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Generalization of the Cover Pebbling Number for Networks

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Pebbling can be viewed as a model of resource transportation for networks. We use a graph to denote the network. A pebbling move on a graph consists of the removal of two pebbles from a vertex and the placement of one pebble on an adjacent vertex. The t-pebbling number of a graph G is the minimum number of pebbles so that we can move t pebbles on each vertex of G regardless of the original distribution of pebbles. Let ω be a positive function on $V(G)$; the ω-cover pebbling number of a graph G is the minimum number of pebbles so that we can reach a distribution with at least $\omega(v)$ pebbles on v for all $v \in V(G)$. In this paper, we give the ω-cover pebbling number of trees for nonnegative function ω, which generalized the t-pebbling number and the traditional weighted cover pebbling number of trees.

Keywords: network, tree, path partition, pebbling, cover pebbling

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1. INTRODUCTION

Pebbling in graphs was introduced by Chung [1]. It can also be viewed as a model of resource transportation for networks. Let G be a simple connected graph; we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. $d(u, v)$ is the distance of u and v, and we write $u \sim v$ if they are adjacent. $N(v) = \{u|u \sim v\}$ is the neighbor of v, and $d(v) = |N(v)|$ is the degree of v. Let H be a subgraph of G; we use $d_H(v)$ to denote the degree of v in H.

A pebble distribution D on G is a function $D: V(G) \to N$ (N is the set of nonnegative integers), where $D(v)$ is the number of pebbles on v, $|D| = \sum_{v \in V(D)} D(v)$ is the *size* of D.

A *pebbling move* consists of the removal of two pebbles from a vertex and the placement of one pebble on an adjacent vertex. Let D and D' be two pebble distributions of G. If so, we say that D *contains* D' if $D(v) \geq D'(v)$ for all $v \in V(G)$, and D' is *reachable* from D if there is some sequence (probably empty) of pebbling moves (a pebbling sequence in short) starting from D and resulting in a distribution, which contains D'. For a graph G and a vertex v, we call v a *root* (or target vertex) if the goal is to place pebbles on v. If t pebbles can be moved to v from D by a sequence of pebbling moves, then we say that D is *t-fold v-solvable*, and v is *t-reachable* from D. If D is t-fold v-solvable for every vertex v, we say that D is *t-solvable.*

The t-pebbling number of a graph G, denoted by $f_t(G)$, is the smallest number m such that every distribution with size m is t-solvable. While $t = 1$, we use $f(G)$ instead of $f_1(G)$, which is called *the pebbling number* of G.

For any two graphs G and H, we define the *Cartesian product* $G \times H$ to be the graph with the vertex set $V(G \times H)$ and edge set the union of $\{((a, v), (b, v))|(a, b) \in E(G), v \in E(H)\}$ and $\{((u, x), (u, y))|u \in V(G), and (x, y) \in E(H)\}$.

To determine the pebbling number of a general graph G is difficult. The problem of whether a distribution is v-solvable for some $v \in V(G)$ was shown to be NP-complete [2, 3]. The problem of deciding whether the pebbling number of a graph G is less than k was shown to be Π_2^P-complete [3]. The pebbling numbers of trees [4], cycles [5], hypercubes [1], and so on are determined. A conjecture called Graham's Conjecture is given by Chung [1].

Conjecture 1.1. (Graham's Conjecture) *Let G and H be two connected graphs; the pebbling number of the Cartesian product of G and H satisfies:*

$$f(G \times H) \leq f(G)f(H).$$

There are many results about Graham's Conjecture [6–10], while this conjecture is still open.

Definition 1.2. Let ω be a nonnegative function on $V(G)$ and D a distribution on $V(G)$. We say D is *ω-solvable* (or D solves ω) if we can reach a distribution D^* from D, by a sequence of pebbling moves, so that $D^*(v) \geq \omega(v)$ for all $v \in V(G)$. The *ω-cover pebbling number* of G, denoted by $\gamma_\omega(G)$, is the minimum number m so that every distribution D with size m is ω-solvable.

Definition 1.3. Let ω be a positive function on $V(G)$; define

$$s_\omega(v) = \sum_{u \in V(G)} \omega(u) 2^{d(u,v)},$$

and

$$s_\omega(G) = \max_{v \in V(G)} s_\omega(v).$$

The ω-cover pebbling number of a graph G has been determined for *positive* ω by [11].

