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l_1-Embeddability Under Gate-Sum Operation of Two l_1-Graphs

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An l_1-graph is one in which the vertices can be labeled by binary vectors such that the Hamming distance between two binary addresses is, to scale, the distance in the graph between the corresponding vertices. This study was designed to determine whether the gate-sum operation can inherit the l_1-embeddability. The subgraph H of a graph G is called a gate subgraph if, for every vertex $v \in V(G)$, there exists a vertex $x \in V(H)$ such that for every vertex u of H, x lies on a shortest path from v to u. The graph G is defined as the *gate-sum* of two graphs G_1 and G_2 with respect to H if H is a gate subgraph of at least one of G_1 and G_2, such that $G_1 \cup G_2 = G$, $G_1 \cap G_2 = H$, and both G_1 and G_2 are isometric subgraphs of G. In this article, we have shown that the gate-sum graph of two l_1-graphs is also an l_1-graph.

Keywords: hypercube, l_1-embeddability, gate subgraph, gate-sum, convex cuts

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1. INTRODUCTION

A computer network is a group of computer systems and other computing hardware devices that are linked together through communication channels to facilitate communication and resource-sharing among a wide range of users. Networks are usually visualized as a graph, with the computers or devices being represented by vertices and the connections between vertices shown as edges. Graham and Pollak [1] were concerned with message switching in interconnected loops of computers, and they studied the problem of addressing graphs with a ternary alphabet $\{0, 1, \delta\}$ such that any graph may be addressed with an edge distance of unity for some address length n. Blake and Gilchrist [2] restricted attention to the binary alphabet. They formulated a routing algorithm for message switching in computer networks that simplifies the computation of the minimum-length path between any two vertices. An l_1-*graph* is one in which the vertices can be labeled by binary vectors such that the Hamming distance between two binary addresses is, to scale, the distance in the graph of corresponding vertices [3]. The graph operation can construct a new graph from a given graph, and some properties can be inherited under these operations. Our motivation for this study was to determine which operations can inherit the l_1-embeddability. Thus, the purpose of this work is to determine the l_1-embeddability of the gate-sum graph of two l_1-graphs.

Let $G = (V, E)$ be a connected simple graph. The *distance* between two vertices u and v of G, denoted by $d_G(u, v)$, is the length of a shortest u–v path in G. Then $[V(G), d_G]$ is a graphic metric space associated with G [3]. A subgraph H of G is an *isometric subgraph* if $d_H(u, v) = d_G(u, v)$ for any $u, v \in H$. A subgraph of G is *convex* if, for any two vertices, it includes all of the shortest paths between them. Obviously, a convex subgraph of G is an isometric subgraph. Let $S \subset V(G)$ be any subset of vertices of G. The *induced subgraph* $G[S]$ is the graph that has the vertex set S and the edge set consisting of all edges in E for which both ends are in S [4].

Bandelt and Chepoi [5] introduced the definition of a gate subgraph. A subgraph H of a graph G is a *gate subgraph* if, for every vertex $v \in V(G)$, there exists a unique vertex $x \in V(H)$ such that x lies

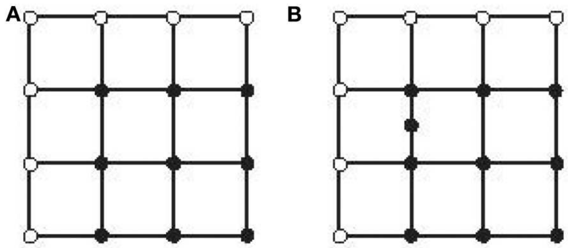

FIGURE 1 | Examples of a convex subgraph **(A)** and a gate subgraph **(B)**.

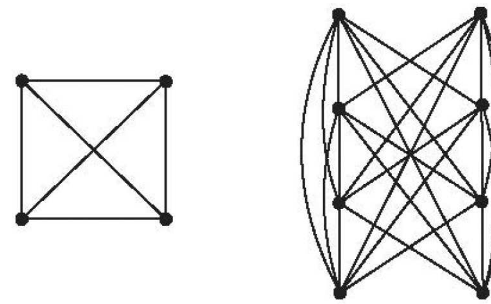

FIGURE 2 | The complete graph K_4 and the cocktail graph $K_{4\times2}$.

on the shortest path between v and any vertex $u \in V(H)$; x is called the *gate* of v. Hammack et al. [6] showed that a gate subgraph is convex, but that a convex subgraph may not be a gate subgraph. For example, each subgraph induced by the black vertices in **Figures 1A,B** is a convex subgraph in each graph. The subgraph shown in **Figure 1A** is a gate subgraph, whereas that in **Figure 1B** is not.

If u and v are two vertices of a path, the subsequence of this path starting with u and ending with v is the *segment* of this path from u to v. The *shortest path* P_{xy} is the path connecting x to y that has the fewest edges. Clearly, the segment of a shortest path is still a shortest path [7].

The l_1-*space* is the metric space of sequences whose series is absolutely convergent, denoted by (X, d_1). Thus, X is the set of all real sequences $x = (x_1, x_2, \ldots)$ such that $\sum_{k=1}^{\infty} |x_k| < \infty$, and the distance function is defined as $d_1(x, y) = \sum_{k=1}^{\infty} |x_k - y_k|$ for any $x, y \in X$. A graph G is an l_1-*graph* if $(V(G), d_G)$ is isometrically embeddable into some l_1-space. That is, there is a distance-preserving mapping φ from $V(G)$ into X such that $d_G(x, y) = d_1(\varphi(x), \varphi(y))$.

The *n-dimensional hypercube* Q_n is the graph whose vertices are ordered n-tuples of 0s and 1s, two vertices being joined if and only if they differ in exactly one coordinate.

Assouad and Deza [8] showed that a graph G is an l_1-graph if and only if G is *scale-λ-embeddable* into a hypercube Q_n for some positive integers λ and n, meaning that there exists a mapping $\phi : V(G) \to V(Q_n)$ such that

$$\lambda \cdot d_G(x, y) = d_{Q_n}(\phi(x), \phi(y))$$

for any $x, y \in V(G)$. The integer λ is the *scale* of G. The smallest such integer λ is called the *minimum scale* of G. According to Shpectorov [9], the minimum scale λ of G is equal to 1 or is even. In particular, if $\lambda = 1$, G is an isometric subgraph of Q_n, also called a *partial cube*.

Shpectorov [9] and Deza and Grishukhin [10] showed that a graph G is an l_1-*graph* if and only if it is an isometric subgraph of the Cartesian product of cocktail party graphs and half-cubes. The *cocktail party graph* $K_{n\times2}$ is a complete multipartite graph with n parts, each of cardinality 2, which is equivalent to a complete graph K_{2n} deleting a perfect matching, as shown in **Figure 2**. The hypercube Q_n is a bipartite graph, and the *half-cube* $\frac{1}{2}Q_n$ is the graph defined on one of two parts of this hypercube, with two vertices being joined if the distance between them in Q_n is 2.

An l_1-*rigid* graph is an l_1-graph that essentially admits a unique l_1-embedding. Shpectorov [9] showed that every l_1-rigid graph G is an isometric subgraph of a half-cube. He also proved that every l_1-*rigid* graph has scale 1 or 2. Deza and Laurent [11] proved that the complete graph K_n $(n \geq 4)$ and the cocktail graph $K_{n\times2}$ $(n \geq 4)$ are not l_1-rigid, where the variety of l_1-embeddings of $K_{n\times2}$ all come from that of the complete graph K_n. The half-cube graph $\frac{1}{2}Q_n$ $(n = 3, 4)$ is l_1-rigid. Hence, they claim that, if G is not l_1-rigid, the variety of its l_1-embeddings arises from that of the complete graph. Deza and Tuma [12] and Chepoi et al. [13] studied the forbidden subgraphs of an l_1-rigid graph. They determined that an l_1-graph is l_1-rigid if and only if it is K_4-free.

