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Recent Progress in Nuclear Lattice Simulations

*Dean Lee**

Facility for Rare Isotope Beams and Department of Physics & Astronomy, Michigan State University, East Lansing, MI, United States

We review several recent results on lattice simulations by the Nuclear Lattice Effective Field Theory Collaboration. In the first part we discuss the implementation of nuclear forces on the lattice using chiral effective field theory. The new development we highlight is the use of non-local lattice operators to achieve a simpler spin channel decomposition, in contrast with previous studies that considered only local interactions. In the second part, we present evidence that nuclear physics is close to a quantum phase transition. This development is also linked to the study of the differences between local and non-local interactions. In the final part we further explore the link between the nuclear forces and nuclear structure. We consider the simplest possible nuclear interaction which can accurately reproduce the ground state energies of neutron matter, light nuclei, and medium-mass nuclei. We discuss what these recent developments say about the emergence of nuclear structure from nuclear forces and the road ahead for nuclear lattice simulations.

Keywords: nuclear structure, nuclear forces, lattice effective field theory, nuclear lattice simulations, quantum phase transition

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***Correspondence:**
Dean Lee
leed@frib.msu.edu

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1. INTRODUCTION

In this article we review several recent advances by the Nuclear Lattice Effective Field Theory Collaboration that combine of chiral effective field theory with lattice methods, an approach that we call nuclear lattice effective field theory. See Epelbaum et al. [1] and Machleidt and Entem [2] for reviews of chiral effective field theory (EFT) and Lee [3] and Lähde and Meißner [4] for reviews of nuclear lattice EFT. We begin the article by presenting some technical developments regarding the nuclear lattice interactions. The new technology that makes this possible is the use of non-local smearing for the operator interactions. These are important for producing interactions that can give accurate reproductions of the empirical nucleon-nucleon phase shifts.

In the second part of the article we consider the features of the nuclear interactions which are responsible for the bulk binding properties of atomic nuclei. From *ab initio* lattice simulations we present numerical data showing that nature sits near a quantum phase transition. It turns out that the differences between local and non-local interactions will again be important. In the last part of the review we continue with our first principles investigations of nuclear forces and nuclear structure. This time we discuss a simple nuclear interaction which gives the correct ground state energies of neutron matter, light nuclei, and medium-mass nuclei with no more than a few percent error. We then conclude by discussing the connection between nuclear structure and nuclear forces and some possible future directions for nuclear lattice simulations.

2. IMPROVED LATTICE INTERACTIONS

Simulations using lattice chiral EFT have been applied to predict the scattering and structure of nuclei [5–7]. The application of nuclear forces at higher chiral orders are problematic on the lattice due to the breaking of rotational invariance produced by the lattice regularization [8, 9]. In this review we focus only on the short-range interactions since this is where the new developments have been made. We use the terms leading order (LO), next-to-leading order (NLO), next-to-next-to-leading order (N^2LO), and next-to-next-to-next-to-leading order (N^3LO) for the successive orders in chiral EFT.

In Li et al. [10], we introduce a set of short-range interactions in lattice chiral EFT that decompose more naturally into spin channels. In the following we explain how to construct two-nucleon operators on the lattice with intrinsic spin S, S_z, orbital angular momentum L, L_z, and isospin I, I_z. Since orbital angular momentum is not exactly conserved on the lattice, we enforce the required orbital angular momentum projection by hand using smeared distributions of annihilation and creation operators whose angular dependence are given by spherical harmonics. The use of these smeared annihilation and creation operators has the effect of controlling the range of the interaction as well as reducing lattice artifacts. Since the annihilation and creation operators are no longer at the same point in space, the resulting interaction depends on the particle velocities and is therefore a non-local interaction. This is in contrast with local interactions, where the annhiliation and creation operators are at the same point in space. Previous lattice studies had only considered local interactions [8].

