)x^4\\
&-(451n-616)x^3+(180n+16)x^2-(28n+96)x+16],\\
&Q(T_1^4(n-6,0,\dots,0)) - Q(T_1^7(n-4,0,\dots,0))\\
=\ &(x-1)^{n-7}[2x^5-(2n+8)x^4+(8n+26)x^3\\
&-(10n+68)x^2+(4n+80)x-32]\\
&Q(T_4^4(n-7,0,\dots,0)) - Q(T_1^7(n-4,0,\dots,0))\\
=\ &(x-1)^{n-8}[(n-1)x^6-(12n-22)x^5+(46n-85)x^4\\
&-(75n-96)x^3+(52n+16)x^2-(12n+64)x+16]\\
&Q(T_{14}^4(n-7,0,\dots,0)) - Q(T_1^7(n-4,0,\dots,0))\\
=\ &(x-1)^{n-8}[(n-1)x^6-(12n-24)x^5+(47n-107)x^4\\
&-(80n-176)x^3+(60n-108)x^2-(16n-16)x].
\end{aligned}$$

Further by Theorem 2.10 and lemmas 3.2–3.4, 3.6–3.8, we have $G \succeq T_1^7(n-4,0,0,0)$. □

Lemma 3.9. *For $n \geq |T_j^6| (j=1,\dots,9,11,\dots,24)$, $T_j^6(n-|T_j^6|,0,\dots,0) \succeq T_1^6(n-4,0,0,0)$.*

Proof: We have

$$\begin{aligned}
&Q(T_2^6(n-6,0,\dots,0)) - Q(T_1^6(n-5,0,\dots,0))\\
=\ &(x-1)^{n-7}[(n-3)x^5-(9n-29)x^4+(25n-76)x^3\\
&-\ (26n-56)x^2+(8n+16)x-16]
\end{aligned}$$

By the results of **Appendix**, the results hold. □

Theorem 3.10. *For $G \in \mathscr{T}_n^6$, $G \succeq T_{10}^6(n-5,0,0,0)$ or $G \succeq T_1^6(n-5,0,0,0)$. The equality holds if and only if $G \cong T_1^6(n-5,0,0,0,0)$ or $G \cong T_{10}^6(n-5,0,0,0)$.*

Proof: If $G \in \mathscr{T}_n^{6,e}$, by Theorem 2.9 and

$$Q(T_1^6(n-5,0,\dots,0)) - Q(T_1^7(n-4,0,\dots,0)) = 0 \quad (1)$$

we have $G \succeq T_1^6(n-5,0,0,0)$.
If $G \in \mathscr{T}_n^{6,o}$, by Theorem 2.10, lemmas 3.5 and 3.9, we have $G \succeq T_{10}^6(n-7,0,\dots,0)$ or $G \succeq T_1^6(n-5,0,0,0)$. □

Remark: By (3.1), $T_1^6(n-5,0,\dots,0)$ and $T_1^7(n-4,0,\dots,0)$ have the same signless Laplacian characteristic polynomials.

Theorem 3.11. *$T_{10}^6(n-7,0,\dots,0)$, $T_1^6(n-5,0,0,0)$, $T_1^7(n-4,0,0,0)$ are the only three minimal elements in the partial set $(\mathscr{T}_n, \succeq)$.*

Proof: By (3.1), theorems 3.8 and 3.10, it is obvious that $T_1^6(n-5,0,0,0)$, $T_1^7(n-4,0,0,0)$ are the minimal elements in the partial set $(\mathscr{T}_n, \succeq)$.
Note that if there is a graph G_0 in $\mathscr{T}_n^3 \cup \mathscr{T}_n^4 \cup \mathscr{T}_n^{6,e} \cup \mathscr{T}_n^7$ such that $T_{10}^6(n-7,0,\dots,0) \succeq G_0$, then by Theorem 3.8 and (3.1), we have $T_{10}^6(n-7,0,\dots,0) \succeq T_1^6(n-5,0,0,0)$. But

$$\begin{aligned}
&Q(T_{10}^6(n-7,0,\dots,0)) - Q(T_1^6(n-5,0,\dots,0))\\
=\ &(x-1)^{n-8}[(2n-6)x^6-(22n-76)x^5+(83n-308)x^4\\
&-(137n-542)x^3\\
&+(98n-448)x^2-(24n-192)x-48],
\end{aligned}$$

it is a contradiction.
Hence the results hold. □

4. THE INCIDENCE ENERGY OF TRICYCLIC GRAPHS

The incidence energy $IE(G)$ of a graph G is defined to be the sum of the square root of all eigenvalues of $Q(G)$[3].