Deza and Laurent [11] proved that the graph obtained by identifying single vertices from two l_1-graphs is also an l_1-graph. Wang and Zhang [14] proved that the graph obtained by gluing two l_1-graphs along an edge is also an l_1-graph if at least one of the original graphs is bipartite. However, for two non-bipartite graphs, this is not always the case. They also determined that even for two bipartite l_1-graphs, gluing a convex subgraph cannot guarantee the l_1-embeddability of the obtained graph. Naturally, we wondered if this result could be generalized.

Suppose that H_i is a subgraph of G_i, $i = 1, 2$. If H_1 is isomorphic to H_2, their vertices can be identified under some isomorphism as a new graph H such that the incidence relationship between vertices and edges remains. The resulting graph is called the *H-sum* of G_1 and G_2, denoted by $G_1 \cup_H G_2$. In particular, if H is a single vertex v or an edge $e = uv$, the H-sum is called the 1-*sum* or the 2-*sum*, denoted by $G_1 \cup_v G_2$ and $G_1 \cup_{uv} G_2$, respectively. Additionally, if G_1 and G_2 are isometric in $G_1 \cup_H G_2$, and H is a gate subgraph of at least one of G_1 and G_2, then $G_1 \cup_H G_2$ is called a *gate-sum* of G_1 and G_2, denoted by $G_1 \cup_H^g G_2$. Both G_1 and G_2 are isometric subgraphs of $G_1 \cup_H G_2$ if and only if $d_{G_1}(x, y) = d_{G_2}(x, y)$ for any $x, y \in H$.

For example, see the graph in **Figure 3**, where the marked K_4 is an isomorphic subgraph of G_1 and G_2. The K_4-sum graph $G_1 \cup_{K_4} G_2$, shown in **Figure 3C**, is obtained by identifying these two marked K_4 as the same subgraph. In particular, in **Figure 3B**, the marked K_4 is a gate subgraph of G_2. Obviously, both G_1 and G_2 are isometric subgraphs of $G_1 \cup_{K_4} G_2$. Therefore, it can be seen as a gate-sum graph $G_1 \cup_{K_4}^g G_2$ of G_1 and G_2 with respect to K_4.

In this paper, we have shown that the gate-sum graph of two l_1-graphs G_1 and G_2 is also an l_1-graph. The remainder of this article is organized as follows. In section 2, we have introduced the

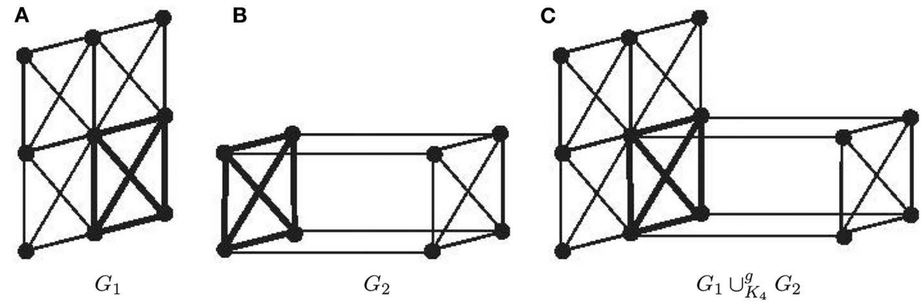

FIGURE 3 | The gate-sum graph $G_1 \cup^g_{K_4} G_2$ of G_1 and G_2 with respect to K_4.

concept of convex cuts of graphs, which are used to characterize the l_1-graphs. We have proven that the collection of convex cuts of the gate-sum graph $G_1 \cup^g_H G_2$ can be expanded by those of G_1 and G_2. We have then proven the main theorem. For the sake of brevity, we obtained the main result by omitting the proofs of certain lemmas. In section 3, we have presented detailed proofs of those lemmas that were not proved in section 2. Finally, we have presented our conclusions to this study in section 4.

2. CONVEX CUTS AND MAIN RESULTS

Deza and Tuma [12] introduced the concept of convex cuts, which can be used to characterize l_1-graphs. A *cut* $\{A, B\}$ of G is a partition of $V(G)$ into two nonempty parts. If both A and B are convex sets, then the cut $\{A, B\}$ is a *convex cut*. A cut $\{A, B\}$ of G *cuts* an edge uv if $u \in A$ and $v \in B$. An *edge cut* of G is a subset of $E(G)$ of the form $[S, \overline{S}]$, where S is a nonempty proper subset of $V(G)$, $\overline{S} = V \backslash S$, and $[S, \overline{S}]$ is the set of edges with one end in S and the other in $\overline{S}$. Similarly, we say that a cut $\{A, B\}$ of G *cuts* a subgraph H if $[A \cap V(H), B \cap V(H)]$ is an edge cut of H.

Deza and Tuma [12] and Deza et al. [15] proved the following theorem.

Theorem 2.1. ([12, 15]) *A graph G is scale-λ-embeddable into a hypercube if and only if there exists a collection $\mathcal{C}(G)$ of (not necessarily distinct) convex cuts of G such that every edge of G is cut by exactly λ cuts from $\mathcal{C}(G)$.*

For example, in the graph K_4 in **Figure 4**, the cuts $\{\{a\}, \{b, c, d\}\}$, $\{\{b\}, \{a, c, d\}\}$, $\{\{c\}, \{a, b, d\}\}$, $\{\{d\}, \{a, b, c\}\}$ are convex cuts. Every edge of K_4 is cut by exactly 2 cuts of $\{\{a\}, \{b, c, d\}\}$, $\{\{b\}, \{a, c, d\}\}$, $\{\{c\}, \{a, b, d\}\}$, and $\{\{d\}, \{a, b, c\}\}$. By Theorem 2.1, the graph K_4 is scale-2-embeddable into the hypercube Q_4.

Furthermore, Wang and Zhang [14] showed that the scale of an l_1-graph can be proportionally amplified.

Lemma 2.2. ([14]) *If a graph G is scale-λ-embeddable into a hypercube, then, for any positive integer r, G is scale-$r\lambda$-embeddable into a hypercube.*

Let G_1 and G_2 be two l_1-graphs and $G_1 \cup^g_H G_2$ be a gate-sum graph of G_1 and G_2. Without loss of generality, suppose that G_1 is scale-λ-embeddable into some hypercube and G_2 is scale-η-embeddable into some hypercube. By Theorem 2.1, there are two collections $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ such that every edge of G_1 and G_2 is cut by exactly λ and η cuts, respectively. According to Theorem 2.1 and Lemma 2.2, to prove $G_1 \cup^g_H G_2$ is an l_1-graph, it is sufficient to construct a collection $\mathcal{C}(G_1 \cup^g_H G_2)$ of convex cuts of $G_1 \cup^g_H G_2$ such that every edge of $G_1 \cup^g_H G_2$ is cut by exactly the same number of cuts. Now, we construct a collection of convex cuts of $G_1 \cup^g_H G_2$ from the convex cuts of $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$.

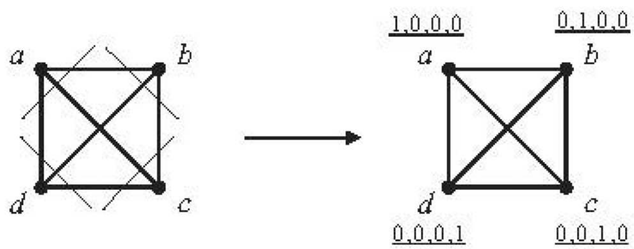

FIGURE 4 | Convex cuts and binary address of K_4.

We now define the *expansion* of convex cuts. Suppose that H is a subgraph of G and $\{A, B\}$ is a convex cut of H. If G has a convex cut $\{A', B'\}$ such that $A \subseteq A'$ and $B \subseteq B'$, then we say that the convex cut $\{A, B\}$ of H *expands* the convex cut $\{A', B'\}$ of G. We say that the collection $\mathcal{C}(H)$ *expands* a collection $\mathcal{C}(G)$ if every convex cut of $\mathcal{C}(H)$ can expand a convex cut of G. We also say that the collection $\mathcal{C}(H)$ is the *restriction* of $\mathcal{C}(G)$ on the subgraph H.

To enhance the readability of this paper, we list the following three lemmas without proofs. Their proofs have been given in section 3.

