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Fuzzy Synchronization Control for Fractional-Order Chaotic Systems With Different Structures

Jin Xu[1], *Ning Li*[1], *Xiulan Zhang*[2]* *and Xiaoli Qin*[3]

[1] Department of Applied Mathematics, Huainan Normal University, Huainan, China, [2] School of Mathematics and Physics, Guangxi University for Nationalities, Nanning, China, [3] Department of Mathematics, Deyang High School, Chengdu Normal University, Deyang, China

This paper discusses the synchronization problem for a class of unknown fractional-order chaotic systems (FOCSs) with indeterminate external disturbances and non-symmetrical control gain. A paralleled adaptive fuzzy synchronization controller is constructed in which indeterminate non-linear functions are approximated by the fuzzy logic systems depending on fractional-order Lyapunov stability criteria and the fractional-order parameter adaptive law is designed to regulate corresponding parameters of the fuzzy systems. The proposed method guarantees the boundedness of all of the signals in the closed-loop system, and at the same time, it ensures the convergence of the synchronization error between the master and slave FOCSs. Finally, the feasibility is demonstrated by simulation studies.

Keywords: adaptive fuzzy control, fractional-order chaotic system, fractional-order adaptation, chaos synchronization, fuzzy logic system

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Anhui University, China

Correspondence:
Xiulan Zhang
xlzhang@gxun.edu.cn

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1. INTRODUCTION

The fractional calculus appeared in the same era as the classical integer-order calculus, but due to the facts that the fractional-order calculus lacks actual background and its theory is complex, the fractional calculus has rarely been investigated by scholars. Recently, it has been shown that fractional calculus not only provides new mathematical methods for practical systems but also is especially well-suited for describing some dynamical behaviors of physical systems [1–5]. Consequently, the fractional-order calculus has been employed to describe phonology and thermal systems, signal processing and system identification [6, 7], control and robotics [8–11], and many other real-world systems. Since the fractional-order calculus has memory ability, in the description of complex dynamic systems, a model built depending on fractional-order calculus is more accurate than an integer-order one. The study for the fractional-order chaotic system (FOCS) has thus slowly become a hot research topic.

It is well-known that chaotic systems (integer-order or fractional-order) are sensitive to initial state values, i.e., the stability of systems will change obviously with small changes in initial values; thus, the synchronization control of FOCSs is challenging work. Some methods, such as PD control [12], PID control [13–15], adaptive fuzzy backstepping control [16–20], sliding mode control [21–25], and Lyapunov direct [26–28] and adaptive neural network control [29–32] have been used to control or synchronize fractional chaotic systems. Chen et al. [21] investigated the adaptive synchronization of FOCSs, where different structures of the master and slave FOCSs and the existence of external disturbances are ignored. In Wang et al. [33], the synchronization of FOCSs accompanied by external disturbances was studied. To handle the unmatched disturbances, in

He et al. [30], a robust synchronization method with non-linear input was proposed, but its control cost was very high. It should be mentioned that, in the above literature, the stability analysis of the synchronization of FOCSs still uses the ideal of linear systems. Generally speaking, the synchronization of FOCS systems with some unknown factors and external disturbances needs to be further researched.

Motivated by the above discussion, this paper aims to design a synchronization controller for a master and slave fractional-order chaotic system (FOCS) subject to different structures and external disturbances. The control gain matrix is assumed to be unknown. Fuzzy logic systems are used to approximate the unknown non-linear functions. Fractional-order parameter adaptive laws are designed to update the fuzzy parameters. The main contributions of this work are summarized as follows. (1) The non-symmetrical control gain matrix and external disturbances in FOCSs are considered. Besides, unlike some prior works, such as Liu et al. [16] and Pan et al. [31], the sequence-leading minor in the control gain matrix is not assumed to be zero. (2) Based on the Lyapunov stability theorem, fractional-order fuzzy parameter adaptive laws are designed.

2. PRELIMINARIES

The ν-th fractional-order integral is defined as:

$$ {}_0^C D_k^{-\nu} f(k) = \frac{1}{\Gamma(\nu)} \int_0^k (k-\tau)^{\nu-1} f(\tau) d\tau, \tag{1} $$

where $\Gamma(\cdot)$ function can be defined as

$$ \Gamma(z) = \int_0^\infty k^{z-1} e^{-k} dk. \tag{2} $$

The ν-th Caputo's derivative can be defined as:

$$ {}_0^C D_k^{\nu} f(k) = \frac{1}{\Gamma(n-\nu)} \int_0^k (k-\tau)^{n-\nu-1} f^{(n)}(\tau) d\tau, \tag{3} $$

clearly, where n is an integer satisfying $n-1 \leq \nu < n$.

The Laplace transform of Caputo's fractional-order derivative (3) can be expressed by Li et al. [2]

$$ \begin{aligned} \mathscr{L}({}_0^C D_k^{\nu} f(k)) &= \int_0^\infty e^{-sk} {}_0^C D_k^{\nu} f(k) dk \\ &= s^\nu F(s) - \sum_{t=0}^{n-1} s^{\nu-t-1} f^{(t)}(0). \end{aligned} \tag{4} $$

When $0 < \nu < 1$, $\mathscr{L}({}_0^C D_k^{\nu} f(k)) = s^\nu F(s) - s^{\nu-1} f(0)$.

For simplicity, we suppose that $\nu \in (0,1)$ in the rest of this paper. The following conclusions will be given in advance.

Definition 1. Pudlubny [3] The Mittag-Leffler function can be given by

$$ E_{\nu,\xi}(z) = \sum_{t=0}^{\infty} \frac{z^t}{\Gamma(\nu t + \xi)}, \tag{5} $$

where $\nu, \xi > 0$, and $z \in C$, the Laplace transform of which is

$$ \mathscr{L}\{k^{\xi-1} E_{\nu,\xi}(-bk^\nu)\} = \frac{s^{\nu-\xi}}{s^\nu + b}. \tag{6} $$

Lemma 1. Pudlubny [3] If $m(k) \in C^1[0,T](T > 0)$ (the symbol C^1 means that a function has a continuous derivative), the following equation satisfies:

$$ {}_0^C D_k^{-\nu} {}_0^C D_k^{\nu} m(k) = m(k) - m(0), \tag{7} $$

$$ {}_0^C D_k^{\nu} {}_0^C D_k^{-\nu} m(k) = m(k). \tag{8} $$

Lemma 2. (Lyapunov's second fractional-order method [34]) Suppose that $\boldsymbol{e}(k) = 0$ is an equilibrium point of the following FOCS:

$$ {}_0^C D_k^{\nu} \boldsymbol{e}(k) = \boldsymbol{h}(k, \boldsymbol{e}(k)), \tag{9} $$

where $\boldsymbol{e}(k) \in R^n$ is a system variable, and $\boldsymbol{h}(\boldsymbol{e}(k)) \in R^n$ is a non-linear function that has a Lipschitz local condition. If there exists a Lyapunov function $V(k, \boldsymbol{e}(k))$ and positive parameters a_1, a_2, a_3 such that

$$ a_1 ||\boldsymbol{e}(k)|| \leq V(k, \boldsymbol{e}(k)) \leq a_2 ||\boldsymbol{e}(k)||, \tag{10} $$

$$ {}_0^C D_k^{\nu} V(k, \boldsymbol{e}(k)) \leq -a_3 ||\boldsymbol{e}(k)||, \tag{11} $$

then system (9) is asymptotically stable.