Theorem 1.4. ([11]) *Let ω be a positive weight function on $V(G)$; the ω-cover pebbling number of G is*

$$\gamma_\omega(G) = s_\omega(G).$$

From Theorem 1.4, we can get

Theorem 1.5. *([11]) Let ω_1 be a positive function on G and ω_2 be a positive function on H. The function ω on $G \times H$ is given by $\omega((g,h)) = \omega_1(g)\omega_2(h)$, where $g \in V(G)$ and $h \in V(H)$, then $\gamma_\omega(G \times H) = \gamma_{\omega_1}(G)\gamma_{\omega_2}(H)$.*

We first generalize the definition of $s_\omega(T)$ while ω is a *nonnegative* function on a tree T. We will give the definition of path partition in the next section.

Definition 1.6. Given a tree T and a nonnegative function ω for each vertex $v \in V(T)$, and let $T_\omega(v)$ be the minimum subtree of T containing v and $W := \{u : \omega(u) > 0\}$. We give each edge in $T \backslash E(T_\omega(v))$ a direction toward $T_\omega(v)$ to get a directed graph, which is denoted by $\overrightarrow{T} \backslash E(T_\omega(v))$, and $(a_1, \ldots, a_n)$ is the size of the maximum path partition of $\overrightarrow{T} \backslash E(T_\omega(v))$. We define

$$s_\omega(v) = \sum_{u \in W} \omega(u) 2^{d(u,v)} + \sum_{i=1}^{n} 2^{a_i} - n.$$

and

$$s_\omega(T) = \max_{v \in V(T)} s_\omega(v).$$

Note that while ω is positive, then the two definitions of $s_\omega(T)$ are the same. Definition 1.6 is thus a generalization of Definition 1.3. In this paper, we generalize Theorem 1.4 while T is a tree and ω is nonnegative. Thus, our main result is as follows

Theorem 1.7. *Let T be a tree with a nonnegative weight function ω on $V(T)$. The ω-cover pebbling number of T is*

$$\gamma_\omega(T) = s_\omega(T).$$

Theorem 1.8. *Let T be a tree with a nonnegative weight function ω on $V(T)$. If $|W| = 1$, then Theorem 1.7 is equivalent to Theorem 2.2.*

Proof. If $|W| = 1$, assume that $\omega(v) = t$, and $\omega(u) = 0$ for $u \neq v$. We will show that $f_t(T, v) = s_\omega(T)$.

Assume the size of a maximum path partition of $\vec{T}_v$ is $(a_0, a_1, \ldots, a_n)$, and $d(v, v_0) = a_0$, P_0 is the longest directed path from v_0 to v. Then $(a_1, \ldots, a_n)$ must be the size of a maximum path partition in $\vec{T}_v \backslash P_0$. So $f_t(T, v) = s_\omega(v_0) \leq s_\omega(T)$.

Assume $s_\omega(T) = s_\omega(v_1)$, and $d(v_1, v) = a_0$. Let P_0 be the path connected v_1 and v, then $T_\omega(v_1) = P_0$; assume $(a_1, \ldots, a_n)$ is the size of the maximum path partition of $T \backslash E(T_\omega(v)) = T \backslash E(P_0)$, so $\alpha = (a_0, a_1, \ldots, a_n)$ is a path partition of $\vec{T}_v$, and $s_\alpha = s_\omega(v_1)$ by Corollary 2.3 and $f_t(T, v) \geq s_\omega(v_1) = s_\omega(T)$. ∎

Definition 1.9. ([12]) Given a sequence S of pebbling moves on G, *the transition digraph* obtained from S is a directed multigraph denoted $T(G, S)$ that has $V(G)$ as its vertex set. Each move $s \in S$ along edge uv (move off two pebbles from u and add one on v) is represented by a directed edge uv.

The following lemma is useful in the following sections.

Lemma 1.10. ([12], No-Cycle Lemma) *Let S be a sequence of pebbling moves on G, reaching a distribution D. Then there exists a sequence S^* of pebbling moves, thus reaching a distribution D^* where*

1. On each vertex v, $D^*(v) \geq D(v)$;

2. $T(G, S^*)$ does not contain any directed cycles.

2. PRELIMINARIES

We first introduce the path partition and the pebbling number of trees.

Definition 2.1. ([4]) Given a root v of a tree T, then we can view T as a directed graph $\overrightarrow{T_v}$ with edges directed toward v. *A path partition* is a set of nonoverlapping directed paths in which the union is $\overrightarrow{T_v}$. A path partition is said to *majorize* another if the non-increasing sequence of the path size majorizes that of the other (that is $(a_1, a_2, \ldots, a_r) > (b_1, b_2, \ldots, b_t)$ if and only if $a_i > b_i$, where $i = \min\{j : a_j \neq b_j\}$). A path partition of a tree $\overrightarrow{T_v}$ is said to be *maximum* if it majorizes all other path partitions. Note that, in this paper, the sequence of the size of a path partition is always non-increasing.