We will work with lattice units where quantities are multiplied by the corresponding power of the spatial lattice spacing a to make the object dimensionless. Let $\langle \mathbf{n}'\mathbf{n}\rangle$ denote the nearest neighbors $\mathbf{n}'$ of lattice site $\mathbf{n}$. Our starting point is $a_{i,j}(\mathbf{n})$, the nucleon annihilation operator for lattice site $\mathbf{n}$ with spin i and isospin j. We then add lattice operators on neighboring lattice sites with coefficient, $s_{\rm NL}$. This defines the smeared annihilation operator

$$a_{{\rm NL},i,j}(\mathbf{n}) = a_{i,j}(\mathbf{n}) + s_{\rm NL}\sum_{\langle \mathbf{n}'\mathbf{n}\rangle} a_{i,j}(\mathbf{n}'). \tag{1}$$

The overall normalization of the smeared annihilation operators is not important since any normalization factors can be absorbed into the definition of the interaction coefficients that we build with these smeared operators.

This process can easily be extended to lattice sites beyond the nearest neighbors of $\mathbf{n}$. After this we construct combinations with spin S, S_z, and isospin I, I_z,

$$\begin{aligned}&[a(\mathbf{n})a(\mathbf{n}')]^{s_{\rm NL}}_{S,S_z,I,I_z}\\ &= \sum_{i,j,i',j'} a_{{\rm NL},i,j}(\mathbf{n})M_{ii'}(S,S_z)M_{jj'}(I,I_z)a_{{\rm NL},i',j'}(\mathbf{n}').\end{aligned} \tag{2}$$

Let us define the lattice derivative ∇_l along the l spatial direction as

$$\nabla_l f(\mathbf{n}) = \frac{1}{2}f(\mathbf{n}+\hat{\mathbf{l}}) - \frac{1}{2}f(\mathbf{n}-\hat{\mathbf{l}}). \tag{3}$$

We can also define the lattice derivative $\nabla_{1/2,l}$ with half steps in the forward and backward directions,

$$\nabla_{1/2,l} f(\mathbf{n}) = f(\mathbf{n}+\frac{1}{2}\hat{\mathbf{l}}) - f(\mathbf{n}-\frac{1}{2}\hat{\mathbf{l}}). \tag{4}$$

This yields a well-defined function on the lattice sites when we take double derivatives,

$$\nabla^2_{1/2,l} f(\mathbf{n}) = f(\mathbf{n}+\hat{\mathbf{l}}) - 2f(\mathbf{n}) + f(\mathbf{n}-\hat{\mathbf{l}}). \tag{5}$$

The lattice Laplacian operator is defined as

$$\nabla^2_{1/2} f(\mathbf{n}) = \sum_l \nabla^2_{1/2,l} f(\mathbf{n}). \tag{6}$$

We select orbital angular momenta with the help of solid spherical harmonics,

$$R_{L,L_z}(\mathbf{r}) = \sqrt{\frac{4\pi}{2L+1}} r^L Y_{L,L_z}(\theta,\phi). \tag{7}$$

We work with polynomials of the lattice derivatives that act on one of the annihilation operators, with coefficients prescribed by the solid spherical harmonics,

$$P^{2M,s_{\rm NL}}_{S,S_z,L,L_z,I,I_z}(\mathbf{n}) = [a(\mathbf{n})\nabla^{2M}_{1/2}R^*_{L,L_z}(\nabla)a(\mathbf{n})]^{s_{\rm NL}}_{S,S_z,I,I_z}. \tag{8}$$

This ensures the lattice operators have the correct rotational properties in the continuum limit. We use Clebsch-Gordan coefficients to put together the required spin and orbital angular momentum combinations,

$$\begin{aligned}&O^{2M,s_{\rm NL}}_{S,L,J,J_z,I,I_z}(\mathbf{n})\\ &\quad= \sum_{S_z,L_z}\langle SS_zLL_z|JJ_z\rangle P^{2M,s_{\rm NL}}_{S,S_z,L,L_z,I,I_z}(\mathbf{n}).\end{aligned} \tag{9}$$