Lemma 2.3. *Suppose that $G_1 \cup^g_H G_2$ is a gate-sum graph of two l_1-graphs G_1 and G_2. Then, a convex cut of G_1 (or G_2) not cutting H can expand a convex cut of $G_1 \cup^g_H G_2$.*

Next, we will prove that two convex cuts of G_1 and G_2 can expand a convex cut of $G_1 \cup^g_H G_2$ if they cut the same edges of H. Suppose that the convex cut $\{A_1, B_1\}$ of G_1 is cutting H and that the cut $\{A_2, B_2\}$ is that of G_2. Then, $\{A_1, B_1\}$ and $\{A_2, B_2\}$ cut the same edges of H. If $A_1 \cap A_2 \neq \emptyset$, then $A_1 \cap B_2 = \emptyset$. If not, $A_1 \cap A_2 \neq \emptyset$ and $A_1 \cap B_2 \neq \emptyset$, which contradicts the assertion that $\{A_1, B_1\}$ and $\{A_2, B_2\}$ cut the same edges of H. Similarly, we have $B_1 \cap B_2 \neq \emptyset$ and $B_1 \cap A_2 = \emptyset$. Because $A_i \cup B_i = V(G_i)$ $(i = 1, 2)$ and $V(G_1) \cap V(G_2) = V(H)$, we know that $A_1 \cap V(H) = A_1 \cap (A_1 \cup B_1) \cap (A_2 \cup B_2) = A_1 \cap A_2$ and $A_2 \cap V(H) = A_1 \cap A_2$. Similarly, $B_1 \cap V(H) = B_1 \cap B_2 = B_2 \cap V(H)$. Furthermore, we have that $V(H) = V(G_1) \cap V(G_2) = (A_1 \cup B_1) \cap (A_2 \cup B_2) = [A_1 \cap (A_2 \cup B_2)] \cup [B_1 \cap (A_2 \cup B_2)] = [A_1 \cap A_2] \cup [B_1 \cap B_2]$. We denote $V(H_A) = A_1 \cap A_2$ and $V(H_B) = B_1 \cap B_2$. Then, $V(H_A) \cup V(H_B) = V(H)$, and we have the following lemma.

Lemma 2.4. *Suppose that $G_1 \cup_H^g G_2$ is a gate-sum graph of two l_1-graphs G_1 and G_2. Assume that $\{A_1, B_1\}$ is a convex cut of G_1 and $\{A_2, B_2\}$ is that of G_2. If H is l_1-rigid, $\{A_1, B_1\}$ and $\{A_2, B_2\}$ cut the same edges of H. Then, $\{A_1, B_1\}$ and $\{A_2, B_2\}$ can together expand a convex cut $\{A_1 \cup_{V(H_A)} A_2, B_1 \cup_{V(H_B)} B_2\}$ of $G_1 \cup_H^g G_2$.*

If H is not l_1-rigid, then it has more than one kind of collection of convex cuts. Any two collections $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ may not be equal on H. Therefore, the convex cuts of $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ may not cut the same edges of H.

To solve this problem, we have proven that any kind of collection of convex cuts of H can expand two new collections of convex cuts of G_1 and G_2, respectively, such that they are equal on H.

Lemma 2.5. *Let H be an isometric subgraph of an l_1-graph G. If H is not l_1-rigid, each collection $\mathcal{C}(H)$ of H can expand a collection $\mathcal{C}(G)$ of G.*

We will now prove the main theorem of this work.

Theorem 2.6. *Suppose that $G_1 \cup_H^g G_2$ is a gate-sum graph of G_1 and G_2. If G_1 and G_2 are l_1-embeddable, then $G_1 \cup_H^g G_2$ is also l_1-embeddable.*

Proof: Without loss of generality, suppose that H is a gate subgraph of G_1. Because a gate subgraph is a convex subgraph, H is a convex subgraph of G_1. Then, H is an l_1-graph. Suppose that G_1 is scale-λ-embeddable into some hypercube and G_2 is scale-η-embeddable into some hypercube. By Theorem 2.1, there are two collections $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ such that every edge of G_1 and G_2 is cut by exactly λ and η cuts, respectively.

If H is l_1-rigid, H has only one kind of collection of convex cuts. Then, $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ have the same restriction on H (which means that $\lambda = \eta$).

If H is not l_1-rigid, the restriction on H of $\mathcal{C}(G_1)$ is not equal to that of $\mathcal{C}(G_2)$. Suppose that $\lambda \neq \eta$. By Lemma 2.2, G_2 is scale-$\lambda\eta$-embeddable into some hypercube. Then, G_2 has a collection $\mathcal{C}'(G_2)$ such that every edge of G_2 is cut by exactly $\lambda\eta$ cuts. By Lemma 2.5, every $\mathcal{C}(H)$ can expand a collection $\mathcal{C}(G_1)$. Obviously, the restriction on H of $\mathcal{C}'(G_2)$ is a kind of $\mathcal{C}(H)$. Thus, it can expand a new collection $\mathcal{C}'(G_1)$ of G_1 such that every edge of G_1 is cut by exactly $\lambda\eta$ cuts.

Hence, there always are two collections $\mathcal{C}'(G_1)$ and $\mathcal{C}'(G_2)$ for which the restrictions of them on H are equal, and every edge of G_1 and G_2 is cut by exactly $\lambda\eta$ cuts.

As $\mathcal{C}'(G_1)$ and $\mathcal{C}'(G_2)$ are equal on H, there are the same number of convex cuts of $\mathcal{C}'(G_1)$ and $\mathcal{C}'(G_2)$ cutting H. Denote the convex cuts of $\mathcal{C}'(G_1)$ that are cutting H as $\{A_1, B_1\}, ..., \{A_h, B_h\}$ and those of $\mathcal{C}'(G_2)$ as $\{A'_1, B'_1\}, ..., \{A'_h, B'_h\}$. Because the restrictions on H of $\mathcal{C}'(G_1)$ and $\mathcal{C}'(G_2)$ are equal, each convex cut of $\{A_1, B_1\}, ..., \{A_h, B_h\}$ must equal one of $\{A'_1, B'_1\}, ..., \{A'_h, B'_h\}$ on H. Without loss of generality, we assume that each pair of $\{A_i, B_i\}$ and $\{A'_i, B'_i\}$ cut the same edges of H $(1 \leq i \leq h)$. By Lemma 2.4, each pair of convex cuts $\{A_i, B_i\}$ and $\{A'_i, B'_i\}$ can together expand a convex cut $\{A_i \cup A'_i, B_i \cup B'_i\}$ of $G_1 \cup_H^g G_2$ $(1 \leq i \leq h)$. Then, every edge of H is cut by $\{A_i \cup A'_i, B_i \cup B'_i\}$ to give exactly $\lambda\eta$ cuts $(1 \leq i \leq h)$.

By Lemma 2.3, the convex cuts of $\mathcal{C}'(G_1)$ and $\mathcal{C}'(G_2)$ that do not cut H can expand the convex cuts of $G_1 \cup_H^g G_2$ that do not cut H.

Now, the convex cuts $\{A_i \cup A'_i, B_i \cup B'_i\}$ for $1 \leq i \leq h$, together with the convex cuts of $\mathcal{C}'(G_1)$ and $\mathcal{C}'(G_2)$ that do not cut H, form a collection of convex cuts of $G_1 \cup_H^g G_2$, such that every edge of $G_1 \cup_H^g G_2$ is cut by $\lambda\eta$ convex cuts. Therefore, by Theorem 2.1, the graph $G_1 \cup_H^g G_2$ is scale-$\lambda\eta$-embedded into some hypercube. This completes the proof. $\square$

Note that, for any graph, a single vertex is a gate subgraph. A *cycle* is a closed path that originates and terminates at the same vertex. A graph is bipartite if and only if it contains no odd cycles [4]. Therefore, for any edge $e = uv$ of a bipartite graph, there is no vertex a such that $d(u, a) = d(v, a)$. The subgraph induced by an edge is then a gate subgraph in a bipartite graph. Obviously, both G_1 and G_2 are isometric subgraphs of the graphs $G_1 \cup_v G_2$ and $G_1 \cup_{uv} G_2$. The following corollaries can be immediately obtained from Theorem 2.6.