Note: By the definition of the maximum path partition, we can give a way to determine the size of the maximum path partition. First, we choose the longest directed path P_1 in $\overrightarrow{T_v}$, with length a_1. Then, we choose the longest directed path P_2 in $\overrightarrow{T_v}\backslash E(P_1)$, with length a_2, and so on. Moreover, it should be noted that the maximum path partition may not be unique, but the size of the maximum path partition must be unique.

Moews [4] found the t-pebbling number of trees by a path partition.

Theorem 2.2. ([4]) *Let T be a tree, $v \in V(T)$, and $(a_1, \ldots, a_n)$ be the size of the maximum path partition of $\overrightarrow{T_v}$. Then,*

$$f_t(T, v) = t2^{a_1} + \sum_{i=2}^{n} 2^{a_i} - n + 1,$$

$$f_t(T) = \max_{v \in V(T)} f_t(T, v).$$

Corollary 2.3. *Let T be a tree, $v \in V(T)$, and $\alpha = (a_1, \ldots, a_n)$ be the size of a path partition of $\overrightarrow{T_v}$, $s_\alpha := t2^{a_1} + \sum_{i=2}^{n} 2^{a_i} - n+1$. Then,*

$$f_t(T, v) = \max_{\alpha} s_\alpha.$$

Proof. Let α_0 be the size of the maximum path partition of $\overrightarrow{T_v}$. Then, $f_t(T, v) = s_{\alpha_0} \leq \max_\alpha s_\alpha$.

Assume $P_1, P_2, \ldots, P_n$ is a path partition of $\overrightarrow{T_v}$, and the length of P_i is a_i for $1 \leq i \leq n$. Note that for each P_i we should assume the two endpoints v_i and v_i' satisfy $d(v_i, v) > d(v_i', v)$. We put $t2^{a_1} - 1$ pebbles on v_1 and $2^{a_i} - 1$ pebbles on v_i for $2 \leq i \leq n$; it is clear that t pebbles cannot be moved to v from this distribution. Thus, for each α, $s_\alpha - 1 < f_t(T, v)$, so $s_\alpha \leq f_t(T, v)$ so $\max_\alpha s_\alpha \leq f_t(T, v)$. ∎

Definition 2.4. Let C be a *generalized distribution* on G, where $C(v)$ is an integer (may be negative) for all $v \in V(G)$. *A pebbling move* on G consists of the removal of two pebbles from a vertex v (with $C(v) \geq 2$) and the placement of one pebble on an adjacent vertex.

In the following, a *distribution D* means that $D(v) \geq 0$, and a *generalized distribution C* means $C(v)$ is an integer for all $v \in V(G)$.

Definition 2.5. A pebbling move from u to v under a distribution D is *executable* if $D(u) \geq 2$. A pebbling sequence S is a finite set of pebbling moves, assuming $S = (S_1, ..., S_k)$, where S_i is a pebbling move for $1 \leq i \leq k$, and the pebbling move S_i is in front of S_j if $1 \leq i < j \leq k$. We say the pebbling sequence S *executable*, if S_i is executable for $1 \leq i \leq k$.

Definition 2.6. Let ω be a nonnegative function on $V(G)$ and C be a generalized distribution on $V(G)$. We say C is *ω-solvable*, if we can reach a distribution C^* from C, by a sequence of pebbling moves so that $C^*(v) \geq \omega(v)$. In particular, if $\omega(v) = 0$ for all $v \in V(G)$, then we say that C is *0-solvable*.

Lemma 2.7. *Let D be a distribution on a graph G and ω be a nonnegative function on $V(G)$, $C := D - \omega$. Then, D is ω-solvable if and only if C is 0-solvable.*

Proof. If C is 0-solvable, let δ be an executable pebbling sequence that reaches a distribution D^* so that $D^* > 0$ from C. It is then clear that δ is also an executable pebbling sequence that can reach a distribution D' so that $D' = D^* + \omega > \omega$ from D. Thus D is ω-solvable.

On the other hand, if D is ω-solvable, by Lemma 1.10, there exists a pebbling sequence S reaching a distribution D^* with $D^*(v) \geq \omega(v)$, and $T(G, S)$ does not contain any direct cycle. We can thus give a sequence of the vertices of G, as $(v_1, v_2, \ldots, v_n)$, so that each directed edge v_iv_j in $T(G, S)$ satisfies $i < j$. We can thus rearrange the sequence of pebbling moves S along the order $(v_1, v_2, \ldots, v_n)$. It means we first choose all pebbling moves in S that remove pebbles from v_1, choose all pebbling moves in S that remove pebbles from v_2, and so on, and we denote this sequence of pebbling moves by S'. We will show that S' is an executable pebbling sequence that reach $D^* - \omega$ from C.