Lemma 3. Aguila-Camacho et al. [35] Suppose that $\boldsymbol{e}(k) \in R^n$ is a continuous and derivable function, then

$$ \frac{1}{2} {}_0^C D_k^{\nu} \boldsymbol{e}^T(k) \boldsymbol{e}(k) \leq \boldsymbol{e}^T(k) {}_0^C D_k^{\nu} \boldsymbol{e}(k). \tag{12} $$

Lemma 4. Costa et al. [36] and Liu et al. [37] Let matrix $\boldsymbol{G} \in R^{n\times n}$ be the non-zero sequence-leading minor, then $\boldsymbol{G}$ can be factorized as $\boldsymbol{G} = \boldsymbol{G}_1 \boldsymbol{A}_g \boldsymbol{T}_g$, where $\boldsymbol{G}_1 \in R^{n\times n}$ is a positive matrix, $\boldsymbol{A}_g \in R^{n\times n}$ is a diagonal matrix whose diagonal line is $+1$ or -1 (signal of each of its elements is determined by corresponding the sequence-leading minor signal of $\boldsymbol{G}$), and $\boldsymbol{T}_g \in R^{n\times n}$ is a upper triangular matrix.

3. PROBLEM DESCRIPTION

3.1. System Dynamics

Suppose that the slave and respond FOCSs are separately defined as

$$ {}_0^C D_k^{\nu} \boldsymbol{x}(k) = \boldsymbol{h}(\boldsymbol{x}(k)), \tag{13} $$

$$ {}_0^C D_k^{\nu} \boldsymbol{y}(k) = \boldsymbol{p}(\boldsymbol{y}(k)) + \boldsymbol{G}\boldsymbol{u}(k) + \boldsymbol{D}(k), \tag{14} $$

where $\boldsymbol{x}(k) = [x_1(k), x_2(k), \cdots, x_n(k)]^T \in R^n$ and $\boldsymbol{y}(k) = [y_1(k), y_2(k), \cdots, y_n(k)]^T \in R^n$ are separately system measurable state variables of the drive system and respond system, $\boldsymbol{h}, \boldsymbol{p} : R^n \rightarrow R^n$ are uncertain non-linear continuous functions, $\boldsymbol{G} \in R^{n\times n}$ is an unknown constant matrix, $\boldsymbol{D}(k) \in R^{n\times n}$ is an indeterminate external disturbance, and $\boldsymbol{u}(k) \in R^n$ is the control input.

3.2. Introduction of a Fuzzy System

A fuzzy logic system includes the knowledge base, fuzzier, fuzzy inference engine based on the fuzzy rules and defuzzier [38–41]. The form of the j-th fuzzy rule is

$\mathcal{R}^{(j)}$: If x_1 is E_1^j, x_2 is E_2^j, $\cdots$, x_n is E_n^j, then $\hat{h}(\boldsymbol{x}(k))$ is C^j $(j=1,2,\cdots,N)$,
where $\boldsymbol{x}(k)=[x_1(k),x_2(k),\cdots,x_n(k)]^T\in\mathbb{R}^n$ and $\hat{h}(\boldsymbol{x}(k))\in\mathbb{R}$ are respectively the input and the output of fuzzy logic systems. The output is

$$\hat{h}(\boldsymbol{x}(k))=\frac{\sum_{j=1}^{N}\theta_j(k)\left[\prod_{i=1}^{n}\mu_{E_i^j}(x_i(k))\right]}{\sum_{j=1}^{N}\left[\prod_{i=1}^{n}\mu_{E_i^j}(x_i(k))\right]}, \tag{15}$$

where $\theta_j(k)$ is a value where the fuzzy membership function μ_{C^j} is maximum. Generally, we can consider that $\mu_{C^j}(\theta_j(k))=1$, and the fuzzy basic function is $\varphi_j(\boldsymbol{x}(k))=\frac{\prod_{i=1}^{n}\mu_{E_i^j}(x_i(k))}{\sum_{j=1}^{N}\left[\prod_{i=1}^{n}\mu_{E_i^j}(x_i(k))\right]}$. Let $\boldsymbol{\varphi}(\boldsymbol{x}(k))=[\varphi_1(\boldsymbol{x}(k)),\varphi_2(\boldsymbol{x}(k)),\cdots,\varphi_N(\boldsymbol{x}(k))]^T$, $\boldsymbol{\theta}(k)=[\theta_1(k),\theta_2(k),\cdots,\theta_N(k)]^T$, then the output of fuzzy logic systems is written as

$$\hat{h}(\boldsymbol{x}(k))=\boldsymbol{\theta}^T(k)\boldsymbol{\varphi}(\boldsymbol{x}(k)). \tag{16}$$

Theorem 1. Suppose that $p(\boldsymbol{x})$ is a continuous function defined on compact set Ω. For any constants $\varepsilon>0$, there exists a fuzzy logic system approximating function $\hat{h}(\boldsymbol{x})$ forming (16) such that

$$\sup_{\Omega}|p(\boldsymbol{x})-\hat{\boldsymbol{\theta}}^T\boldsymbol{\varphi}(\boldsymbol{x})|\le\varepsilon, \tag{17}$$

where $\hat{\boldsymbol{\theta}}$ is an estimator of optimal vector $\boldsymbol{\theta}^*$.

3.3. Control Objective

The synchronization error can be defined as $\boldsymbol{e}(k)=\boldsymbol{y}(k)-\boldsymbol{x}(k)$. Our control objective is to design an adaptive controller such that the synchronization error tends to zero asymptotically (i.e., $\lim_{k\to\infty}||\boldsymbol{e}(k)||=0$).

4. CONTROLLER DESIGN AND STABILITY ANALYSIS

Assumption 1. The control gain matric $\boldsymbol{G}$ has a non-zero sequence-leading minor whose signal is known.

Remark 1. *Assumption* 1 is not strict. In fact, the gain matrix of some actual systems (such as a visual servo and vehicle thermal management system [42]) is non-symmetrical. According to Lemma 4, one can factorize $\boldsymbol{G}$ as $\boldsymbol{G}=\boldsymbol{G}_1\boldsymbol{AT}$, where $\boldsymbol{G}_1$ is an unknown positive definite matrix, $\boldsymbol{A}$ is a known matrix whose diagonal line is +1 or -1, $\boldsymbol{AA}=\boldsymbol{I}_{n\times n}$ ($\boldsymbol{I}_{n\times n}$ is a n-order unitary matrix), and $\boldsymbol{T}$ is an uncertain upper triangle matrix.