Corollary 2.7. ([11]). *Let G_1 and G_2 be two l_1-graphs. $G_1 \cup_v G_2$ is an l_1-graph.*

Corollary 2.8. ([14]). *Let G_1 and G_2 be two l_1-graphs. If at least one of them is bipartite, $G_1 \cup_{uv} G_2$ is an l_1-graph.*

3. PROOFS OF LEMMAS 2.3–2.5

3.1. Proof of Lemma 2.3

First, we need the following lemma.

Lemma 3.1. *Suppose that $G_1 \cup_H^g G_2$ is a gate-sum graph of G_1 and G_2. If H is a gate subgraph of G_1, then G_2 is a convex subgraph of $G_1 \cup_H^g G_2$.*

Proof: If G_2 is not a convex subgraph of $G_1 \cup_H^g G_2$, there are two vertices x_1 and x_2 lying in G_2 such that the shortest path $P_{x_1x_2}$ passes through a vertex v_3 of G_1. As this shortest path must pass through the vertices of the gate subgraph H of G_1, there are two vertices x'_1 and x'_2 of H on the x_1, v_3-path and x_2, v_3-path of $P_{x_1x_2}$, respectively. Note that both G_1 and G_2 are isometric subgraphs of $G_1 \cup_H^g G_2$. It is clear that the segment from x'_1 to x'_2 of $P_{x_1x_2}$ is a shortest path $P_{x'_1x'_2}$. Then, we have $P_{x'_1x'_2} = P_{x'_1v_3} + P_{v_3x'_2}$.

As H is a gate subgraph of G_1, there exists a unique gate a_3 of v_3 in H such that $P_{x'_1v_3} = P_{x'_1a_3} + P_{a_3v_3}$ and $P_{x'_2v_3} = P_{x'_2a_3} + P_{a_3v_3}$. Then, we have that $P_{x'_1x'_2} = P_{x'_1v_3} + P_{v_3x'_2} = P_{x'_1a_3} + P_{a_3v_3} + P_{x'_2a_3} + P_{a_3v_3} > P_{x'_1a_3} + P_{a_3x'_2}$, which contradicts the assertion that $P_{x'_1x'_2}$ is a shortest path. Thus, G_2 is a convex subgraph of $G_1 \cup_H^g G_2$. $\square$

Proof of Lemma 2.3. Without loss of generality, suppose that H is a gate subgraph of G_1. We need only prove that a convex cut of G_1 or G_2 that does not cut H can expand a convex cut of $G_1 \cup_H^g G_2$.

Case 1. A convex cut of G_1 that does not cut H can expand that of $G_1 \cup_H^g G_2$.

Suppose $\{A, B\}$ is a convex cut of G_1 that does not cut H. Without loss of generality, we assume that $V(H) \subseteq B$. We now prove that $\{A, B \cup_{V(H)} V(G_2)\}$ is a convex cut of $G_1 \cup_H^g G_2$ that does not cut H and is expanded by $\{A, B\}$.

If A is not a convex set of $G_1 \cup_H^g G_2$, there are two vertices v_1 and v_2 belonging to A such that $P_{v_1v_2}$ of $G_1 \cup_H^g G_2$ passes through a vertex v_3 of $G[B] \cup_H G_2$. Therefore, $P_{v_1v_2} = P_{v_1v_3} + P_{v_3v_2}$. If all vertices of $P_{v_1v_2}$ lie entirely in G_1, A cannot be a convex set of G_1. Without loss of generality, suppose that v_3 lies in G_2. Note that H is a gate subgraph of G_1. There are two gates x_1 of v_1 and x_2 of v_2 in H, and these two gates lie in $P_{v_1v_3}$ and $P_{v_3v_2}$, respectively. Then, we have that $P_{v_1v_2} = P_{v_1x_1} + P_{x_1v_3} + P_{v_3x_2} + P_{x_2v_2}$. As G_1 is an isometric subgraph of $G_1 \cup_H^g G_2$, there is some $P_{x_1x_2}$ that lies entirely in G_1, and its length equals that of $P_{x_1x_2}$ of G_2. Then, $P_{v_1v_2} = P_{v_1x_1} + P_{x_1x_2} + P_{x_2v_2}$, and $P_{v_1v_2}$ lies entirely in G_1. As $P_{v_1v_2}$ passes through the vertices of H, and H belongs to B, A cannot be a convex set of G_1. Therefore, A is a convex set of $G_1 \cup_H^g G_2$.

If $B \cup_{V(H)} V(G_2)$ is not a convex set of $G_1 \cup_H^g G_2$, there are two vertices v_4 and v_5 belonging to $B \cup_{V(H)} V(G_2)$ such that $P_{v_4v_5}$ passes through a vertex v_6 in A and $P_{v_4v_5} = P_{v_4v_6} + P_{v_6v_5}$. Obviously, the path $P_{v_4v_6}$ does not intersect with $P_{v_6v_5}$ at any internal vertices. The segment of a shortest path is still a shortest path. This means that v_6 has two internally disjoint paths $P_{v_6v_4}$ and $P_{v_6v_5}$ that connect with the vertices in H. Thus, v_6 has at least two gates, which contradicts the statement that the gate is unique.

Both A and $B \cup_{V(H)} V(G_2)$ are convex sets of $G_1 \cup_H^g G_2$, and they contain all vertices of $G_1 \cup_H^g G_2$. Thus, $\{A, B \cup_{V(H)} V(G_2)\}$ is a convex cut of $G_1 \cup_H^g G_2$. Furthermore, note that $A = A$ and $B \subseteq B \cup_{V(H)} V(G_2)$, and so the convex cut $\{A, B \cup_{V(H)} V(G_2)\}$ is expanded by the convex cut $\{A, B\}$ of G_1.

Case 2. A convex cut of G_2 that does not cut H can expand that of $G_1 \cup_H^g G_2$.

Suppose that $\{C, D\}$ is a convex cut of G_2 that does not cut H. Without loss of generality, we assume that $H \subseteq D$. We now prove that the cut $\{C, D \cup_{V(H)} V(G_1)\}$ is a convex cut of $G_1 \cup_H^g G_2$ and is expanded by $\{C, D\}$.

By Lemma 3.1, it is obvious that C is a convex set of $G_1 \cup_H^g G_2$ because C is a convex set of G_2 and G_2 a convex subgraph of $G_1 \cup_H^g G_2$.

Suppose that the vertex set $D \cup_{V(H)} V(G_1)$ is not a convex set of $G_1 \cup_H^g G_2$. There will be two vertices v_1, v_2 of $D \cup_{V(H)} V(G_1)$ such that the shortest path $P_{v_1v_2}$ passes through a vertex v_3 of C. Let $P_{v_1v_3}$ and $P_{v_3v_2}$ denote the two segments of $P_{v_1v_2}$ divided by v_3. Because the vertices v_1, v_2 belong to $D \cup_{V(H)} V(G_1)$, we can find two vertices v_1', v_2' of D such that $v_1' \in P_{v_1v_3}$ and $v_2' \in P_{v_3v_2}$. Note that both G_1 and G_2 are isometric subgraphs of $G_1 \cup_H^g G_2$. It is clear that the segment from v_1' to v_2' of the path $P_{v_1v_2}$ is a shortest path, and it passes through the vertex v_3 of C, which contradicts the assertion that D is a convex set of G_2. Therefore, $D \cup_{V(H)} V(G_1)$ is a convex set of G.

As C and $D \cup_{V(H)} V(G_1)$ are convex sets of $G_1 \cup_H^g G_2$, $\{C, D \cup_{V(H)} V(G_1)\}$ is a convex cut of G and its two convex sets contain C and D, respectively. It follows that the convex cut $\{C, D\}$ of G_2 that does not cut H expands the convex cut $\{C, D \cup_{V(H)} V(G_1)\}$ of $G_1 \cup_H^g G_2$. This completes the proof. □

3.2. Proof of Lemma 2.4

Proof: Let G_1 and G_2 be two l_1-graphs and $G_1 \cup_H^g G_2$ be the gate-sum graph of G_1 and G_2. By Theorem 2.1, there are two collections $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ such that every edge of G_1 and G_2 is cut by exactly λ and η cuts, respectively, as H is l_1-rigid, $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ must be equal on H. For any convex cut $\{A_1, B_1\}$ of $\mathcal{C}(G_1)$, we can find a convex cut $\{A_2, B_2\}$ of $\mathcal{C}(G_2)$ that cuts the same edge of H.