In S', for each vertex $v \in V(G)$, the pebbling moves that move pebbles to v are in front of the pebbling moves that remove pebbles from v. We may thus assume that, for each vertex v_i, we first move α_i pebbles from other vertices to v_i and then remove β_i pebbles from v_i.

We only need to show that, for each $v_i \in V(G)$, the sequence of pebbling moves that removes β_i pebbles from v_i in S', denoted by σ_i, is executable. We use induction on i. If $i = 1$, and we can then get $D(v_1) - \beta_1 = D^*(v_1) \geq \omega(v_1) \Rightarrow D(v_1) - \omega(v_1) \geq \beta_1 \Rightarrow C(v_1) \geq \beta_1$, and so σ_1 is executable.

Assume σ_h is executable for $h < i$. By induction, the pebbling sequence that moves α_i pebbles to v_i is executable. Moreover, we can get $D(v_i) + \alpha_i - \beta_i = D^*(v_i) \Rightarrow D(v_i) + \alpha_i - \omega(v_i) - \beta_i = D^*(v_i) - \omega(v_i) \geq 0 \Rightarrow D(v_i) - \omega(v_i) + \alpha_i \geq \beta_i \Rightarrow C(v_i) + \alpha_i \geq \beta_i$. Thus σ_i is executable.

So S' is an executable pebbling sequence that reaching $D^* - \omega$ from C. Note that $D^* - \omega \geq 0$, and this completes the proof. ∎

Definition 2.8. Let D be a distribution on a tree T and ω be a nonnegative function on $V(T)$. $C := D - \omega$ is called the *induced generalized distribution* from D and ω of T. Let v be a leaf of T and u be its neighbor in T. The *induced generalized distribution* C' on $T \setminus v$ is given: if $C(v) \geq 0$, then $C'(u) = C(u) + \lfloor C(v)/2 \rfloor$, and if $C(v) < 0$, then $C'(u) = C(u) + 2C(v)$, keeping $C'(x) = C(x)$ unchanged for all $x \neq u$.

Lemma 2.9. *Let D be a distribution on a tree T and ω be a nonnegative function on $V(T)$. $C := D - \omega$, v is a leaf of T, and C' is the induced generalized distribution from D and ω of $T\backslash v$. Then, C is 0-solvable in T if and only if C' is 0-solvable in $T\backslash v$.*

Proof. Firstly, we assume C is 0-solvable in T, and there is a pebbling sequence σ reaching a distribution C^* from C with $C^*(x) \geq 0$ for each $x \in V(T)$.

Case 1.1. $C(v) \geq 0$. By Lemma 1.10, we may assume that no pebble has been moved from u to v; at most, therefore, $\lfloor C(v)/2 \rfloor$

pebbles can be moved from v to u. We may assume the first step of σ is to move $\lfloor C(v)/2 \rfloor$ pebbles from v to u, so the left steps makes C' solve 0 on $T\backslash v$.

Case 1.2. $C(v) < 0$. By Lemma 1.10, we may assume that no pebble has been moved from v to u. So we may assume the last step of σ is to move $-C(v)$ pebbles from u to v, and so the steps before it makes C' solve 0 on $T\backslash v$.

Secondly, we assume C' is 0-solvable in $T\backslash v$, and there is a pebbling sequence δ reaching a distribution C^* from C' with $C^*(x) \geq 0$ for each $x \in V(T\backslash v)$.

Case 2.1. $C(v) \geq 0$. First, we move $\lfloor C(v)/2 \rfloor$ pebbles from v to u, and the left steps are just δ; this sequence makes C solve 0.

Case 2.2. $C(v) < 0$. After the pebbling sequence δ, we move $-C(v)$ pebbles from u to v; this sequence makes C solve 0. ∎

Notations: Assume T^* is a subtree of T, then T^* can be obtained from T by deleting the leaves of the subtree of T (the vertex with degree one) one by one. For each subtree T^* of T, therefore, we can get an induced generalized distribution C^*. In particular, for each vertex $v \in V(T)$, let T_v be a subtree containing v and all of its neighbors. We use C_v to denote the induced generalized distribution from D and ω of T_v and $\widehat{C}(v)$ to denote the induced generalized distribution of $\{v\}$.