Assumption 2. *The product of the external disturbance $\boldsymbol{D}(k)$ and the positive definite matrix $\boldsymbol{G}_1^{-1}$ is a function that is bounded, i.e., there exists an uncertain constant $M_i>0$ so that*

$$|(\boldsymbol{G}_1^{-1}\boldsymbol{D}(k))_i|\le M_i\ \ (\forall k>0). \tag{18}$$

Remark 2. *Assumption 2 is not restrictive, and it is used in some similar literature, for example, in* Liu et al. [9], Rahmani et al. [10], and Ferdaus et al. [11]. *In fact, most commonly used disturbances satisfy Assumption 2.*

The dynamic equation of synchronization error is expressed as

$$\begin{aligned}{}_0^CD_k^{\nu}\boldsymbol{e}(k)&={}_0^CD_k^{\nu}\left(\boldsymbol{y}(k)-\boldsymbol{x}(k)\right)\\&={}_0^CD_k^{\nu}\boldsymbol{y}(k)-{}_0^CD_k^{\nu}\boldsymbol{x}(k)\\&=\boldsymbol{p}(\boldsymbol{y}(k))-\boldsymbol{h}(\boldsymbol{x}(k))+\boldsymbol{Gu}(k)+\boldsymbol{D}(k)\\&=\boldsymbol{p}(\boldsymbol{y}(k))-\boldsymbol{h}(\boldsymbol{x}(k))+\boldsymbol{G}_1\boldsymbol{ATu}(k)+\boldsymbol{D}(k).\end{aligned} \tag{19}$$

Let $\boldsymbol{Q}=\boldsymbol{G}_1^{-1}$, then

$$\begin{aligned}\boldsymbol{Q}{}_0^CD_k^{\nu}\boldsymbol{e}(k)=&\boldsymbol{Qp}(\boldsymbol{y}(k))-\boldsymbol{Qh}(\boldsymbol{x}(k))+(\boldsymbol{AT}-\boldsymbol{A})\boldsymbol{u}(k)\\&+\boldsymbol{Au}(k)+\boldsymbol{QD}(k).\end{aligned} \tag{20}$$

Denote $\boldsymbol{\gamma}(\boldsymbol{z}(k))=\boldsymbol{\gamma}(\boldsymbol{x}(k),\boldsymbol{y}(k),\boldsymbol{u}(k))=\boldsymbol{Qp}(\boldsymbol{y}(k))-\boldsymbol{Qh}(\boldsymbol{x}(k))+(\boldsymbol{AT}-\boldsymbol{A})\boldsymbol{u}(k)=[\gamma_1(\boldsymbol{z}(k)),\ \gamma_2(\boldsymbol{z}(k)),\cdots,\gamma_n(\boldsymbol{z}(k))]^T$ as an indeterminate non-linear function; then, Equation (20) is expressed as

$$\boldsymbol{Q}{}_0^CD_k^{\mu}\boldsymbol{e}(k)=\boldsymbol{\gamma}(\boldsymbol{z}(k))+\boldsymbol{Au}(k)+\boldsymbol{QD}(k). \tag{21}$$

The indeterminate function $\boldsymbol{\gamma}(\boldsymbol{z}(k))$ can be approximated by the fuzzy logic system (16) as

$$\hat{\gamma}_i(\theta_i(k),\boldsymbol{z}(k))=\theta_i(k)^T\varphi_i(\boldsymbol{z}(k)), i=1,2,\cdots,n. \tag{22}$$

Assume that the errors of the optimal parameter and the optimal estimated errors be respectively

$$\tilde{\theta}_i(k)=\theta_i(k)-\theta_i^*, \tag{23}$$

$$\varepsilon_i(\boldsymbol{z}(k))=\gamma_i(\boldsymbol{z}(k))-\hat{\gamma}_i(\theta_i^*,\boldsymbol{z}(k)). \tag{24}$$

From Boulkroune et al. [42] and Tong et al. [43] and Theorem 1, we assume that $|\varepsilon_i(\boldsymbol{z}(k))|\le\varepsilon_i^*$ ($\varepsilon_i^*>0$ is an uncertain constant). If we denote $\boldsymbol{\varepsilon}(\boldsymbol{z}(k))=[\varepsilon_1(\boldsymbol{z}(k)),\varepsilon_2(\boldsymbol{z}(k)),\cdots,\varepsilon_n(\boldsymbol{z}(k))]^T$ and $\boldsymbol{\varepsilon}^*=[\varepsilon_1^*,\varepsilon_2^*,\cdots,\varepsilon_n^*]^T$, the estimated error of the indeterminate non-linear function can be written as

$$\begin{aligned}\boldsymbol{\gamma}(\boldsymbol{z}(k))-\hat{\boldsymbol{\gamma}}(\boldsymbol{\theta}(k),\boldsymbol{z}(k))&=\boldsymbol{\gamma}(\boldsymbol{z}(k))-\hat{\boldsymbol{\gamma}}(\boldsymbol{\theta}^*,\boldsymbol{z}(k))+\hat{\boldsymbol{\gamma}}(\boldsymbol{\theta}^*,\boldsymbol{z}(k))\\&\quad-\hat{\boldsymbol{\gamma}}(\boldsymbol{\theta}(k),\boldsymbol{z}(k))\\&=\boldsymbol{\varepsilon}(\boldsymbol{z}(k))+\hat{\boldsymbol{\gamma}}(\boldsymbol{\theta}^*,\boldsymbol{z}(k))-\hat{\boldsymbol{\gamma}}(\boldsymbol{\theta}(k),\boldsymbol{z}(k))\\&=\boldsymbol{\varepsilon}(\boldsymbol{z}(k))-(\boldsymbol{\theta}(k)-\boldsymbol{\theta}^*)^T\boldsymbol{\varphi}(\boldsymbol{z}(k))\\&=\boldsymbol{\varepsilon}(\boldsymbol{z}(k))-\tilde{\boldsymbol{\theta}}(k)^T\boldsymbol{\varphi}(\boldsymbol{z}(k)).\end{aligned} \tag{25}$$

From the above discussion, the controller $\boldsymbol{u}(k)$ can be designed as

$$\boldsymbol{u}(k)=-\boldsymbol{A}\left[\boldsymbol{L}\boldsymbol{e}(k)+\boldsymbol{\theta}(k)^T\boldsymbol{\varphi}(\boldsymbol{z}(k))+\boldsymbol{H}\mathrm{sign}(\boldsymbol{e}(\mathrm{k}))+\hat{\boldsymbol{M}}\mathrm{sign}(\boldsymbol{e}(\mathrm{k}))\right], \tag{26}$$

where $\boldsymbol{L}=\mathrm{diag}[l_1,l_2,\cdots,l_n]\in R^{n\times n}, l_1,l_2,\cdots,l_n>0$ are parameters that need to be designed, $\boldsymbol{H}=\mathrm{diag}[\hat{\varepsilon}_1^*(k), \hat{\varepsilon}_2^*(k),\cdots,\hat{\varepsilon}_n^*(k)]$, $\hat{\varepsilon}_i^*(k)(i=1,2,\cdots,n)$ are estimated values of uncertain constants ε_i^*, and $\hat{\boldsymbol{M}}=\mathrm{diag}[\hat{M}_1(k),\hat{M}_2(k),\cdots,\hat{M}_n(k)]$, $\hat{M}_i(k)(i=1,2,\cdots,n)$ are estimated values of unknown constants M_i. For the sake of achieving the synchronization objective, this paper designs the following fractional-order parameter adaptive laws:

$$ {}_0^CD_k^\nu\theta_i(k)=\lambda_ie_i(k)\varphi_i(\boldsymbol{z}(k)), \tag{27}$$

$$ {}_0^CD_k^\nu\hat{\varepsilon}_i^*(k)=\xi_i|e_i(k)|, \tag{28}$$

$$ {}_0^CD_k^\nu\hat{M}_i(k)=\mu_i|e_i(k)|, \tag{29}$$

where $\lambda_i,\xi_i,\mu_i>0, i=1,2,\cdots,n$ are designed parameters.