Theorem 4.1. *[11] Let G and G′ be two graphs of order n, if $\varphi_k(G) \preceq \varphi_k(G')$ for $1 \leq k \leq n$, then $IE(G) \leq IE(G')$. In particular, if at least one of inequalities is strict, then $IE(G) < IE(G')$.*

Theorem 4.2. *If $G \in \mathscr{T}_n$, then $IE(G) \geq IE(T_1^6(n-5,0,0,0,0)) = IE(T_1^7(n-4,0,0,0))$. The equality holds if and only if $G \cong T_1^6(n-5,0,0,0,0)$, or $G \cong T_1^7(n-4,0,0,0)$.*

Proof: By Theorem 3.11, we have
$IE(G) \geq min\{IE(T_{10}^6(n-7,0,\cdots,0)), IE(T_1^6(n-5,0,0,0,0)), IE(T_1^7(n-4,0,0,0))\}$.
Note that

$$\begin{aligned}
Q(T_1^7(n-4,0,\dots,0)) &= (x-1)^{n-5}[(x^5-(n+9)x^4\\
&\quad+(9n+24)x^3-(24n+32)x^2\\
&\quad+(20n+48)x-48]\\
&= (x-1)^{n-5}(x-2)^2[x^3-(n+5)x^2\\
&\quad+5nx-12],\\
Q(T_{10}^6(n-7,0,\dots,0)) &= x(x-1)^{n-8}[(x^7-(n+12)x^6\\
&\quad+(14n+48)x^5-(76n+56)x^4+\\
&\quad(203n-83)x^3-(278n-230)x^2\\
&\quad+(182n-128)x-44n]\\
&= x(x-1)^{n-7}(x-2)[x^5-(n+9)x^4\\
&\quad+(11n+19)x^3-(41n-19)x^2\\
&\quad+(58n-64)x-22n].
\end{aligned}$$