Corollary 2.10. *Let D be a distribution on a tree T, ω be a nonnegative function on $V(T)$, and $\widehat{C}(v)$ be the induced generalized distribution from D and ω of $\{v\}$. D is not ω-solvable is equivalent to $\widehat{C}(v) < 0$ for each $v \in V(T)$.*

Proof. From Lemma 2.7 and Lemma 2.9, the result follows by induction. ∎

Lemma 2.11. *Let D be a distribution on a tree T, which is not ω-solvable with $|D| = \gamma_\omega(T) - 1$. For each vertex $x \in V(T)$, which is not a leaf of T, there exists a vertex $y \in N(x)$, so that $C_x(y) \geq 0$.*

Proof. If $C_x(x') < 0$, for all $x' \in N(x)$, assume $y, z \in N(x)$ with $C_x(z) \leq C_x(y) < 0$. We delete all other vertices to the left $T_1 = yxz$ and its induced generalized distribution C_1. Then, $C_1(y) = C_x(y)$, $C_1(z) = C_x(z)$ and $\widehat{C}(x) = C_1(x) + 2C_1(y) + 2C_1(z) \leq -1$ by Corollary 2.10. Note that $C_1(x) = D(x) - w(x) + \sum_{x' \in N(x), x' \notin \{y,z\}} 2C_x(x')$. Thus, $C_1(x) - D(x) \leq 0$ and $C_1(x) + 2C_1(z) - D(x) \leq 0$. Now, we remove $D(x)$ pebbles from x and put $D(x) + 1$ pebbles on y to get a new distribution D' with $|D'| = |D| + 1$. The induced generalized distribution from D' and ω of $\{y\}$ is denoted by $\widehat{C'}(y)$. Then, $\widehat{C'}(y) = (C_1(y) + D(x) + 1) + 2(C_1(x) + 2C_1(z) - D(x)) = (C_1(x) + 2C_1(y) + 2C_1(z)) + (C_1(z) - C_1(y)) + C_1(z) + (C_1(x) - D(x)) + 1 \leq -1 + 0 - 1 + 0 + 1 = -1$, and so D' is not ω-solvable by Corollary 2.10, a contradiction to $|D'| = \gamma_\omega(T)$, and we are done. ∎

Theorem 2.12. *Let ω be a nonnegative function on $V(T)$ and D be a distribution that is not ω-solvable with $|D| = \gamma_\omega(T) - 1$. All pebbles are then distributed on the leaves of T.*

Proof. If $D(x) > 0$ for some vertex $x \in V(T)$, which is not a leaf, then $N(x)$ has at least two vertices. By Lemma 2.11, there exists a vertex $y \in N(x)$ with $C_x(y) \geq 0$. We first show that there exists a vertex $z \in N(x)$ with $C_x(z) < 0$.

If not, that means for all $v \in N(x)$, $C_x(v) \geq 0$. Note that $D(x) > 0$, and thus $\widehat{C}(x) = D(x) + \sum_{v \in N(x)} \lfloor C_x(v)/2 \rfloor > 0$. By Corollary 2.10, D is ω-solvable, a contradiction. Thus, there exists a vertex $z \in N(x)$ with $C_x(z) < 0$.

Let $T_1 = yxz$ be the subtree of T, with induced generalized distribution C_1. Then, $C_1(z) = C_x(z)$, $C_1(y) = C_x(y)$, and $\widehat{C}(x) = C_1(x) + \lfloor C_1(y)/2 \rfloor + 2C_1(z) < 0$.

Now, consider the new distribution D^*, with $D^*(y) = D(y) + D(x) + 1$, $D^*(x) = 0$, and $D^*(v) = D(v)$; $|D^*| = \gamma_\omega(T)$. The induced generalized distribution from D^* and ω of $\{x\}$ is given by $\widehat{C^*}(x) = (C_1(x) - D(x)) + \lfloor (C_1(y) + D(x) + 1)/2 \rfloor + 2C_1(z)$.

If $D(x) = 1$, then $\widehat{C^*}(x) = C_1(x) + \lfloor C_1(y)/2 \rfloor + 2C_1(z) = \widehat{C}(x) < 0$;

If $D(x) \geq 2$, then $\widehat{C^*}(x) \leq C_1(x) - D(x) + \lfloor C_1(y)/2 \rfloor + D(x)/2 + 1 + 2C_1(z) = \widehat{C}(x) + 1 - D(x)/2 \leq \widehat{C}(x) < 0$.

By Corollary 2.10, D^* is not ω-solvable, a contradiction to $|D^*| = \gamma_\omega(T)$. This completes the proof. ∎

From Theorem 2.12, for a given integer p with $p < \gamma_\omega(T)$, there must exist a distribution D, which is not ω-solvable with $|D| = p$, and all pebbles are distributed on the leaves of T.