To facilitate the coming stability analysis, let us display some results in advance.

Lemma 5. Suppose that ${}_0^CD_k^\nu e(k)\le 0$, then we have $e(k)\le e(0)$ on $[0,+\infty)$. On the contrary, ${}_0^CD_k^\nu e(k)\ge 0$ implies that $e(k)\ge e(0)$ on $[0,+\infty)$.

Proof. We only verify the first condition (the second condition is the same). Because ${}_0^CD_k^\nu e(k)\le 0$, there exists a non-negative function $h(k)=-[{}_0^CD_k^\nu e](k)$ satisfying

$$ {}_0^CD_k^\nu e(k)+h(k)=0. \tag{31}$$

Taking the Laplace transform on both sides of equation (31), we get $s^\nu E(s)-s^{\nu-1}e(0)+F(s)=0$, where $E(s)$ and $F(s)$ are separately the Laplace transform of $e(k)$ and $h(k)$. It is simplified to

$$E(s)=\frac{e(0)}{s}-\frac{F(s)}{s^\nu}. \tag{32}$$

Taking the inverse Laplace transform on both sides of equation (32), we obtain

$$e(k)=e(0)-[D^{-\nu}h](k). \tag{33}$$

By the fractional-order integral (1), we have $[D^{-\nu}h](k)\ge 0$. Further, we have $e(k)\le e(0)$ on $[0,+\infty)$.

Remark 3. Lemma 5 shows the difference between a fractional-order derivative and an integer-order derivative, but it cannot be described as: if ${}_0^CD_k^\nu e(k)\le 0$, then $e(k)$ is monotonically decreasing on the interval $[0,+\infty)$; if ${}_0^CD_k^\nu e(k)\ge 0$, then $e(k)$ is monotonically increasing on the interval $[0,+\infty)$. To explain this, a counterexample is given as follows.

Example 1. Consider that $x(0)\ge 0$ is an initial value of the differential equation: ${}_0^CD_k^\nu x(k)=h(k,x)=\mu k^{\mu-1}$, where $0<\mu<1, 0<\nu<1$, and $k>0$. Obviously, $h(k,x)\ge 0$, and the solution of the differential equation is $x(k)=x(0)+\frac{\mu\Gamma(\mu)k^{\mu-1+\nu}}{\Gamma(\mu+\nu)}$. It is clear that $\lim_{k\to+\infty}x(k)=x(0)$ when $0<\mu\le 1-\nu$. Therefore, $x(k)$ is not monotonically increasing, defined on $k\in[0,+\infty)$.

Lemma 6. Suppose that $\boldsymbol{e}(k)\in R^n$ be a continuous one-order derivative, then

$$\frac{1}{2}{}_0^CD_k^\nu\boldsymbol{e}^T(k)\boldsymbol{Q}\boldsymbol{e}(k)\le\boldsymbol{e}^T(k)\boldsymbol{Q}{}_0^CD_k^\nu\boldsymbol{e}(k), \tag{34}$$

where $\boldsymbol{Q}$ is an arbitrary n-order positive definite matrix.

Proof. Since $\boldsymbol{Q}$ is a positive definite matrix, there exists an n-order non-singular symmetric matrix $\boldsymbol{B}=\boldsymbol{B}^T$ so that $\boldsymbol{Q}=\boldsymbol{B}^T\boldsymbol{B}$. From Lemmas 1, 2, and 3, we obtain

$$\begin{aligned}\frac{1}{2}{}_0^CD_k^\nu\boldsymbol{e}^T(k)\boldsymbol{Q}\boldsymbol{e}(k)&=\frac{1}{2}{}_0^CD_k^\nu\boldsymbol{e}^T(k)\boldsymbol{B}^T\boldsymbol{B}\boldsymbol{e}(k)\\&=\frac{1}{2}{}_0^CD_k^\nu\left(\boldsymbol{B}\boldsymbol{e}(k)\right)^T\boldsymbol{B}\boldsymbol{e}(k)\\&\le\left(\boldsymbol{B}\boldsymbol{e}(k)\right)^T{}_0^CD_k^\nu\boldsymbol{B}\boldsymbol{e}(k)\\&=\left(\boldsymbol{B}\boldsymbol{e}(k)\right)^T\boldsymbol{B}{}_0^CD_k^\nu\boldsymbol{e}(k)\\&=\boldsymbol{e}^T(k)\boldsymbol{Q}{}_0^CD_k^\nu\boldsymbol{e}(k).\end{aligned} \tag{35}$$

Lemma 7. Suppose that $V(k)=\frac{1}{2}\boldsymbol{x}^T(k)\boldsymbol{x}(k)+\frac{1}{2}\boldsymbol{y}^T(k)\boldsymbol{y}(k)$, where $\boldsymbol{x}(k)$ and $\boldsymbol{y}(k)\in R^n$ are continuous one-order derivatives. If there exists a constant $q>0$ satisfying the following inequality

$$ {}_0^CD_k^\nu V(k)\le -q\boldsymbol{x}^T(k)\boldsymbol{x}(k), \tag{36}$$

then $||\boldsymbol{x}(k)||$ and $||\boldsymbol{y}(k)||$ are both bounded, and $\boldsymbol{x}(k)$ tends to zero asymptotically, where $||\cdot||$ represents the Euclidian norm.

Proof. According to inequality (36), the following inequality holds:

$$ {}_0^CD_k^\nu V(k)\le -q\boldsymbol{x}^T(k)\boldsymbol{x}(k)\le 0. \tag{37}$$

From Lemma 5, we know that $V(k)\le V(0)(\forall t\ge 0)$ when $V(k)$ defines on $[0,\infty)$. So, $||\boldsymbol{x}(k)||\le\sqrt{2V(k)}\le\sqrt{2V(0)}$ and $||\boldsymbol{y}(k)||\le\sqrt{2V(k)}\le\sqrt{2V(0)}$. Thereby, $||\boldsymbol{x}(k)||$ and $||\boldsymbol{y}(k)||$ are bounded.

Taking the ν-th integral on both sides of inequality ${}_0^CD_k^\nu V(k)\le -q\boldsymbol{x}^T(k)\boldsymbol{x}(k)$, we have

$$V(k)-V(0)\le -q{}_0^CD_k^{-\nu}\boldsymbol{x}^T(k)\boldsymbol{x}(k). \tag{38}$$

From the structure of $V(k)$, we have $\boldsymbol{x}^T(k)\boldsymbol{x}(k)\le 2V(k)$, and furthermore,

$$\boldsymbol{x}^T(k)\boldsymbol{x}(k)\le 2V(0)-2q{}_0^CD_k^{-\nu}\boldsymbol{x}^T(k)\boldsymbol{x}(k). \tag{39}$$