Without loss of generality, suppose that H is a gate subgraph of an l_1-graph G_1. Suppose that x_1 of $V(H)$ is the gate of v_1 in G_1. If v_1 and x_1 belong to different convex sets, assume that v_1 lies in A_1 and x_1 belongs to $B_1 \cap V(H)$. There will be a vertex u in $A_1 \cap V(H)$ such that the shortest path P_{v_1u} must pass through the vertices of B_1, which contradicts the assertion that A_1 is a convex set. Then, both v_1 and x_1 belong to the same convex set A_1 or B_1.

Without loss of generality, suppose that v_1 and x_1 belong to A_1. We now show that $\{A_1 \cup_{V(H_A)} A_2, B_1 \cup_{V(H_B)} B_2\}$ is a convex cut of $G_1 \cup_H^g G_2$. First, we prove that $A_1 \cup_{V(H_A)} A_2$ is a convex set of $G_1 \cup_H^g G_2$. Consider two vertices v_1 and v_2 that belong to $A_1 \cup_{V(H_A)} A_2$.

Case 1. Both v_1 and v_2 lie in A_2.

As A_2 is a convex subset of G_2 and G_2 is a convex subgraph of $G_1 \cup_H^g G_2$, A_2 is a convex subset of $G_1 \cup_H^g G_2$. Obviously, $P_{v_1v_2}$ lies entirely in A_2.

Case 2. The vertex v_1 lies in A_1 and v_2 lies in A_2.

Because v_1 lies in A_1 and v_2 lies in A_2, the gate x_1 of v_1 belongs to $A_1 \cap V(H)$. As $\{A_1, B_1\}$ and $\{A_2, B_2\}$ cut the same edges of H, we have that $A_1 \cap V(H) = A_2 \cap V(H)$ and x_1 also belongs to A_2. Therefore, the shortest path $P_{v_1v_2}$ must pass through the vertices of H.

If $P_{v_1v_2}$ passes through the gate x_1 of v_1, we have that $P_{v_1v_2} = P_{v_1x_1} + P_{x_1v_2}$. Note that both G_1 and G_2 are isometric subgraphs of $G_1 \cup_H^g G_2$. As both v_1 and x_1 belong to A_1 and A_1 is a convex set, the path $P_{v_1x_1}$ lies entirely in A_1. Similarly, v_2 and x_1 belong to A_2, which is a convex set. Hence, $P_{v_2x_1}$ lies entirely in A_2. Thus, the shortest path $P_{v_1v_2}$ lies entirely in $A_1 \cup_{V(H_A)} A_2$.

If there is a shortest path $P_{v_1v_2}$ that does not pass through the gate x_1 of v_1, $P_{v_1v_2}$ will pass through a vertex x_3 of $V(H)$, which is not the gate of v_1, and $P_{v_1v_2} = P_{v_1x_3} + P_{x_3v_2}$.

We now prove that x_3 belongs to $A_1 \cap V(H)$. If this is not the case, then x_3 lies in $B_1 \cap V(H)$, and so $P_{v_1x_3} = P_{v_1x_1} + P_{x_1x_3}$ and $P_{x_1v_2} < P_{x_1x_3} + P_{x_3v_2}$. Furthermore, $P_{v_1x_3} + P_{x_3v_2} = P_{v_1x_1} + P_{x_1x_3} + P_{x_3v_2} > P_{v_1x_1} + P_{x_1v_2}$, which contradicts the assertion that $P_{v_1v_2}$ passes through x_3, but does not pass through the gate x_1.

As v_1 and x_3 belong to A_1, and x_3 and v_2 belong to A_2, we have that $P_{v_1x_3}$ lies entirely in A_1 and $P_{x_3v_2}$ lies entirely in A_2. Therefore, $P_{v_1v_2} = P_{v_1x_3} + P_{x_3v_2}$ lies entirely in $A_1 \cup_{V(H_A)} A_2$.

Hence, for any vertex v_1 of A_1 and any vertex v_2 of A_2, $P_{v_1v_2}$ lies entirely in $A_1 \cup_{V(H_A)} A_2$. This proves case 2.

Case 3. Both v_1 and v_2 lie in A_1.

If $P_{v_1v_2}$ does not pass through the vertices of G_2, then $P_{v_1v_2}$ lies in G_1. Note that A_1 is a convex subgraph of G_1, and $P_{v_1v_2}$ lies in A_1. If $P_{v_1v_2}$ passes through the vertices of G_2, it must pass through a vertex v_3 of A_2. From case 2, we know that both $P_{v_1v_3}$ and $P_{v_3v_2}$ lie in $A_1 \cup_{V(H_A)} A_2$ and that $P_{v_1v_2}$ lies entirely in $A_1 \cup_{V(H_A)} A_2$.

Summarizing the above three cases, for any two vertices v_1 and v_2 of $A_1 \cup_{V(H_A)} A_2$, we have that the shortest path $P_{v_1v_2}$ lies entirely in $A_1 \cup_{V(H_A)} A_2$. It follows that $A_1 \cup_{V(H_A)} A_2$ is a convex set of $G_1 \cup_H^g G_2$.

A similar proof shows that the set $B_1 \cup_{V(H_B)} B_2$ is also a convex set of $G_1 \cup_H^g G_2$. Then, $\{A_1 \cup_{V(H_A)} A_2, B_1 \cup_{V(H_B)} B_2\}$ is a convex cut of $G_1 \cup_H^g G_2$, and its two convex sets contain vertex sets A_1, A_2 and B_1, B_2, respectively. Thus, $\{A_1, B_1\}$ of G_1 and $\{A_2, B_2\}$ of G_2 together expand the convex cut $\{A_1 \cup_{V(H_A)} A_2, B_1 \cup_{V(H_B)} B_2\}$ of $G_1 \cup_H^g G_2$.

□

3.3. Proof of Lemma 2.5

To study the expansion of the collection of convex cuts, we have introduced a new characteristic of l_1-graphs. Shpectorov [9] and Deza and Grishukhin [10] characterized l_1-graphs as follows:

Theorem 3.2. ([9, 10]) *A graph G is an l_1-graph if and only if it is an isometric subgraph of the Cartesian product of cocktail party graphs and half-cubes.*

Suppose that H is an isometric subgraph of an l_1-graph G; H is also an l_1-graph. By Theorem 3.2, H is an isometric subgraph of the Cartesian product of some cocktail party graphs and half-cubes, and G is that of larger cocktail party graphs and larger half-cubes. To expand the collection of convex cuts of H to G, we need only expand the collection of convex cuts of the cocktail party graph and half-cube to a larger cocktail party graph and a larger half-cube, respectively. As the half-cube is l_1-rigid, it has a unique collection of convex cuts. Note that $\frac{1}{2}Q_m$ is a subgraph of $\frac{1}{2}Q_n$. Thus, we have that any collection $\mathcal{C}(\frac{1}{2}Q_m)$ of $\frac{1}{2}Q_m$ can expand a collection $\mathcal{C}(\frac{1}{2}Q_n)$ of $\frac{1}{2}Q_n$ ($m \leq n$). We need only examine whether any collection $\mathcal{C}(K_{m\times 2})$ can expand a collection $\mathcal{C}(K_{n\times 2})$ ($m \leq n$).

We require the definition of a vertex-transitive graph. An *automorphism* of a (simple) graph G is a permutation π of $V(G)$ that has the property that (u, v) is an edge of G if and only if $(\pi(u), \pi(v))$ is an edge of G. The set of all automorphisms of G, with the composition operation, is a group. This group is called the *automorphism group of* G. A graph G is *vertex-transitive* if the automorphism group of G acts transitively on $V(G)$ [16, 17].