3. THE GENERALIZATION OF THE COVER PEBBLING NUMBER ON TREES

Assume that $s_\omega(v_0) = s_\omega(T)$ for some $v_0 \in V(T)$; it should be noted that $\overrightarrow{T} \backslash E(T_\omega(v_0))$ is a directed graph. We define $d_\omega(u, l)$ to be the length of the maximal path containing u in all maximum path partitions of $\overrightarrow{T} \backslash E(T_\omega(v_0))$. If ω is clear, then we use $d(u, l)$ for short (note that $d(u, l)$ maybe 0). Let P_α be a maximal path partition of $\overrightarrow{T} \backslash E(T_\omega(v_0))$; then, $d_\omega(u, l) = \max_{P_\alpha}\{|P| : u \in P, P \in P_\alpha\}$.

Lemma 3.1. *Assume that $s_\omega(v_0) = s_\omega(T)$ for some $v_0 \in V(T)$; then for each vertex $u \in V(T)$ and $d(u, v_0) \geq d(u, l)$.*

Proof. Assume $u, v \in V(T)$. There is exactly one subpath of T with endpoints u and v, and we denote this path by P_{uv}. We thus have $P_{uv} = P_{vu}$.

If $|W| = 1$, we may assume that $\omega(v) = t$, and $\omega(u) = 0$ for $u \neq v$. By the proof of Theorem 1.8, we know that $f_t(T, v) = s_\omega(v_0)$. Let $(a_1, a_2, \ldots, a_n)$ be the size of the maximum path partition of $\overrightarrow{T_v}$. Then $d(v, v_0) = \max_{u \in V(T)} d(v, u) = a_1$. Assume P_1 is the maximal path containing u in $\overrightarrow{T_v} \backslash P_{v_0, v}$, and $P_1 \cap P_{v_0 v} = v_1$. The length of $P_{v_0 v}$ (P_1) is thus a_1 $(d(u, l))$ and $d(v_1, v_0) \leq d(u, v_0)$. If $d(u, v_0) < d(u, l)$, then $d(v_1, v_0) < d(u, l)$, and we get a path $P_1 \cup P_{v_1 v}$ with length $a_1 - d(v_1, v_0) + d(u, l) > a_1$, a contradiction to the maximum of a_1, and thus $d(u, v_0) \geq d(u, l)$.

If $|W| \geq 2$, we only need to show it while $u \in V(T_\omega(v_0))$.

If $d(u, v_0) < d(u, l)$ for some $u \in V(T_\omega(v_0))$, there exists a leaf v_1 in $\overrightarrow{T} \backslash E(T_\omega(v_0))$ so that $d(u, l) = d(u, v_1)$, and we will show that $s_\omega(v_1) > s_\omega(v_0)$.

Let $TC(v)$ be the component of $T\backslash u$ containing the vertex v. We thus have $TC(v_1) \cap W = \emptyset$.

Case 1. $TC(v_0) \cap W \neq \emptyset$.

Assume $w_1 \in TC(v_0) \cap W$, then $d(w_1, v_1) \geq d(u, v_1) + 1$ and

$$d(w_1, v_1) - d(w_1, v_0) \geq d(u, v_1) - d(u, v_0) + 2 \geq 3.$$

Note that $\overrightarrow{T}\backslash E(T_\omega(v_0) \cup P_{v_1 u}) \subseteq \overrightarrow{T}\backslash E(T_\omega(v_1))$. So

$$\begin{aligned} & s_\omega(v_1) - s_\omega(v_0) \\ & \geq \sum_{x\in W} \omega(x)(2^{d(x,v_1)} - 2^{d(x,v_0)}) - 2^{d(u,v_1)} \\ & \geq \omega(w_1)(2^{d(w_1,v_1)} - 2^{d(w_1,v_0)}) - 2^{d(u,v_1)} \\ & \geq 2^{d(w_1,v_1)} - 2^{d(w_1,v_0)} - 2^{d(u,v_1)} \\ & \geq 2^{d(w_1,v_1)} - \frac{2^{d(w_1,v_1)}}{8} - \frac{2^{d(w_1,v_1)}}{2} \\ & = \frac{3 \cdot 2^{d(w_1,v_1)}}{8} > 0. \end{aligned}$$

Hence, $s_\omega(v_1) > s_\omega(v_0)$, which is a contradiction to $s_\omega(v_0) = s_\omega(T)$.

Case 2. $TC(v_0) \cap W = \emptyset$.

Let $\tau_\omega(v) = \sum_{x\in W} \omega(x)2^{d(x,v)}$. If so, then $\tau_\omega(v_0) = 2^{d(u,v_0)}\tau_\omega(u)$, and $\tau_\omega(v_1) = 2^{d(u,v_1)}\tau_\omega(u)$. For $|W| \geq 2$, $\tau_\omega(u) \geq 2^0 + 2^1 = 3$.