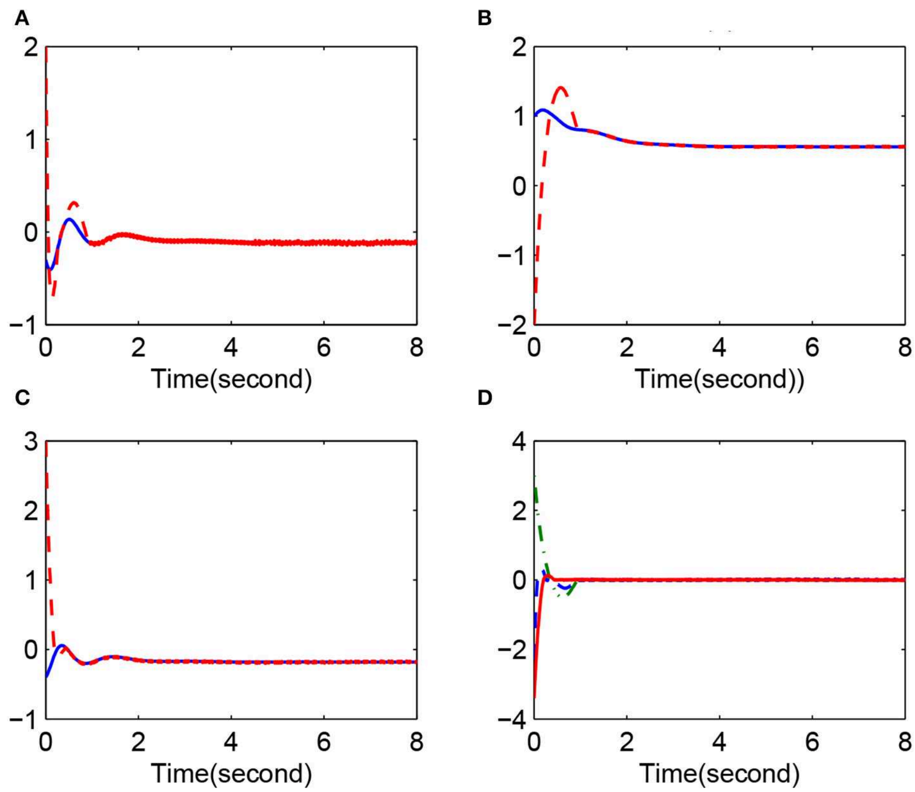

FIGURE 1 | Synchronization results of Case 1.

It follows from (39) that there exists a non-negative function $M(k)$ such that

$$\boldsymbol{x}^T(k)\boldsymbol{x}(k) + M(k) = 2V(0) - 2q_0^C D_k^{-\nu}\boldsymbol{x}^T(k)\boldsymbol{x}(k). \tag{40}$$

Taking the Laplace transform on (40), we obtain:

$$\boldsymbol{X}^T(s)\boldsymbol{X}(s) = 2V(0)\frac{s^{\nu-1}}{s^\nu + 2q} - \frac{s^\nu}{s^\nu + 2b}\boldsymbol{M}(s), \tag{41}$$

where $\boldsymbol{X}(s)$ and $\boldsymbol{M}(s)$ are respectively the Laplace transforms of $\boldsymbol{x}(k)$ and $M(k)$. Taking the inverse Laplace transform on both sides of equation (41), the solution is

$$\boldsymbol{x}^T(k)\boldsymbol{x}(k) = 2V(0)E_{\nu,1}(-2qk^\nu) - M(k) * \left[k^{-1}E_{\nu,0}(-2qk^\nu)\right], \tag{42}$$

where * is the convolution. Since k^{-1} and $E_{\nu,0}(-2qk^\nu)$ are both non-negative functions, $\boldsymbol{x}^T(k)\boldsymbol{x}(k) \le 2V(0)E_{\nu,1}(-2qk^\nu)$. From Li et al. [2], we know that $\boldsymbol{x}(k)$ is M-L stable and $\boldsymbol{x}(k)$ tends to zero asymptotically, i.e., $\lim_{k\to\infty} ||\boldsymbol{x}(k)|| = 0$.

Lemma 8. Suppose that $V_0(k) = \frac{1}{2}\boldsymbol{z}^T(k)\boldsymbol{Q}_1\boldsymbol{z}(k) + \frac{1}{2}\boldsymbol{d}^T(k)\boldsymbol{Q}_2\boldsymbol{d}(k)$, where $\boldsymbol{z}(k), \boldsymbol{d}(k) \in R^n$ and $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in R^{n\times n}$ are both positive definite matrixes. If there exists a positive definite matrix $\boldsymbol{Q}_3$ and a constant $q_0 > 0$ satisfying

$${}_0^C D_k^\nu V_0(k) \le -q_0\boldsymbol{z}^T(k)\boldsymbol{Q}_3\boldsymbol{z}(k), \tag{43}$$

then $||\boldsymbol{z}(k)||$ and $||\boldsymbol{d}(k)||$ are bounded, and $\boldsymbol{z}(k)$ tends to zero asymptotically (i.e., $\lim_{k\to\infty} ||\boldsymbol{z}(k)|| = 0$).

The main results of the paper are given as follows.

Theorem 2. Under Assumption 1 and Assumption 2, the synchronization between the drive system (13) and the respond system (14) can be achieved on the work of the adaptive fuzzy controller (26) and fractional-order adaptive laws (27), (28), and (29). In addition, all the signals of the closed-loop system are bounded.

Proof. Since $\boldsymbol{A} = \boldsymbol{I}_{n\times n}$, substituting the controller (26) into the error dynamic equation (21) gives

$$\begin{aligned}\boldsymbol{Q}_0^C D_k^\nu \boldsymbol{e}(k) = &-\boldsymbol{L}\boldsymbol{e}(k) + \boldsymbol{\gamma}(\boldsymbol{z}(k)) - \boldsymbol{\theta}^T(k)\boldsymbol{\varphi}(\boldsymbol{z}(k)) \\ &- \boldsymbol{H}\mathrm{sign}(\boldsymbol{e}(k)) - \hat{\boldsymbol{M}}\mathrm{sign}(\boldsymbol{e}(k)) + \boldsymbol{Q}\boldsymbol{D}(k).\end{aligned} \tag{44}$$

It is simplified as

$$\begin{aligned}\boldsymbol{Q}_0^C D_k^\nu \boldsymbol{e}(k) = &-\boldsymbol{L}\boldsymbol{e}(k) + \boldsymbol{\varepsilon}(\boldsymbol{z}(k)) - \tilde{\boldsymbol{\theta}}(k)^T\boldsymbol{\varphi}(\boldsymbol{z}(k)) \\ &- \boldsymbol{H}\mathrm{sign}(\boldsymbol{e}(k)) - \hat{\boldsymbol{M}}\mathrm{sign}(\boldsymbol{e}(k)) + \boldsymbol{Q}\boldsymbol{D}(k).\end{aligned} \tag{45}$$

Let $\tilde{\varepsilon}_i^*(k) = \hat{\varepsilon}_i^*(k) - \varepsilon_i^*$ and $\tilde{M}_i(k) = \hat{M}_i(k) - M_i, i = 1, 2, \cdots, n$. Multiplying both sides of equation (45) by $\boldsymbol{e}^T(k)$ yields

$$\boldsymbol{e}^T(k)\boldsymbol{Q}_0^C D_k^\nu \boldsymbol{e}(k) = -\boldsymbol{e}^T(k)\boldsymbol{L}\boldsymbol{e}(k) + \boldsymbol{e}^T(k)\boldsymbol{\varepsilon}(\boldsymbol{z}(k)) + \boldsymbol{e}^T(k)\boldsymbol{Q}\boldsymbol{D}(k)$$