In other words, a *vertex-transitive graph* is a graph G such that, given any two vertices v_1 and v_2 of G, there is some automorphism $f : V(G) \to V(G)$ such that $f(v_1) = v_2$.

For a complete graph K_n, we constructed its collection of convex cuts. Without loss of generality, assume that $V(K_n) = \{v_1, ..., v_n\}$. From Theorem 3.2, K_n is an l_1-graph. Suppose that K_n is scale-λ-embeddable into a hypercube. Theorem 2.1 implies that there is a collection $\mathcal{C}(K_n)$ such that every edge uv is cut by λ cuts (u, v belong to K_n and λ is even). We assume that $\{S_1, V(K_n) - S_1\}$ is a convex cut of $\mathcal{C}(K_n)$, and that both S_1 and $V(K_n) - S_1$ are convex sets of $V(K_n)$ ($|S_1| = q$). As the complete graph is vertex-transitive, each S_i constructs a convex cut of K_n of the form $S_i \subseteq V(K_n), |S_i| = q$ $(1 \leq i \leq \binom{n}{q})$. Then, we have that all convex cuts $\{S_i, V(K_n) - S_i\}$, $|S_i| = q$ $(1 \leq i \leq \binom{n}{q})$, form a collection of convex cuts of K_n such that every edge of K_n is cut by the same cuts.

Obviously, there are $\binom{n}{q}$ different convex cuts, and each convex cut acts on $q(n - q)$ edges. Note that the complete graph K_n has $\frac{n(n-1)}{2}$ edges and is vertex-transitive. Thus, we have that $\lambda = \frac{\binom{n}{q} q(n-q)}{\frac{n(n-1)}{2}} = 2\binom{n-2}{q-1}$.

For $m \leq n$, we can now prove that the collection $\mathcal{C}(K_{m\times 2})$ of $K_{m\times 2}$ can expand a collection $\mathcal{C}(K_{n\times 2})$ of $K_{n\times 2}$.

Theorem 3.3. *Let $K_{n\times 2}$ be a cocktail party graph and $K_{m\times 2}$ be a cocktail party subgraph of $K_{n\times 2}$. Every collection $\mathcal{C}(K_{m\times 2})$ of $K_{m\times 2}$ can expand a collection $\mathcal{C}(K_{n\times 2})$ of $K_{n\times 2}$.*

Proof: Obviously, the cocktail party graph $K_{n\times 2}$ has a complete subgraph K_n. Without loss of generality, assume that $V(K_n) = \{v_1, ..., v_n\}$, $V(K'_n) = \{v'_1, ..., v'_n\}$, and $V(K_{n\times 2}) = \{v_1, ..., v_n, v'_1, ..., v'_n\}$ such that $d_{K_{n\times 2}}(v_j, v'_j) = 2$ $(1 \leq j \leq n)$, $d_{K_{n\times 2}}(v_i, v_j) = d_{K_{n\times 2}}(v_i, v'_j) = 1$ $(i \neq j)$. If the vertex set S is a subset of $V(K_{n\times 2})$, then the vertex set $S' = \{x' | x \in S\}$ is a subset of $V(K'_{n\times 2})$.

First, we prove that every convex cut of $K_{n\times 2}$ has only two forms: $\{S \cup (V(K'_n) - S'), S' \cup (V(K_n) - S)\}$ and $\{V(K_n), V(K'_n)\}$.

Suppose that $\{A, B\}$ is a convex cut of $K_{n\times 2}$. If x belongs to A, x' will belong to B. If not, both x and x' belong to A, and A is a convex subset of $V(K_{n\times 2})$; all vertices of $V(K_{n\times 2})$ will then belong to A. Furthermore, B is an empty set, which contradicts both A and B being nonempty. We now have that the vertex sets S and S' belong to different convex sets of $\{A, B\}$. Without loss of generality, suppose that $S \subseteq A$ and $S' \subseteq B$. If $V(K_n) - S \subseteq A$, then $V(K'_n) - S' \subseteq B$ and $\{A, B\} = \{V(K_n), V(K'_n)\}$. If $V(K_n) - S \subseteq B$, then $V(K'_n) - S' \subseteq A$ and $\{A, B\} = \{S \cup (V(K'_n) - S'), S' \cup (V(K_n) - S)\}$.

Thus, the convex cut of $K_{n\times 2}$ has only two forms, $\{S \cup (V(K'_n) - S'), S' \cup (V(K_n) - S)\}$ and $\{V(K_n), V(K'_n)\}$.

Second, we prove that the collection of convex cuts $\{S_i \cup (V(K'_n) - S'_i), S'_i \cup (V(K_n) - S_i)\}$, $|S_i| = q$ $(1 \leq i \leq \binom{n}{q})$, together with some $\{V(K_n), V(K'_n)\}$ make the cocktail graph $K_{n\times 2}$ embeddable into some cubes.

For every edge uv in K_n, uv is cut by the convex cut $\{S_i \cup (V(K'_n) - S'_i), S'_i \cup (V(K_n) - S_i)\}$. We have that $u \in (S_i \cup (V(K'_n) - S'_i)) \cap V(K_n) = S_i$ and $v \in (S'_i \cup (V(K_n) - S_i)) \cap V(K_n) = V(K_n) - S_i$, or $u \in V(K_n) - S_i$ and $v \in S_i$. Note that $|S_i| = q$ and $V(K_n)$ has n vertices, so the number of convex cuts that cut edge uv is $2\binom{n-2}{q-1}$. This is similar to each edge $u'v'$ of K'_n.

If $u \in K_n$ and $v' \in K'_n$, uv' is cut by the convex cut $\{S_i \cup (V(K'_n) - S'_i), S'_i \cup (V(K_n) - S_i)\}$. We have that $u \in (S_i \cup (V(K'_n) - S'_i)) \cap V(K_n) = S_i$, $v' \in (S'_i \cup \{V(K_n) - S_i\}) \cap V(K'_n) = S'_i$, $u' \in S'_i$, and $v \in S_i$, or $u \in (S'_i \cup \{V(K_n) - S_i\}) \cap V(K_n) = V(K_n) - S_i$, $v' \in (S_i \cup (V(K'_n) - S'_i)) \cap V(K'_n) = V(K'_n) - S'_i$, $u' \in V(K'_n) - S'_i$, and $v \in V(K_n) - S_i$. Note that $|S_i| = |S'_i| = q$ and $|V(K_n) - S_i| =$

$|V(K'_n) - S'_i| = n - q$, so the number of convex cuts that cut edge uv' is $\binom{n-2}{q-2} + \binom{n-2}{n-q-2} = \binom{n-2}{q-2} + \binom{n-2}{q}$.

As $\frac{n-\sqrt{n}}{2} \leq q \leq \frac{n+\sqrt{n}}{2}$, $2\binom{n-2}{q-1} \geq \binom{n-2}{q-2} + \binom{n-2}{q}$. If $2\binom{n-2}{q-1} = \binom{n-2}{q-2} + \binom{n-2}{q}$, every edge of $K_{n\times 2}$ is cut by $2\binom{n-2}{q-1}$ cuts.

If $2\binom{n-2}{q-1} > \binom{n-2}{q-2} + \binom{n-2}{q}$, then $\frac{n-\sqrt{n}}{2} < q < \frac{n+\sqrt{n}}{2}$. Obviously, $\{V(K_n), V(K'_n)\}$ is a convex cut of $K_{n\times 2}$, which only cuts the edges with one end vertex in K_n and the other one in K'_n. Then, the collection $\{S_i \cup (V(K'_n) - S'_i), S'_i \cup (V(K_n) - S_i)\}$ $(1 \leq i \leq \binom{n}{q})$ together with $\{V(K_n), V(K'_n)\}$ form a new collection $\mathcal{C}'(K_{n\times 2})$ such that every edge of $K_{n\times 2}$ is cut by $2\binom{n-2}{q-1}$ cuts.