In Li et al. [10], we consider the neutron-proton system up to N^3LO. The chosen lattice spacings are 1.97, 1.64, 1.32, and 0.99 fm. In **Figure S1** we show the phase shifts and mixing angles for neutron-proton scattering vs. relative momenta for lattice spacing $a = 1.32$ fm. In **Figure S2** we show the phase shifts and mixing angles for neutron-proton scattering for lattice spacing $a = 0.99$ fm. The estimated uncertainties at NLO, N^2LO, and N^3LO are labeled with the blue, green, and red bands, respectively. The Nijmegen partial wave analysis results are shown with black solid lines, and the lattice results at N^3LO are shown with diamonds. These results show marked improvement over previous studies of the lattice chiral EFT interactions [8]. We see that in most cases the error bands at LO, NLO, N^2LO, and N^3LO are overlapping with each other and decreasing in width with each successive order. This indicates the consistency of the effective field theory expansion as well as its convergence rate. There are however some exceptions such as the high-momentum region of the $^3P_2$ and $^3D_2$ phase shifts where the convergence is not yet optimal. We have work in progress now that indicates these problems are resolved with a better lattice treatment of

the tensor force in the one-pion exchange potential and will be discussed in a forthcoming publication. As one would expect, we also see a general improvement of the lattice phase shifts as we decrease the lattice spacing. We see both better agreement with the empirical data as well as smaller error bands.

3. NUCLEAR PHYSICS NEAR A QUANTUM PHASE TRANSITION

In Elhatisari et al. [11], we investigate two different LO interactions in chiral EFT, which we call interactions A and B. Both are defined with lattice spacing $a = 1.97$ fm and have nearly the same nucleon-nucleon scattering phase shifts and binding energies for the three- and four-nucleon systems. However, for heavier nuclei the binding energies are very different. Interactions A and B each have the same one-pion exchange interaction as well as the same Coulomb potential between protons. The only distinction between the interactions A and B is the form of the short-range interactions, and we limit our discussion here to the short-range interactions.

We write σ_S with $S = 1, 2, 3$ for the spin Pauli matrices, and τ_I with $I = 1, 2, 3$ for the isospin Pauli matrices. We will write $a(\mathbf{n})$ for the column vector of annihilation operators $a_{i,j}(\mathbf{n})$, and we will write $a^\dagger(\mathbf{n})$ for the row vector of creation operators $a^\dagger_{i,j}(\mathbf{n})$. For our chosen non-local smearing parameter $s_{\rm NL}$, we construct the smeared operators as in Equation (1),

$$a_{\rm NL}(\mathbf{n}) = a(\mathbf{n}) + s_{\rm NL} \sum_{\langle \mathbf{n}'\mathbf{n}\rangle} a(\mathbf{n}'), \tag{10}$$

$$a^\dagger_{\rm NL}(\mathbf{n}) = a^\dagger(\mathbf{n}) + s_{\rm NL} \sum_{\langle \mathbf{n}'\mathbf{n}\rangle} a^\dagger(\mathbf{n}'). \tag{11}$$

The point-like density operators are defined as

$$\rho(\mathbf{n}) = a^\dagger(\mathbf{n})a(\mathbf{n}), \tag{12}$$

$$\rho_S(\mathbf{n}) = a^\dagger(\mathbf{n})[\sigma_S]a(\mathbf{n}), \tag{13}$$

$$\rho_I(\mathbf{n}) = a^\dagger(\mathbf{n})[\tau_I]a(\mathbf{n}), \tag{14}$$

$$\rho_{S,I}(\mathbf{n}) = a^\dagger(\mathbf{n})[\sigma_S \otimes \tau_I]a(\mathbf{n}). \tag{15}$$