Note that $\overrightarrow{T}\backslash E(T_\omega(v_0) \cup P_{v_1 u}) \subseteq \overrightarrow{T}\backslash E(T_\omega(v_1))$. So

$$\begin{aligned} & s_\omega(v_1) - s_\omega(v_0) \\ & \geq 2^{d(u,v_1)}\tau_\omega(u) - 2^{d(u,v_0)}\tau_\omega(u) - 2^{d(u,v_1)} \\ & = \tau_\omega(u)(2^{d(u,v_1)} - 2^{d(u,v_0)}) - 2^{d(u,v_1)} \\ & \geq 3(2^{d(u,v_1)} - 2^{d(u,v_0)}) - 2^{d(u,v_1)} \\ & \geq 3(2^{d(u,v_1)} - \frac{2^{d(u,v_1)}}{2}) - 2^{d(u,v_1)} \\ & = \frac{2^{d(u,v_1)}}{2} > 0. \end{aligned}$$

Hence, $s_\omega(v_1) > s_\omega(v_0)$, which is a contradiction to $s_\omega(v_0) = s_\omega(T)$, and this completes the proof. ∎

Corollary 3.2. *Let ω be a nonnegative function in $V(T)$, for some $v \in W$, and ω' be a nonnegative function satisfying $\omega'(v) = \omega(v) - 1$, $\omega'(u) = \omega(u)$ for other vertices in T. If so, then*

$$s_\omega(T) \geq s_{\omega'}(T) + 2^{d_\omega(v,l)}.$$

Proof. Assume that there exist v_1 and v_2, so that $s_\omega(v_1) = s_\omega(T)$ and $s_{\omega'}(v_2) = s_{\omega'}(T)$.

By the definition of $s_\omega(v)$, if $\omega(v) \geq 2$, then $d_\omega(v, l) = d_{\omega'}(v, l)$, we have

$$\begin{aligned} s_\omega(T) = s_\omega(v_1) \geq s_\omega(v_2) & \\ & = s_{\omega'}(v_2) + 2^{d(v,v_2)} \\ & \geq s_{\omega'}(v_2) + 2^{d_{\omega'}(v,l)} \ \text{(by Lemma 3.1)} \\ & = s_{\omega'}(T) + 2^{d_\omega(v,l)}. \end{aligned}$$

If $\omega(v) = 1$, the difference between $\overrightarrow{T}\backslash T_\omega(v_1)$ and $\overrightarrow{T}\backslash T_{\omega'}(v_2)$ is just the length of the maximal path containing v, we have

$$\begin{aligned} s_\omega(T) = s_\omega(v_1) \geq s_\omega(v_2) & \\ & = s_{\omega'}(v_2) + 2^{d(v,v_2)} + 2^{d_\omega(v,l)} - 2^{d_{\omega'}(v,l)} \\ & \geq s_{\omega'}(v_2) + 2^{d_\omega(v,l)} \ \text{(by Lemma 3.1)} \\ & = s_{\omega'}(T) + 2^{d_\omega(v,l)}. \end{aligned}$$

∎

The proof of Theorem 1.7:

The lower bound holds clearly, as we put $2^{a_i} - 1$ pebbles on the leaf of each path for $1 \leq i \leq n$ (no pebble can then be moved to $T_\omega(v)$), and $\sum_{u\in S} w(u)2^{d(u,v)} - 1$ pebbles on v, obviously it is not ω-solvable.

For the upper bound, it holds if $|\omega| = 1$ or $|W| = 1$ by the proof of Theorem 1.8. It also holds for $|T| \leq 2$ by Theorem 2.2 and Theorem 1.4. We may thus assume that $|\omega| \geq 2$, $|W| \geq 2$, and $|T| \geq 3$.

If the result is false for some T and ω, then we choose one counterexample T and its weight ω so that $|T|$ and $|\omega|$ are both minimal. It means the upper bound holds for T' and its weight ω' if $|T'| < |T|$ or $|\omega'| < |\omega|$.

Let D be a distribution on T, which is not ω-solvable with size $s_\omega(T)$. By Theorem 2.12, we may assume that all pebbles are distributed on the leaves of T.

Assume $s_\omega(v_0) = s_\omega(T)$. There exists $x \in W\backslash v_0$ satisfying $d_{T_\omega(v_0)}(x) = 1$. If $d_T(x) \neq 1$, we can get $d(x, l) > 0$, and there exists a nonempty component in $T\backslash E(T_\omega(v_0))$, which is connected with x. Say T_1 and $b_1 \geq b_2 \geq \ldots \geq b_m$ is the size of the maximum path partition of T_1.