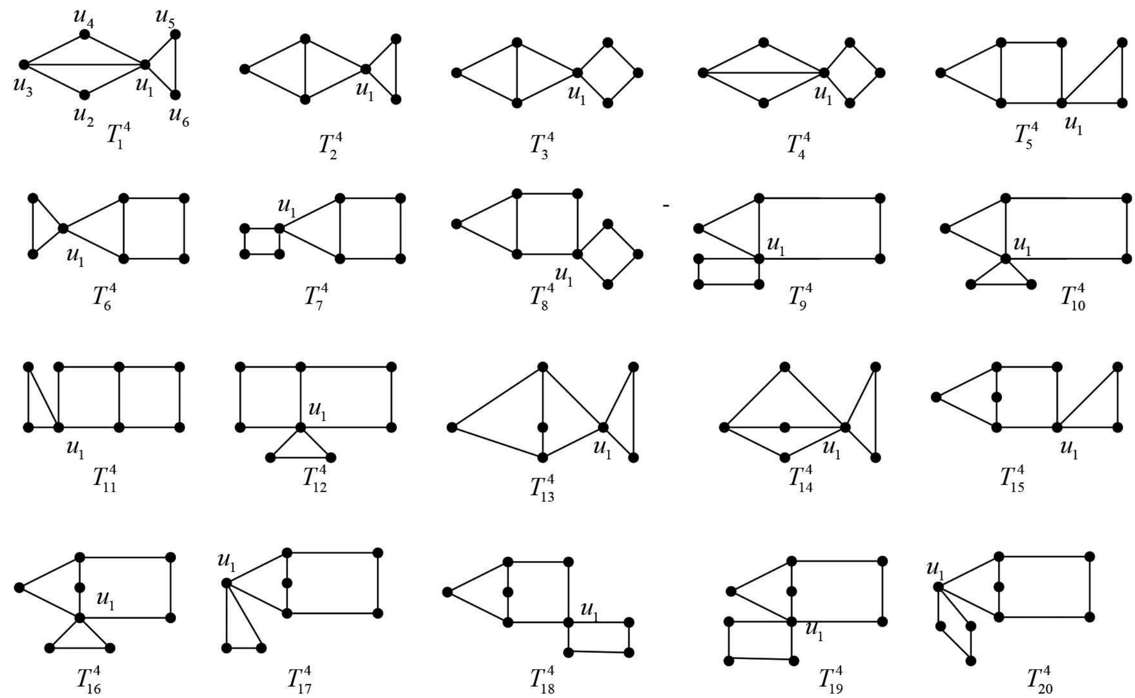

FIGURE 7 | The graphs $T_i^4 (i = 1, 2, \dots, 20)$.

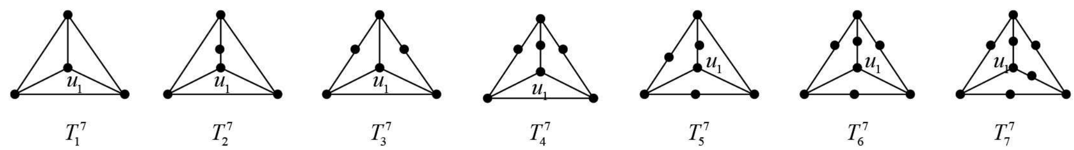

FIGURE 8 | The graphs $T_i^7 (i = 1, 2, \dots, 7)$.

Let $\alpha_1 \geq \alpha_2 \geq \alpha_3$ be the roots of $x^3 - (n+5)x^2 + 5nx - 12 = 0$, and $\beta_1 \geq \beta_2 \geq \beta_3 \geq \beta_4 \geq \beta_5$ be the roots of $x^5 - (n+9)x^4 + (11n+19)x^3 - (41n-19)x^2 + (58n-64)x - 22n = 0$, then

$$\begin{aligned} IE(T_1^7) &= (n-5) + 2\sqrt{2} + \sqrt{\alpha_1} + \sqrt{\alpha_2} + \sqrt{\alpha_3} \\ IE(T_{10}^6) &= (n-7) + \sqrt{2} + \sqrt{\beta_1} + \sqrt{\beta_2} + \sqrt{\beta_3} + \sqrt{\beta_4} + \sqrt{\beta_5}. \end{aligned}$$

If $n \leq 40$, by Matlab7.0 it is easy to see $IE(T_{10}^6) > IE(T_1^7)$ holds.

If $n \geq 40$, it is easy to see that $n - 0.07 \leq \alpha_1 \leq n + 0.07$, $4.93 \leq \alpha_2 \leq 5$, $0 \leq \alpha_3 \leq 0.07$ and $n - 1.98 \leq \beta_1 \leq n - 1.9$, $4.58 \leq \beta_2 \leq 4.62$, $3.41 \leq \beta_3 \leq 3.42$, $2.37 \leq \beta_4 \leq 2.39$, $0.54 \leq \beta_5 \leq 0.55$.

$$6.261 + \sqrt{n - 1.98} \leq \sum_{i=1}^{5} \sqrt{\beta_i} \leq 6.286 + \sqrt{n - 1.9},$$

$$2.22 + \sqrt{n - 0.07} \leq \sum_{i=1}^{3} \sqrt{\alpha_i} \leq 2.5 + \sqrt{n + 0.07},$$

$$3.597 \leq \sum_{i=1}^{5} \sqrt{\beta_i} - \sum_{i=1}^{3} \sqrt{\alpha_i} \leq 3.92.$$

It is easy to see that

$$\begin{aligned} IE(T_{10}^6) - IE(T_1^7) &= \sum_{i=1}^{5} \sqrt{\beta_i} - \sum_{i=1}^{3} \sqrt{\alpha_i} - 2 - \sqrt{2} \\ &\geq 3.597 - 2 - \sqrt{2} > 0. \end{aligned}$$