The smeared non-local densities are defined as

$$\rho_{\rm NL}(\mathbf{n}) = a^\dagger_{\rm NL}(\mathbf{n})a_{\rm NL}(\mathbf{n}), \tag{16}$$

$$\rho_{S,{\rm NL}}(\mathbf{n}) = a^\dagger_{\rm NL}(\mathbf{n})[\sigma_S]a_{\rm NL}(\mathbf{n}), \tag{17}$$

$$\rho_{I,{\rm NL}}(\mathbf{n}) = a^\dagger_{\rm NL}(\mathbf{n})[\tau_I]a_{\rm NL}(\mathbf{n}), \tag{18}$$

$$\rho_{S,I,{\rm NL}}(\mathbf{n}) = a^\dagger_{\rm NL}(\mathbf{n})[\sigma_S \otimes \tau_I]a_{\rm NL}(\mathbf{n}), \tag{19}$$

while the smeared local densities for local smearing parameter $s_{\rm L}$ are

$$\rho_{\rm L}(\mathbf{n}) = a^\dagger(\mathbf{n})a(\mathbf{n}) + s_{\rm L} \sum_{\langle \mathbf{n}'\,\mathbf{n}\rangle} a^\dagger(\mathbf{n}')a(\mathbf{n}'), \tag{20}$$

$$\rho_{S,{\rm L}}(\mathbf{n}) = a^\dagger(\mathbf{n})[\sigma_S]a(\mathbf{n}) + s_{\rm L} \sum_{\langle \mathbf{n}'\,\mathbf{n}\rangle} a^\dagger(\mathbf{n}')[\sigma_S]a(\mathbf{n}'), \tag{21}$$

$$\rho_{I,{\rm L}}(\mathbf{n}) = a^\dagger(\mathbf{n})[\tau_I]a(\mathbf{n}) + s_{\rm L} \sum_{\langle \mathbf{n}'\,\mathbf{n}\rangle} a^\dagger(\mathbf{n}')[\tau_I]a(\mathbf{n}'), \tag{22}$$

$$\rho_{S,I,{\rm L}}(\mathbf{n}) = a^\dagger(\mathbf{n})[\sigma_S \otimes \tau_I]a(\mathbf{n}) + s_{\rm L} \sum_{\langle \mathbf{n}'\,\mathbf{n}\rangle} a^\dagger(\mathbf{n}')[\sigma_S \otimes \tau_I]a(\mathbf{n}'). \tag{23}$$

The non-local short-range interactions have the form

$$V_{\rm NL} = \frac{c_{\rm NL}}{2} \sum_{\mathbf{n}} :\rho_{\rm NL}(\mathbf{n})\rho_{\rm NL}(\mathbf{n}): + \frac{c_{I,{\rm NL}}}{2} \sum_{\mathbf{n},I} :\rho_{I,{\rm NL}}(\mathbf{n})\rho_{I,{\rm NL}}(\mathbf{n}):, \tag{24}$$

while the local short-range interactions are given by

$$\begin{aligned} V_{\rm L} =& \frac{c_{\rm L}}{2} \sum_{\mathbf{n}} :\rho_{\rm L}(\mathbf{n})\rho_{\rm L}(\mathbf{n}): + \frac{c_{S,{\rm L}}}{2} \sum_{\mathbf{n},S} :\rho_{S,{\rm L}}(\mathbf{n})\rho_{S,{\rm L}}(\mathbf{n}): \\ &+ \frac{c_{I,{\rm L}}}{2} \sum_{\mathbf{n},I} :\rho_{I,{\rm L}}(\mathbf{n})\rho_{I,{\rm L}}(\mathbf{n}): + \frac{c_{S,I,{\rm L}}}{2} \sum_{\mathbf{n},S,I} :\rho_{S,I,{\rm L}}(\mathbf{n})\rho_{S,I,{\rm L}}(\mathbf{n}):. \end{aligned} \tag{25}$$

The : : symbol indicates normal ordering where the annihilation operators are pushed to the right and the creation operators are pulled to the left.