Let $2\binom{n-2}{q-1} < \binom{n-2}{q-2} + \binom{n-2}{q}$. If n is even, choose $T_i \subseteq V(K_n)$ such that $|T_i| = \frac{n}{2}$ $(1 \leq i \leq \binom{n}{\frac{n}{2}})$. If n is odd, choose $T_i \subseteq V(K_n)$ such that $|T_i| = \frac{n+1}{2}$ $(1 \leq i \leq \binom{n}{\frac{n+1}{2}})$. Then, $\{T_i \cup (V(K'_n) - T'_i), T'_i \cup (V(K_n) - T_i)\}$ is a convex cut of $K_{n\times 2}$. Obviously, the number of edges with both vertices in $V(K_n)$ (or $V(K'_n)$) that are cut by $\{T_i \cup (V(K'_n) - T'_i), T'_i \cup (V(K_n) - T_i)\}$ is greater than the number of edges that are cut by the same cut with one end vertex in K_n and the other vertex in K'_n. Thus, the collection $\{S_i \cup (V(K'_n) - S'_i), S'_i \cup (V(K_n) - S_i)\}$ $(1 \leq i \leq \binom{n}{q})$ together with $\{T_i \cup (V(K'_n) - T'_i), T'_i \cup (V(K_n) - T_i)\}$ $(1 \leq i \leq \binom{n}{\frac{n}{2}}$ (or $\binom{n}{\frac{n+1}{2}}$)) and $\{V(K_n), V(K'_n)\}$ form a new collection $\mathcal{C}'(K_{n\times 2})$ such that every edge of $K_{n\times 2}$ is cut by $2\binom{n-2}{q-1} + 2a\binom{n-2}{\frac{n}{2}-1}$ cuts. The constant a is the minimal number such that $2\binom{n-2}{q-1} + 2a\binom{n-2}{\frac{n}{2}-1} \geq \binom{n-2}{q-2} + \binom{n-2}{q} + a(\binom{n-2}{\frac{n}{2}-2} + \binom{n-2}{\frac{n}{2}})$.

Third, we prove that every collection of convex cuts of $K_{m\times 2}$ can expand that of $K_{n\times 2}$ $(m \leq n)$.

Similarly, each convex cut of $K_{m\times 2}$ has only two forms: $\{A \cup (V(K'_m) - A'), A' \cup (V(K_m) - A)\}$, and $\{V(K_m), V(K'_m)\}$.

Obviously, $(V(K_m) - A) \subseteq (V(K_n) - A)$ and $(V(K'_m) - A') \subseteq (V(K'_n) - A')$. Then, each convex cut $\{A \cup (V(K'_m) - A'), A' \cup (V(K_m) - A)\}$ of $\mathcal{C}(K_{m\times 2})$ can expand a convex cut $\{A \cup (V(K'_n) - A'), A' \cup (V(K_n) - A)\}$ of $\mathcal{C}(K_{n\times 2})$. Similarly, the convex cut $\{V(K_m), V(K'_m)\}$ expands the cut $\{V(K_n), V(K'_n)\}$.

Assume that $|A_i| = |A_j|$ is true for all convex cuts of $\mathcal{C}(K_{m\times 2})$ except the convex cut $\{V(K_m), V(K'_m)\}$. This means that $\{A_i \cup (V(K'_m) - A'_i), A'_i \cup (V(K_m) - A_i)\}, |A_i| = q$ $(1 \leq i \leq \binom{m}{q})$. Then, all of the cuts together with $\{V(K_m), V(K'_m)\}$ expand a collection of convex cuts of $K_{n\times 2}$, in the form $\{A_i \cup (V(K'_n) - A'_i), A'_i \cup (V(K_n) - A_i)\}$, $|A_i| = q$ $(1 \leq i \leq \binom{n}{q})$, together with $\{V(K_n), V(K'_n)\}$. By the second part, $\{A_i \cup (V(K'_n) - A'_i), A'_i \cup (V(K_n) - A_i)\}$, $|A_i| = q$ $(1 \leq i \leq \binom{n}{q})$, together with $\{V(K_n), V(K'_n)\}$ ensure that every edge of the graph $K_{n\times 2}$ is cut by the same cuts.

Let $|A_i| \neq |A_j|$ for some i and j of the convex cuts of $\mathcal{C}(K_{m\times 2})$. Without loss of generality, suppose that $\mathcal{C}(K_{m\times 2})$ has three kinds of convex cuts, formed as $\{A_i \cup (V(K'_n) - A'_i), A'_i \cup (V(K_n) - A_i)\}$, $|A_i| = q$ $(1 \leq i \leq \binom{n}{q})$, and $\{B_i \cup (V(K'_n) - B'_i), B'_i \cup (V(K_n) - B_i)\}$, $|B_i| = p$ $(1 \leq i \leq \binom{n}{p})$, together with $\{V(K_n), V(K'_n)\}$. By the above discussion, all of the convex cuts $\{A_i \cup (V(K'_n) - A'_i), A'_i \cup (V(K_n) - A_i)\}$, $|A_i| = q$ $(1 \leq i \leq \binom{n}{q})$, together with $\{V(K_n), V(K'_n)\}$ expand a collection $\mathcal{C}_1(K_{n\times 2})$ of convex cuts of $K_{n\times 2}$ such that every edge of $K_{n\times 2}$ is cut by the same cuts. Similarly, all of the convex cuts $\{B_i \cup (V(K'_n) - B'_i), B'_i \cup (V(K_n) - B_i)\}$, $|B_i| = p$ $(1 \leq i \leq \binom{n}{p})$, together with $\{V(K_n), V(K'_n)\}$ expand a collection $\mathcal{C}_2(K_{n\times 2})$ of convex cuts of $K_{n\times 2}$ such that every edge of $K_{n\times 2}$ is cut by the same cuts.

Obviously, the collection $\mathcal{C}_1(K_{n\times 2})$ together with the collection $\mathcal{C}_2(K_{n\times 2})$ is still a collection of convex cuts of $K_{n\times 2}$ such that every edge of $K_{n\times 2}$ is cut by the same cuts.

Therefore, every collection $\mathcal{C}(K_{m\times 2})$ of $K_{m\times 2}$ can expand a collection $\mathcal{C}(K_{n\times 2})$ such that every edge of $K_{n\times 2}$ is cut by the same number of cuts. $\square$

We have that, for each cocktail party graph and half-cube, the collection $\mathcal{C}(\frac{1}{2}Q_m)$ can expand a collection $\mathcal{C}(\frac{1}{2}Q_n)$, and the collection $\mathcal{C}(K_{m\times 2})$ can expand a collection $\mathcal{C}(K_{n\times 2})$ $(m \leq n)$. By Theorem 3.2, we can prove that the collection of convex cuts of an l_1-graph can expand that of a larger l_1-graph.

Hammack et al. [6] introduced the *Cartesian product* $G\square H$ of two graphs G and H as the graph whose vertex set is the Cartesian product $V(G) \times V(H)$. Two vertices (u, v) and (u', v') are adjacent in $G\square H$ if and only if $u = u'$ and v is adjacent to v' in H, or $v = v'$ and u is adjacent to u' in G. Thus,

$V(G\square H) = \{(u, v)|u \in V(G) \text{ and } v \in V(H)\}$

$E(G\square H) = \{(u, v)(u', v')|u = u', vv' \in E(H), \text{ or } uu' \in E(G), u = u'\}$

The graphs G and H are called *factors* of the product $G\square H$. Hammack et al. proved the following lemmas.

Lemma 3.4. ([6]) *A subgraph W of $G = G_1\square \cdots \square G_n$ is convex if and only if $W = W_1\square \cdots \square W_n$, where each W_i is convex in G_i.*

Lemma 3.5. ([6]) *If $G = G_1\square \cdots \square G_n$ and $x, y \in V(G)$, then*

$$d_G(x, y) = \sum_{i=1}^{n} d_{G_i}(p_i(x), p_i(y))$$

For any index $1 \leq i \leq n$, p_i is a projection map $p_i : G_1\square \cdots \square G_n \to G_i$, defined as $p_i(x_1, x_2, ..., x_n) = x_i$.

We can now prove that the convex cut of a Cartesian product can be represented by the convex cuts of all factors.