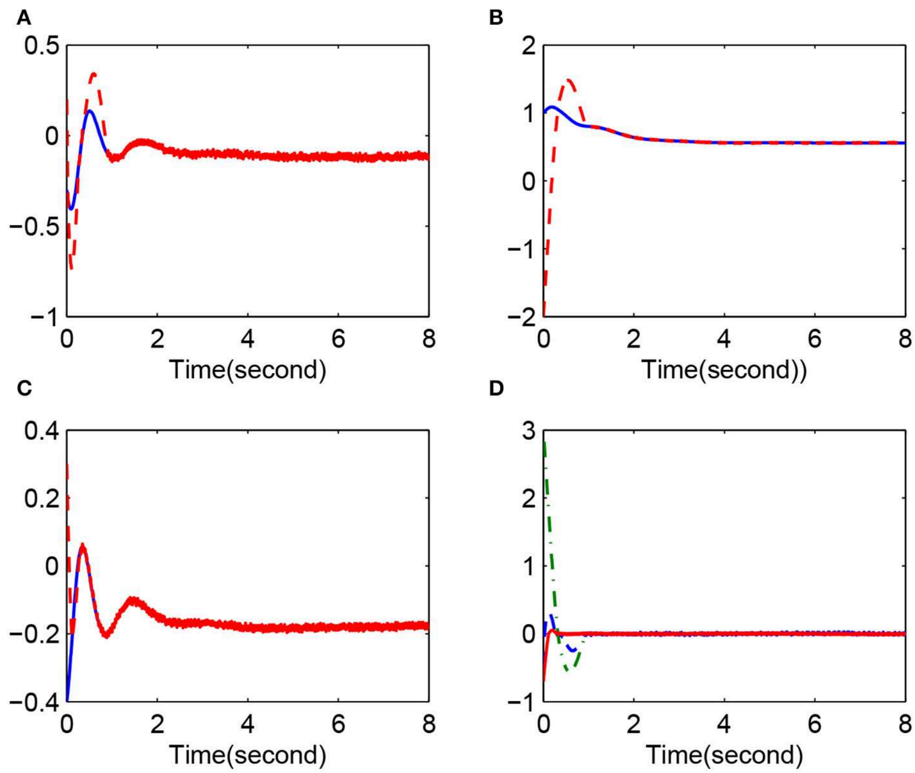

FIGURE 2 | Synchronization results of Case 2.

$$
\begin{aligned}
&- \boldsymbol{e}^T(k)\tilde{\boldsymbol{\theta}}(k)^T\boldsymbol{\varphi}(\boldsymbol{z}(k)) - \boldsymbol{e}^{\mathrm{T}}(k)\boldsymbol{H}\mathrm{sign}(\boldsymbol{e}(\mathrm{k})) - \boldsymbol{e}^{\mathrm{T}}(\mathrm{k})\hat{\boldsymbol{M}}\mathrm{sign}(\boldsymbol{e}(\mathrm{k}))\\
&\leq -\boldsymbol{e}^T(k)\boldsymbol{L}\boldsymbol{e}(k) + \sum_{i=1}^{n}|e_i(k)|\varepsilon_i^* - \sum_{i=1}^{n}e_i(k)\tilde{\theta}_i(k)^T\varphi_i(\boldsymbol{z}(k)) -\\
&\sum_{i=1}^{n}|e_i(k)|\hat{\varepsilon}_i^*(k) - \sum_{i=1}^{n}|e_i(k)|\hat{M}_i(k) + \sum_{i=1}^{n}|e_i(k)|M_i\\
&= -\boldsymbol{e}^T(k)\boldsymbol{L}\boldsymbol{e}(k) - \sum_{i=1}^{n}e_i(k)\tilde{\theta}_i(k)^T\varphi_i(\boldsymbol{z}(k)) - \sum_{i=1}^{n}|e_i(k)|\tilde{M}_i(k)\\
&- \sum_{i=1}^{n}|e_i(k)|\tilde{\varepsilon}_i^*(\boldsymbol{z}(k)). \qquad (46)
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
V(k) &= \frac{1}{2}\boldsymbol{e}^T(k)\boldsymbol{Q}\boldsymbol{e}(k) + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\lambda_i}\tilde{\theta}_i(k)^T\tilde{\theta}_i(k) +\\
&\frac{1}{2}\sum_{i=1}^{n}\frac{1}{\xi_i}\left(\tilde{\varepsilon}_i^*(k)\right)^T\tilde{\varepsilon}_i^*(k) + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\mu_i}(\tilde{M}_i(k))^T\tilde{M}_i(k). \qquad (47)
\end{aligned}
$$

Because the ν-order Caputo derivative of a constant is zero, we have ${}_0^CD_k^{\nu}\theta_i(k) = {}_0^CD_k^{\nu}\tilde{\theta}_i(k)$, ${}_0^CD_k^{\nu}\hat{\varepsilon}_i^*(k) = {}_0^CD_k^{\nu}\tilde{\varepsilon}_i^*(k)$, and ${}_0^CD_k^{\nu}\hat{M}_i(k) = {}_0^CD_k^{\nu}\tilde{M}_i(k)$, $i = 1, 2, \cdots, n$. By Lemma 3 and Lemma 6, taking the ν-order derivative of $V(k)$ in equality (47) yields

$$
\begin{aligned}
&{}_0^CD_k^{\nu}V(k) \leq e^T(k)\boldsymbol{Q}{}_0^CD_k^{\nu}e(k) + \sum_{i=1}^{n}\frac{1}{\lambda_i}\tilde{\theta}_i(k)^T{}_0^CD_k^{\nu}\tilde{\theta}_i(k) +\\
&\sum_{i=1}^{n}\frac{1}{\xi_i}\left(\tilde{\varepsilon}_i^*(k)\right)^T{}_0^CD_k^{\nu}\tilde{\varepsilon}_i^*(k) + \sum_{i=1}^{n}\frac{1}{\mu_i}(\tilde{M}_i(k))^T{}_0^CD_k^{\nu}\tilde{M}_i(k)\\
&\leq -\sum_{i=1}^{n}|e_i(k)|\tilde{\varepsilon}_i^*(k) - \sum_{i=1}^{n}e_i(k)\tilde{\theta}_i(k)^T\varphi_i(\boldsymbol{z}(k)) -\\
&\boldsymbol{e}^T(k)\boldsymbol{L}\boldsymbol{e}(k) - \sum_{i=1}^{n}|e_i(k)|\tilde{M}_i(k) + \sum_{i=1}^{n}\frac{1}{\lambda_i}\tilde{\theta}_i(k)^T{}_0^CD_k^{\nu}\tilde{\theta}_i(k)\\
&+ \sum_{i=1}^{n}\frac{1}{\xi_i}\left(\tilde{\varepsilon}_i^*(k)\right)^T{}_0^CD_k^{\nu}\tilde{\varepsilon}_i^*(k) + \sum_{i=1}^{n}\frac{1}{\mu_i}(\tilde{M}_i(k))^T{}_0^CD_k^{\nu}\tilde{M}_i(k). \quad (48)
\end{aligned}
$$