For interaction A we take the short-range interaction to have the purely non-local form described in Equation (24). For interaction B we take the short-range interaction to have a combination of the non-local and local forms described in Equations (24), (25), respectively. In **Table 1**, we show results for the ground state energies of ^{8}Be, ^{12}C, ^{16}O, and ^{20}Ne. We present results for interactions A and B at LO, including Coulomb interactions, and the comparison with experimental data. While the ground state energies for B are close to the experimental data, the binding energies for A are not.

To identify the reason for the disagreement, we compute the ground state energies of ^{8}Be, ^{12}C, ^{16}O, and ^{20}Ne for interaction A with the Coulomb interactions turned off. When we turn off Coulomb and divide the ground state energy for each nucleus with that of ^{4}He, we find the ratios

$$\frac{E_{^8{\rm Be}}}{E_{^4{\rm He}}} = 1.997(6), \quad \frac{E_{^{12}{\rm C}}}{E_{^4{\rm He}}} = 3.00(1), \tag{26}$$

$$\frac{E_{^{16}{\rm O}}}{E_{^4{\rm He}}} = 4.00(2), \quad \frac{E_{^{20}{\rm Ne}}}{E_{^4{\rm He}}} = 5.03(3). \tag{27}$$

TABLE 1 | Results for the ground energies of ^{8}Be, ^{12}C, ^{16}O, and ^{20}Ne using interactions A and B at LO with Coulomb interactions and the comparison with experimental data.

Nucleus	A (LO + Coulomb)	B (LO + Coulomb)	Experiment
^{8}Be	−56.51(14)	−57.29(7)	−56.591
^{12}C	−84.0(3)	−89.9(5)	−92.162
^{16}O	−110.5(6)	−126.0(7)	−127.619
^{20}Ne	−137(1)	−164(1)	−160.645

These consecutive integer numbers are an indication of a Bose condensate of alpha particles filling the periodic box. In Elhatisari et al. [11] this interpretation is confirmed by showing that the alpha-alpha interactions are indeed very weak for interaction A.

These results shows that nature sits near a quantum phase transition. Along the $N = Z$ line where the number of neutrons equals the number of protons, there is a first-order quantum transition separating a Bose gas of alpha particles from a nuclear liquid. The strength of the alpha-alpha interactions determines whether one has a Bose gas of alpha particles or a nuclear liquid. In turn, the alpha-alpha interactions are impacted by the strength and range of the local part of the nucleon-nucleon interactions. The local part of interaction B is stronger than the local part of interaction A, and hence the different behaviors that we see.

The phase diagram of nuclear matter at zero temperature is in **Figure S3**. The parameter λ indicates the strength of the local part of the nucleon-nucleon interactions. This can be constructed explicitly by taking a linear combination of the interactions used for interactions A and B. Nuclear matter is in a Bose gas phase of alpha particles when λ is below the critical value, and nuclear matter is in a nuclear liquid phase if λ crosses above the critical value. When λ increases even further, finite A-body nuclei become stable as their energy fall below the threshold value $E_\alpha A/4$ associated with multi-alpha continuum states.

4. ESSENTIAL ELEMENTS FOR NUCLEAR BINDING

In Lu et al. [12], we investigate a basic question connecting nuclear forces and nuclear structure. What are the essential elements for nuclear binding? For this analysis we work with a simple leading order effective field theory without pions. This simple pionless EFT theory is SU(4)-invariant, where the SU(4) is Wigner's approximate symmetry where the four nucleon degrees of freedom can be rotated into each other [13]. The lattice Hamiltonian has the form