So the assertions hold. □

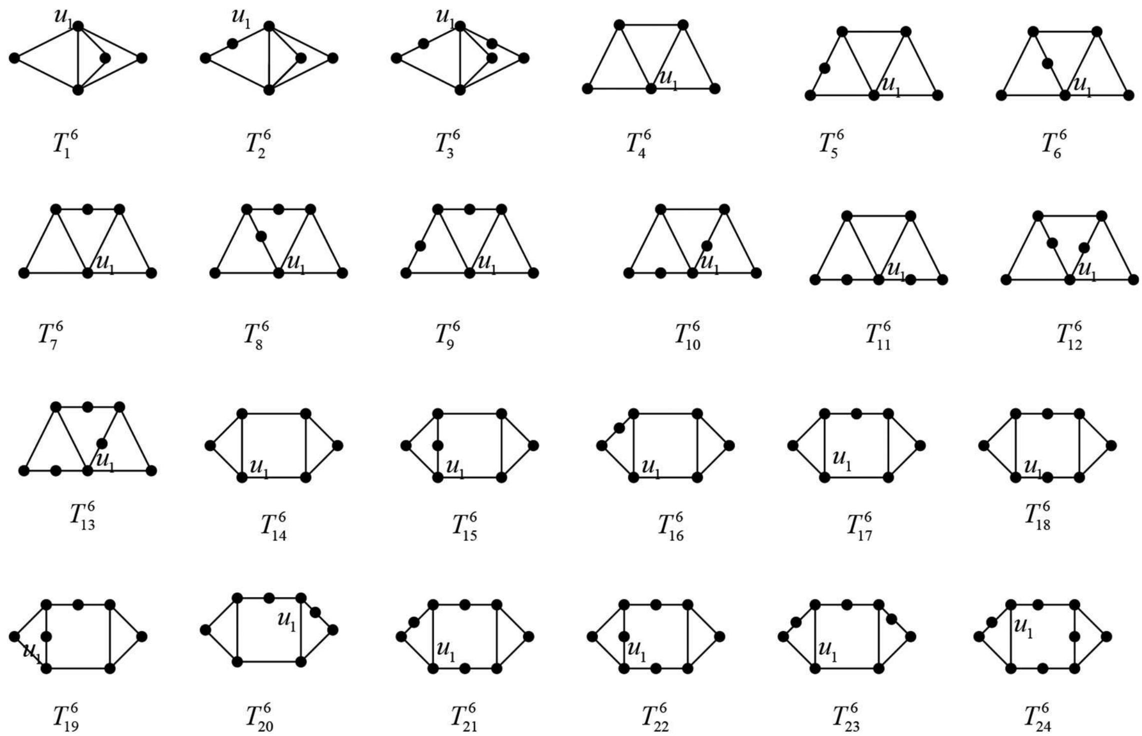

FIGURE 9 | The graphs $T_i^6 (i = 1, 2, \ldots, 24)$.

5. CONCLUSION AND EXTENSION

This paper propose an appropriate graph transformation to reflect the monotonicity of the coefficients, and give a sharp lower bound for incidence energy in the class of tricyclic graphs and characterize the extremal structures. The study on boundary of the incidence energy and its extremum structure of tricyclic graphs enriches and develops the study of the graph structure, but also connects the mathematical branch with other disciplines such as biology, physics and chemistry. It promotes the development of some theories of graph theory. It promotes the development of graph structure, the development of graph theory, and the study of graph theory and its application. For example, mathematical biology, application of graph theory in power system, molecular structure based on graph theory. Furthermore, similar to the graph energy, the incidence energy also reflects some physical and chemical properties of conjugated molecules, such as melting point and boiling point, this provides a theoretical reference for the researchers of the synthesis of new materials and new materials, and saves the cost for the development of new materials and new materials to a certain extent. Based on the extensive application of graph theory in many fields, the findings of this study have many important implications for future practice.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/**Supplementary Materials**, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

ZZ contributed to the conception of the study, performed the data analyses, and wrote the manuscript. HL contributed significantly to analysis and manuscript preparation and helped to perform the analysis with constructive discussions.

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SUPPLEMENTARY MATERIAL

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fphy.2020.00208/full#supplementary-material

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.