Substituting (27), (28), and (29) into (48) gives

$$
{}_0^CD_k^{\nu}V(k) \leq -\boldsymbol{e}^T(k)\boldsymbol{L}\boldsymbol{e}(k) \leq -\frac{l_0}{\lambda_{\max}}\boldsymbol{e}^T(k)\boldsymbol{Q}\boldsymbol{e}(k), \qquad (49)
$$

where $l_0 = \min\{l_1, l_2, \cdots, l_n\}$ and $\lambda_{\max}$ is a maximal eigenvalue in positive definite matrix $\boldsymbol{Q}$. From Lemma 8 and inequality (49), we know that the synchronization error satisfies $\lim_{k\to\infty}||\boldsymbol{e}(k)|| = 0$, and if ${}_0^CD_k^{\nu}\tilde{\theta}_i(k)$, ${}_0^CD_k^{\nu}\tilde{\varepsilon}_i^*(k)$, and ${}_0^CD_k^{\nu}\tilde{M}_i(k)$ are bounded, then ${}_0^CD_k^{\nu}\theta_i(k)$, ${}_0^CD_k^{\nu}\hat{\varepsilon}_i^*(k)$, and ${}_0^CD_k^{\nu}\hat{M}_i(k)$ are both bounded. Since

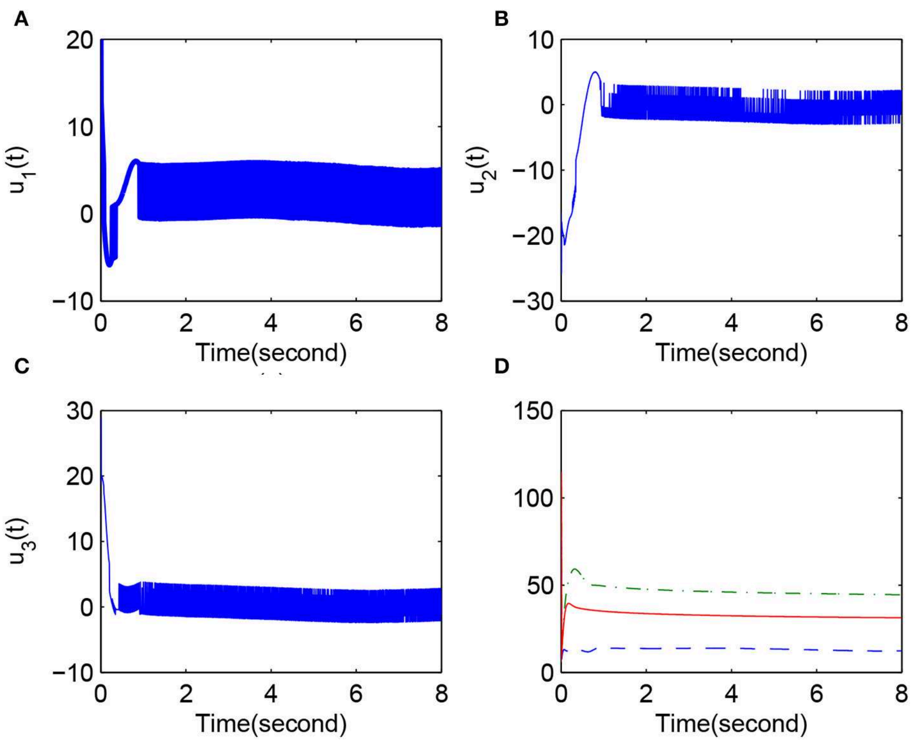

FIGURE 3 | Control variables and fuzzy logic system parameters of Case 3.

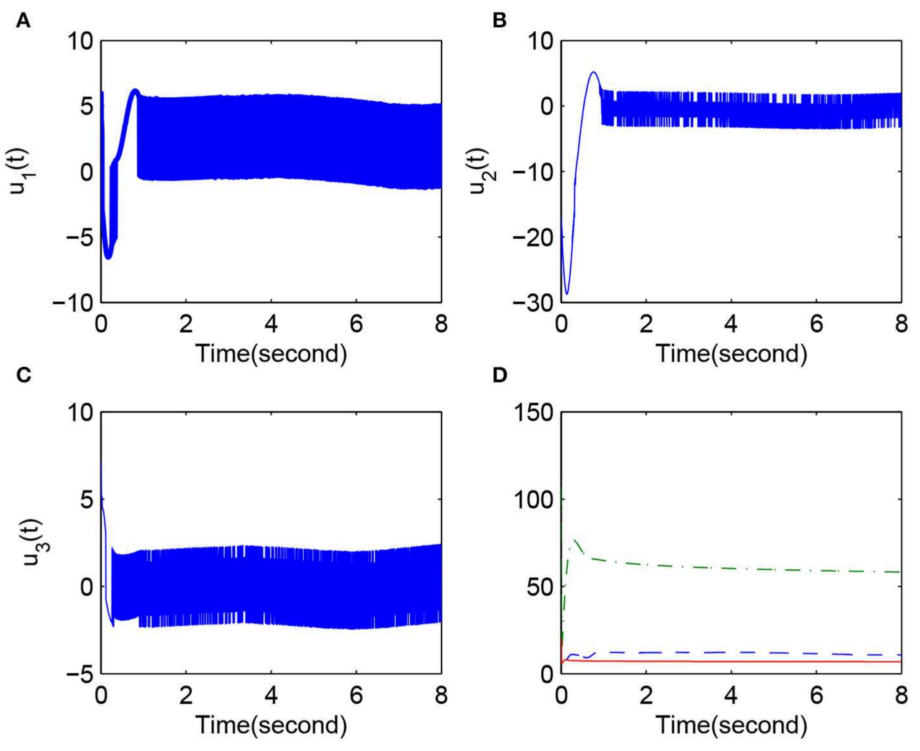

FIGURE 4 | Control variables and fuzzy logic system parameters of Case 4.

system (13) is a chaotic system, we know that $\boldsymbol{x}(k)$ is also bounded. Thus, $\boldsymbol{e}(k)$ is also bounded, which implies that $\boldsymbol{y}(k)$ is bounded, too. Consequently, by using (26), we get that $\boldsymbol{u}(k)$ is bounded. Thereby, all the signals in the closed-loop system are bounded.

Remark 4. For respond system (14), when $\boldsymbol{G} = \boldsymbol{E}$, the synchronization between the uncertain FOCSs was solved in Liu et al. [44]. However, this solution cannot solve the synchronization question for systems with uncertain non-symmetrical control gain; when $\boldsymbol{D}(k) = \boldsymbol{0}$, Ha et al. [45] researched the synchronization of FOCSs with indeterminate non-symmetrical control gain but also did not solve the synchronization question of systems with unknown disturbances. In contrast, by considering the above two conditions, this paper addresses the synchronization question for systems with uncertain non-symmetrical control gain and unknown disturbances.

5. NUMERICAL SIMULATION

In the simulation, the effectiveness of the controller is tested by researching the synchronization between the fractional-order Newton-Leipnik system [46, 47] and the fractional-order Lü system [48, 49].