$$H_{\mathrm{SU(4)}} = H_{\mathrm{free}} + \frac{1}{2!}C_2 \sum_{\boldsymbol{n}} \tilde{\rho}^2(\mathbf{n}) + \frac{1}{3!}C_3 \sum_{\mathbf{n}} \tilde{\rho}^3(\mathbf{n}), \qquad (28)$$

where H_{free} is the free nucleon Hamiltonian and

$$\tilde{\rho}(\mathbf{n}) = \rho_{\mathrm{NL}}(\mathbf{n}) + s_{\mathrm{L}} \sum_{\langle \mathbf{n}'\mathbf{n} \rangle} \rho_{\mathrm{NL}}(\mathbf{n}), \qquad (29)$$

and $\rho_{\mathrm{NL}}(\mathbf{n})$ was defined in Equation (16). The local part of the interaction is adjusted using the parameter s_{L}, while the non-local part of the interaction is controlled by the parameter s_{NL}, which appears implicitly in the definition of $\rho_{\mathrm{NL}}(\mathbf{n})$. The parameter C_2 controls the strength of the two-nucleon interaction, and C_3 controls the three-nucleon interaction. For these calculations the lattice spacing is $a = 1.32$ fm.

The two-nucleon interaction coefficient C_2 and interaction range, controlled jointly by s_{NL} and s_{L}, are set by fitting the scattering length a_0 and effective range r_0 averaged over the two S-wave channels, $^1S_0$ and $^3S_1$. The three-nucleon coupling strength C_3 is set according to the binding energy of ^{3}H. At the empirical binding energy $B(^3\mathrm{H}) = 8.48$ MeV, the ^{4}He binding energy is 28.9 MeV, and this is near the empirical value $B(^4\mathrm{He}) = 28.3$ MeV. This fitting process is carried out for several pairs of values for s_{NL} and s_{L}. We calculate several nuclear ground states for each pair using auxiliary-field lattice Monte Carlo simulations. We find that $s_{\mathrm{NL}} = 0.5$ and $s_{\mathrm{L}} = 0.061$ gives the best agreement overall. We note that the rather large value of s_{L} as compared with s_{NL} is an indication that the local interaction plays an important role in nuclear binding. The corresponding couplings are $C_2 = -3.41 \times 10^{-7}$ MeV^{-2} and $C_3 = -1.4 \times 10^{-14}$ MeV^{-5}. Overall about 20% of the binding is coming from the three-body interaction. This is consistent with the expected hierarchy of forces in effective field theory, with three-body forces being less important than two-body forces.

We present binding energies and charge radii in **Table 2** for selected nuclei together with the experimental values and the computed Coulomb energy. Although the ^{3}H energy is a constraint in the fitting process, the other values are predictions. We have included the charge radius of the proton for calculations of the nuclear charge radii, but have not included smaller effects arising from the charge distribution of the neutron, relativistic corrections, and spin-orbit terms. In **Figure S4** we show the binding energies for 86 bound nuclei with up to $A = 48$ nucleons in comparison with empirical data. The Monte Carlo error bars are not visible as they are smaller than the symbol size. The errors associated with Euclidean time extrapolation and volume extrapolation are less than 1% relative error, and these errors are not shown. In **Figure S4** one can see that the agreement with empirical results is fairly good. The remaining discrepancies are a sign of missing effects such as interactions which are spin dependent.

In **Figure S5** we show the charge densities of ^{16}O and ^{40}Ca. These densities are calculated using the pinhole algorithm [16] where a barrier is placed in the middle of the Euclidean time evolution, and the amplitude vanishes unless each nucleon passes through a pinhole. As the number of pinholes are set to equal the number of nucleons, the sampling over pinholes yields a classical distribution of the nucleon positions. We show the comparison

TABLE 2 | *Left side:* Computed binding energies of several nuclei compared with experimental values.