Case 1. $D(T_1)$ cannot move a pebble to x. $|D(T_1)| \leq \sum_{i=1}^m 2^{b_i} - m$, and we consider D on $T\backslash T_1$, $|D(T\backslash T_1)| \geq s_\omega(T) - D(T_1) \geq s_\omega(T\backslash T_1)$, and $D(T\backslash T_1)$ is not ω-solvable, a contradiction to the minimum of $|T|$.

Case 2. $D(T_1)$ can move one pebble to x. It costs us at most $2^{b_1} = 2^{d_\omega(x,l)}$ pebbles on T_1. The left pebbles on T is not ω'-solvable (ω' satisfies $\omega'(x) = \omega(x) - 1$, and it is unchanged for other vertices in T). From the minimum of $|\omega|$ and Corollary 3.2, we thus have $|D| < s_{\omega'}(T) + 2^{d_\omega(x,l)} \leq s_\omega(T)$, a contradiction to $|D| = s_\omega(T)$.

We may therefore assume $d_T(x) = 1$.

We claim that $D(x) = 0$. Otherwise, let ω' satisfy $\omega'(x) = \omega(x) - 1$ and $\omega'(v) = \omega(v)$ for $v \neq x$. Regardless of one pebble being on x, we know that $|D| - 1$ other pebbles cannot solve ω'. From the minimum of $|\omega|$, we have $|D| - 1 \leq s_{\omega'}(T) - 1$. By Corollary 3.2, $s_{\omega'}(T) + 1 \leq s_\omega(T)$, so $|D| \leq s_\omega(T) - 1$, a contradiction to $|D| = s_\omega(T)$, so $D(x) = 0$.

Assuming that $x' \sim x$ in T, we then delete x. Let $C'(x') = C(x') + 2C(x)$ and $C'(v) = C(v)$ otherwise. Note that all pebbles are distributed on the leaves of T, so $C'(x) = D(x') - \omega(x') - 2(D(x) - \omega(x)) = -\omega(x') - 2\omega(x)$. By Lemma 2.9, D is not ω-solvable in T is equivalent to D is not ω'-solvable in $T\backslash x$, where $\omega'(x') = \omega(x') + 2\omega(x)$ and $\omega'(v) = \omega(v)$ for $v \neq x$. By the minimum of $|T|$, we have $|D| \leq s_{\omega'}(T\backslash x) - 1$, note that $x \neq v_0$, we have $s_{\omega'}(T\backslash x) = s_\omega(T)$, a contradiction to $|D| = s_\omega(T)$. This completes the proof. ∎

Moreover, by Theorem 1.7, we can immediately get

Corollary 3.3. *Let T be a tree, and let ω be a nonnegative function on $V(T)$, $W = \{v \in V(T) : \omega(v) > 0\}$, $L = \{v \in V(T) : d(v) = 1\}$, then if $L \subseteq W$,*

$$\gamma_\omega(T) = \max_{v \in V(T)} \sum_{u \in V(T)} \omega(u) 2^{d(u,v)}.$$

Theorem 1.4 gives a sufficient condition of a nonnegative weight function ω on $V(G)$ for a graph G so that the ω-cover pebbling number of G is

$$\gamma_\omega(G) = \max_{v \in V(G)} \sum_{u \in V(G)} \omega(u) 2^{d(u,v)}.$$

Corollary 3.3 gives a weaker sufficient condition of a nonnegative weight function ω on $V(T)$ for a tree T so that the ω-cover pebbling number of T is

$$\gamma_w(T) = \max_{v \in V(T)} \sum_{u \in V(T)} \omega(u) 2^{d(u,v)}.$$

Here, we explore some problems.

Problem 3.4. *Give a weaker sufficient condition of a nonnegative function ω on $V(G)$ for a graph G so that the ω-cover pebbling number of G is*

$$\gamma_\omega(G) = \max_{v \in V(G)} \sum_{u \in V(G)} \omega(u) 2^{d(u,v)}.$$

Problem 3.5. *For a nonnegative function ω, determine the ω-cover pebbling number of more graphs, such as cycles, hypercubes, and so on.*

We also give a conjecture which is similar to Graham's Conjecture.

Conjecture 3.6. *Let ω_1 be a nonnegative function on G and ω_2 be a nonnegative function on H. The function ω on $G \times H$ is given by $\omega((g,h)) = \omega_1(g)\omega_2(h)$, where $g \in V(G)$ and $h \in V(H)$, then $\gamma_\omega(G \times H) \leq \gamma_{\omega_1}(G)\gamma_{\omega_2}(H)$.*

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary materials, further inquiries can be directed to the corresponding authors.

AUTHOR CONTRIBUTIONS

Z-JX provided this topic and wrote the paper. Z-JX and Z-MH solved the problem. Z-MH reviewed and edited the manuscript. All authors contributed to the article and approved the submitted version.

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