The fractional-order Newton-Leipnik system is given as follows.

$$\boldsymbol{h}(\boldsymbol{x}(k)) = \begin{pmatrix} -0.4x_1(k) + x_2(k) + 10x_2(k)x_3(k) \\ -x_1(k) - 0.4x_2(k) + 5x_1(k)x_3(k) \\ 0.175x_3(k) - 5x_1(k)x_2(k) \end{pmatrix}. \tag{50}$$

The fractional-order Lü system can be written as:

$$\boldsymbol{p}(\boldsymbol{y}(k)) = \begin{pmatrix} -36y_1(k) + 36y_2(k) \\ 20y_2(k) - y_1(k)y_3(k) \\ -3y_3(k) + y_1(k)y_2(k) \end{pmatrix}. \tag{51}$$

The gain matrix $\boldsymbol{G}$ (which is non-symmetric) and its factorization (by Lemma 4) are as follows.

$$\begin{aligned} \boldsymbol{G} &= \begin{pmatrix} 1 & a & 0.3 \\ 0 & -0.4 & 0.2 \\ 0 & 0 & b \end{pmatrix} = \boldsymbol{G}_1\boldsymbol{A}\boldsymbol{T} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & -0.3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -0.2 & 0.3 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{pmatrix}, \end{aligned} \tag{52}$$

where parameters a and b are separately $a = -0.2$ and $b = 0.9$. The external disturbance is

$$\boldsymbol{D}(t) = \begin{pmatrix} 0.15\sin(k) \\ 0.05\cos(k) \\ 0.1\cos(k) \end{pmatrix}. \tag{53}$$

It is easy to gather that the following inequality holds:

$$|\boldsymbol{G}_1^{-1}\boldsymbol{D}(k)| \le \begin{pmatrix} 1 \\ 0.5 \\ 0.8 \end{pmatrix}. \tag{54}$$

The initial values of the drive system and respond system can be respectively $\boldsymbol{x}(0) = [-0.3, 1, -0.4]^T$ and $\boldsymbol{y}(0) = [2, -2, 3]^T$. When $\boldsymbol{u}(k) \equiv \boldsymbol{0}, \boldsymbol{D}(k) = \boldsymbol{0}$ and $\mu = 0.95$, the above two systems exhibit chaotic phenomena.

In the numerical simulation, the input variables of the fuzzy system are $\boldsymbol{x}(k), \boldsymbol{y}(k)$, and $\boldsymbol{u}(k)$. For inducing calculation of the fuzzy logic system, we will replace $\boldsymbol{x}(k)$ and $\boldsymbol{y}(k)$ by $\boldsymbol{e}(k)$. For $e_1(k), e_2(k)$, and $e_3(k)$, we can select five Gaussian membership functions whose mathematical expectations are respectively $-4, -2, 0, 2$, and 4 and whose parameters are ([1.2],[-4, -2, 0, 2, 4]), uniformly distributed in the interval $[-4, 4]$ for each input. Therefore, the number of the rules that are produced by the fuzzy logic system approximating function is $5^3 = 125$. In order to better test the effectiveness of the controller, we can chose adjustable parameters, which are represented by $\theta_1(0), \theta_2(0)$, and $\theta_3(0)$, as random vectors in 125 dimensions.

The other parameters of the controller are defined as $l_i = 5, \lambda_i = 500, \xi_i = 0.5$, and $\mu_i = 0.5$, and the estimated values of the fuzzy logic system approximating error are $\hat{\varepsilon}_1^*(0) = \hat{\varepsilon}_3^*(0) = 1.8$ and $\hat{\varepsilon}_2^*(0) = 1.5$. Estimators of the product between the uncertain external disturbance and unknown constant matrix are $\hat{M}_1(0) = 1, \hat{M}_2(0) = 0.4$, and $\hat{M}_3(0) = 0.3$. For the sake of better showing the simulation results, the initial value of the respond system is chosen as $\boldsymbol{y}(0) = [0.2, -2, 0.3]^T$, which is compared to $\boldsymbol{y}(0) = [2, -2, 3]^T$. The simulation results are shown separately in **Figures 1–4**, detailed explanations of which follow.

Case 1, in **Figure 1**, synchronization result: $x(0) = [-0.3, 1, -0.4]^T$ and $y(0) = [2, -2, 3]^T$. (**Figure 1A**) $x_1(k)$ (solid line) and $y_1(k)$ (dotted line); (**Figure 1B**) $x_2(k)$ (solid line) and $y_2(k)$ (dotted line); (**Figure 1C**) $x_3(k)$ (solid line) and $y_3(k)$ (dotted line); (**Figure 1D**) synchronization error $e_1(k)$ (dotted line), $e_2(k)$ (dashed line), and $e_3(k)$ (solid line).

Case 2, in **Figure 2**, synchronization result: $x(0) = [-0.3, 1, -0.4]^T$ and $y(0) = [0.2, -2, 0.3]^T$. (**Figure 2A**) $x_1(k)$ (solid line) and $y_1(k)$ (dotted line); (**Figure 2B**) $x_2(k)$ (solid line) and $y_2(k)$ (dotted line); (**Figure 2C**) $x_3(k)$ (solid line) and $y_3(k)$ (dotted line); (**Figure 2D**) synchronization error $e_1(k)$ (dotted line), $e_2(k)$ (dashed line), and $e_3(k)$ (solid line).

Case 3, in **Figure 3**, control variables and fuzzy logic system parameters: $x(0) = [-0.3, 1, -0.4]^T$ and $y(0) = [2, -2, 3]^T$. (**Figure 3A**) $u_1(t)$; (**Figure 3B**) $u_2(k)$; (**Figure 3C**) $u_3(k)$; (**Figure 3D**) $||\theta_1(k)||$ (dotted line), $||\theta_2(k)||$ (dashed line), and $||\theta_3(k)||$ (solid line).

Case 4, in **Figure 4**, control variables and fuzzy logic system parameters: $x(0) = [-0.3, 1, -0.4]^T$ and $y(0) = [0.2, -2, 0.3]^T$. (**Figure 4A**) $u_1(k)$; (**Figure 4B**) $u_2(k)$; (**Figure 4C**) $u_3(k)$; (**Figure 4D**) $||\theta_1(k)||$ (dotted line), $||\theta_2(k)||$ (dashed line), and $||\theta_3(k)||$ (solid line).

The simulation results clearly show that the convergence rate of synchronization error is fast when l_i is reasonable. **Figures 1, 4** give the error change trend that the error is large at first and then gets smaller and smaller after a time, finally tending to zero asymptotically. Furthermore, from case 1 and case 2, we know that a minimal change in initial values can have obvious effects on the error but cannot affect the eventual convergence of error. This implies that the fuzzy system proposed in this paper has good approximation performance. **Figures 3**, **4** display the changing situation of control variables and fuzzy logic system parameters, and it conforms to our expectations. In addition, from the above simulation results, we can see a chattering phenomenon because a discontinuous sign function is used in the synchronization controller.

6. CONCLUSION

In this paper, a robust adaptive fuzzy controller for indeterminate FOCSs with uncertain external disturbances and non-symmetrical control gain is proposed. The proposed method has good ability on the condition that each sequence-leading minor of the uncertain non-symmetrical gain matrix is non-zero, and the upper bound of the product of the positive definite matrix factorized by gain matrix and external disturbance is known. The stability of the closed-loop system is successfully discussed by using a fractional-order Lyapunov method and quadratic Lyapunov functions.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

JX and XZ contributed the conception and design of the study. XQ and NL organized the literature. JX performed the design of figures and wrote the first draft of the manuscript. All authors contributed to manuscript revision and read and approved the submitted version.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.