	B	Exp.	Coulomb	B/Exp.	R_{ch}	Exp.	R_{ch}/Exp.
^{3}H	8.48 (2)(0)	8.48	0.0	1.00	1.90 (1)(1)	1.76	1.08
^{3}He	7.75 (2)(0)	7.72	0.73 (1)(0)	1.00	1.99 (1)(1)	1.97	1.01
^{4}He	28.89 (1)(1)	28.3	0.80 (1)(1)	1.02	1.72 (1)(3)	1.68	1.02
^{16}O	121.9 (1)(3)	127.6	13.9 (1)(2)	0.96	2.74 (1)(1)	2.70	1.01
^{20}Ne	161.6 (1)(1)	160.6	20.2 (1)(1)	1.01	2.95 (1)(1)	3.01	0.98
^{24}Mg	193.5 (02)(17)	198.3	28.0 (1)(2)	0.98	3.13 (1)(2)	3.06	1.02
^{28}Si	235.8 (04)(17)	236.5	37.1 (2)(3)	1.00	3.26 (1)(1)	3.12	1.04
^{40}Ca	346.8 (6)(5)	342.1	71.7 (4)(4)	1.01	3.42 (1)(3)	3.48	0.98

Right side: Computed charge radii of several nuclei compared with empirical values. The first parenthesis denotes the Monte Carlo error, and the second parenthesis is the time extrapolation error. All energies are in MeV, and all lengths are in fm. Experimental data is from Wang et al. [14]. Table from Lu et al. [12].

with electron scattering data, as well as the lattice results with the Coulomb interaction included via first order perturbation theory. The Coulomb force suppresses the densities near the nucleus center, bringing the results in better agreement with the experimental data. The results are surprisingly accurate in view of the simple nuclear interaction.

The ground state energy of pure neutron matter is shown in **Figure S6** as a function of the neutron density. We show the results of several other calculations for comparison. For the lattice results the number of neutrons ranges from 14 to 66. The calculations are done with three box sizes $L = 5, 6, 7$. Our lattice results, shown as the filled red polygons, are generally in good agreement with the other calculations for densities higher than 0.05 fm^{-3}. For lower densities the disagreement is larger due to the neutron-neutron scattering length being incorrect. The open red polygons correspond to an improved calculation where a contact interaction is included to give the correct neutron-neutron scattering length. There is also a correction included to restore invariance with respect to Galilean boosts [17]. In spite of the simplicity of the interaction, the results are quite good.

5. DISCUSSION

We have reviewed several recent results by the Nuclear Lattice EFT Collaboration. We presented the improved description of scattering phase shifts in chiral effective field theory up to N^{3}LO. We then showed evidence that nuclear physics is close to a quantum phase transition. After this we described the minimal nuclear interaction that can accurately reproduce the ground state energies of neutron matter, light nuclei, and medium-mass nuclei. A common theme flowing through all aspects of our review was the notable difference between local and non-local interactions. We now put some of these findings into context.

Numerous calculations show the reliability of chiral EFT for the properties of light nuclei [18–22]. However, the binding energies as well as the charge radii of medium mass nuclei are sometimes not well reproduced [20, 23–29]. One notable case is that the charge radius of ^{16}O, which is often too small [23, 25–28]. Without further input, chiral EFT calculations do not yet provide accurate predictions at higher nuclear densities.

One practical approach is to put constraints on the nuclear force by fitting the properties of medium mass nuclei and nuclear matter saturation [29]. This strategy has been used in several calculations [30–33]. The approach we have pursued in nuclear lattice studies is to focus on the microscopic origins of the problem. In Elhatisari et al. [11], we find that nuclear matter resides near a quantum phase transition that lies at the boundary between a Bose condensate of alpha particles and a nuclear liquid. There we show that local interactions are especially important for nuclear binding. In Lu et al. [12], we construct a simple nuclear interaction that can produce, with no more than a few percent error, the ground state energies of neutron matter, light nuclei, and medium-mass nuclei. From that analysis we see the importance of the range and locality of the SU(4)-invariant forces in determining the bulk properties of nuclei.

The dominance of SU(4)-invariant interactions can be explained in terms of coherent enhancement. Upon summation over nucleonic spin configurations, much of the effect of spin-dependent forces cancel. In a similar manner, isospin-dependent forces will cancel in symmetric nuclear matter due to the protons and neutrons being equal in number. One important exception though is the Coulomb interaction. SU(4) symmetry and large scattering length universality have a long history in nuclear physics. It is well-known that the Tjon line connecting ^{3}H and ^{4}He binding energies is a manifestation of universality in nuclear systems [9, 34–36].

SU(4)-symmetric short-range interactions are now being used with local and non-local smearing and one-pion exchange. These improved calculations of light and medium-mass nuclei will use chiral forces up to N^{3}LO. One of the central questions that we seek to address is why the straightforward application of chiral EFT does not give reliable and accurate predictions at higher nuclear densities. While more investigations are needed, it seems that part of the answer to this question is related to the emergence of a physical length scale relevant to many-body nuclear systems, the size of the alpha particle. The amount of fine tuning needed to reproduce the alpha-alpha and alpha-nucleon scattering phase shifts can be understood as the competition between Pauli repulsion and attractive nucleon-nucleon interactions. This important and delicate balance is amplified by the fact that the range of the nucleon-nucleon interaction is comparable to the size of the alpha particle. Hence small differences in the range or locality of the interaction can have a large impact on the interactions of the alpha particle. This is turn has consequences for nuclear systems with increasing numbers of nucleons.

DATA AVAILABILITY STATEMENT

The datasets analyzed in this article are not publicly available. Requests to access the datasets should be directed to leed@frib.msu.edu.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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SUPPLEMENTARY MATERIAL

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fphy.2020.00174/full#supplementary-material

Figure S1 | Results for the neutron-proton scattering phase shifts and mixing angles vs. the relative momenta for lattice spacing a = 1.32 fm. The blue, green and red bands show the estimated uncertainties at NLO, N^2LO and N^3LO respectively. The black solid line and diamonds denote phase shift or mixing angle from the Nijmegen partial wave analysis and lattice calculation at N^3LO, respectively. Figure from Li et al. [10].

Figure S2 | Results for the neutron-proton scattering phase shifts and mixing angles vs. the relative momenta for lattice spacing a = 0.99 fm. The blue, green and red bands show the estimated uncertainties at NLO, N^2LO and N^3LO respectively. The black solid line and diamonds denote phase shift or mixing angle from the Nijmegen partial wave analysis and lattice calculation at N^3LO, respectively. Figure from Li et al. [10].

Figure S3 | Phase diagram of nuclear matter at zero temperature. λ controls the strength of the local part of the interactions. Figure from Elhatisari et al. [11].

Figure S4 | Calculated binding energies from ^{3}H to ^{48}Ca. The solid symbols are lattice results and the open symbols are experimental values. The experimental values are from Wang et al. [14]. Figure adapted from Lu et al. [12].

Figure S5 | Computed ^{16}O and ^{40}Ca charge densities compared with experimental results. The circles denote the results without Coulomb interaction. The squares are the results with the Coulomb interaction included perturbatively. Experimental results are from De Vries et al. [15]. Figure adapted from Lu et al. [12].

Figure S6 | Ground state energy of pure neutron matter as a function of neutron density for box sizes L = 5 (upright triangles), L = 6 (squares), L = 7 (right-pointing triangles). The filled red polygons are results for the leading-order interaction. The open red polygons show an improved calculation with a short-range interaction tuned to the physical neutron-neutron scattering length as well as a correction to restore Galilean invariance. Also shown are results calculated with N^3LO chiral interactions (EM 500 MeV, EGM 450/500 MeV and EGM 450/700 MeV) [37], as well as results from variational (APR) [38] and auxiliary-field diffusion MC calculations (GCR) [39]. Figure adapted from Lu et al. [12].

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