Theorem 3.6. *The cut $\{A, B\}$ is a convex cut of a graph $G = G_1\square \cdots \square G_n$ if and only if $\{A, B\}$ has the form $\{V(G_1) \times \cdots \times V(G_{i-1}) \times A_i \times V(G_{i+1}) \times \cdots \times V(G_n), V(G_1) \times \cdots \times V(G_{i-1}) \times B_i \times V(G_{i+1}) \times \cdots \times V(G_n)\}$ in which $\{A_i, B_i\}$ is a convex cut of G_i for $1 \leq i \leq n$.*

Proof: $\Longleftarrow$ Suppose that $G = G_1\square \cdots \square G_n$. If $\{A_i, B_i\}$ is a convex cut of G_i, then $G_i[A_i]$ and $G_i[B_i]$ are convex subgraphs of G_i $(1 \leq i \leq n)$. By Lemma 3.4, $G[A_i] = G_1\square \cdots \square G_{i-1}\square G_i[A_i]\square G_{i+1}\square \cdots \square G_n$ is a convex subgraph of G. Similarly, $G[B_i] = G_1\square \cdots \square G_{i-1}\square G_i[B_i]\square G_{i+1}\square \cdots \square G_n$ is also a convex subgraph of G.

Without loss of generality, suppose that

$$\begin{aligned} V(G) &= \{(x_1, ..., x_i, ..., x_n)|x_i \in V(G_i)\} \\ V(G[A_i]) &= \{(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)|x_j \end{aligned}$$

$$
\begin{aligned}
&\in V(G_j), j \neq i, y_i \in A_i\} \\
&= \{V(G_1) \times \cdots \times V(G_{i-1}) \times A_i \times V(G_{i+1}) \\
&\quad \times \cdots \times V(G_n)\} \\
V(G[B_i]) &= \{(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n) | x_j \\
&\in V(G_j), j \neq i, y_i \in B_i\} \\
&= \{V(G_1) \times \cdots \times V(G_{i-1}) \times B_i \\
&\quad \times V(G_{i+1}) \times \cdots \times V(G_n)\}.
\end{aligned}
$$

As $\{A_i, B_i\}$ is a convex cut of G_i and the vertex y_i belongs to either A_i or B_i, we have that the cut $\{V(G[A_i]), V(G[B_i])\} = \{A, B\}$ is a partition of $V(G)$, and $\{A, B\}$ is a convex cut of G.

$\Longrightarrow$ Suppose that $\{A, B\}$ is a convex cut of G. Then, both $G[A]$ and $G[B]$ are convex subgraphs of G, and $B = \bar{A} = V(G) - A$. By Lemma 3.4, $G[A] = G_1[A_1]\square \cdots \square G_n[A_n]$ and each $G_i[A_i]$ is a convex subgraph of G_i $(1 \leq i \leq n)$.

We now prove that only one A_i is a proper subset of $V(G_i)$. If there are two proper subsets, without loss of generality, suppose that A_1 is a proper subset of $V(G_1)$, A_2 is that of $V(G_2)$, and $A_i = G_i$ $(3 \leq i \leq n)$, $V(G_j) - A_j = B_j$ $(1 \leq j \leq n)$. Then, we have that

$$
\begin{aligned}
A &= \{(x_1, x_2, ..., x_n) | x_i \in V(G_i), i \neq 1, 2, x_1 \in A_1, x_2 \in A_2\} \\
&= \{A_1 \times A_2 \times V(G_3) \times \cdots \times V(G_n)\}
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{A} = B =& \{(x_1, x_2, ..., x_n) | x_i \in V(G_i), i \neq 1, 2, x_1 \notin A_1, x_2 \in A_2, \\
&\text{or } x_1 \in A_1, x_2 \notin A_2, \text{ or } x_1 \notin A_1, x_2 \notin A_2\} \\
=& \{[(B_1 \times A_2) \cup (A_1 \times B_2) \cup (B_1 \times B_2)] \times V(G_3) \\
&\times \cdots \times V(G_n)\}.
\end{aligned}
$$

Suppose that $x_1 \in A_1$, $x_2 \in A_2$, $y_1 \in B_1$, $y_2 \in B_2$, and $x_i \in G_i$ $(3 \leq i \leq n)$. We have two vertices $(y_1, x_2, x_3, x_4, ..., x_n) \in B_1 \times A_2 \times V(G_3) \times \cdots \times V(G_n)$ and $(x_1, y_2, x_3, x_4, ..., x_n) \in A_1 \times B_2 \times V(G_3) \times \cdots \times V(G_n)$. By Lemma 3.5, the distance between them is

$$
\begin{aligned}
&d_G((y_1, x_2, x_3, x_4, ..., x_n), (x_1, y_2, x_3, x_4, ..., x_n)) = d_{G_1}(y_1, x_1) \\
&\quad + d_{G_2}(x_2, y_2) \\
&\quad = d_G((y_1, x_2, x_3, x_4, ..., x_n), (x_1, x_2, x_3, x_4, ..., x_n)) \\
&\quad + d_G((x_1, x_2, x_3, x_4, ..., x_n), (x_1, y_2, x_3, x_4, ..., x_n)).
\end{aligned}
$$

However, vertex $(x_1, x_2, x_3, x_4, ..., x_n)$ belongs to $A_1 \times A_2 \times V(G_3) \times \cdots \times V(G_n)$, which means that there are two vertices in B and a shortest path between them through a vertex in A. Therefore, B is not a convex subset of $V(G)$, which contradicts the assertion that $\{A, B\}$ is a convex cut of G.

Thus, only one A_i is a proper subset of $V(G_i)$, and we have that

$$
\begin{aligned}
A &= \{(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n) | x_j \in V(G_j), j \neq i, y_i \in A_i\} \\
&= \{V(G_1) \times \cdots \times V(G_{i-1}) \times A_i \times V(G_{i+1}) \times \cdots \times V(G_n)\}.
\end{aligned}
$$

Similarly, note that $V(G_j) - A_j = B_j$ $(1 \leq j \leq n)$, and so

$$
\begin{aligned}
B &= \{(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n) | x_j \in V(G_j), j \neq i, y_i \notin A_i\} \\
&= \{V(G_1) \times \cdots \times V(G_{i-1}) \times B_i \times V(G_{i+1}) \times \cdots \times V(G_n)\}.
\end{aligned}
$$

As $G[A]$ and $G[B]$ are convex subgraphs of G, by Lemma 3.4, both $G_i[A_i]$ and $G_i[B_i]$ are convex subgraphs of G_i. Then, A_i and B_i are convex subsets of $V(G_i)$, and $\{A_i, B_i\}$ is a convex cut of G_i $(1 \leq i \leq n)$. □

Proof of Lemma 2.5. Let G be an l_1-graph and H be an isometric subgraph of G. By Theorem 2.1, there is a collection $\mathcal{C}(G)$ such that every edge of G is cut by exactly λ cuts.

As H is not l_1-rigid, H has another l_1-embedding. By Theorem 3.2, G is an isometric subgraph of the Cartesian product of cocktail party graphs and half-cubes. Let $\hat{G} = K_{m_1\times 2}\square \cdots \square K_{m_p\times 2}\square \frac{1}{2}Q_{n_1}\square \cdots \square \frac{1}{2}Q_{n_q}$ be a Cartesian product that contains G as an isometric subgraph, such that each factor of $\hat{G}$ is minimal and the number of factors is minimal. Without loss of generality, we assume that $m_i \leq m_j$ and $n_i \leq n_j$ $(i < j)$.

Because H is an isometric subgraph of G and G is an l_1-graph, H is an l_1-graph. By Theorem 3.2, H has a minimal Cartesian product $\hat{H} = K_{m'_1\times 2}\square \cdots \square K_{m'_s\times 2}\square \frac{1}{2}Q_{n'_1}\square \cdots \square \frac{1}{2}Q_{n'_t}$.

As H is an isometric subgraph of G and G is an isometric subgraph of $\hat{G}$, H is an isometric subgraph of $\hat{G}$. Because $\hat{H}$ may not be equal to $\hat{G}$, we have that $s \leq p$, $t \leq q$, and $m'_i \leq m_i$, $n'_j \leq n_j$ $(1 \leq i \leq s, 1 \leq j \leq t)$.

It is obvious that $\frac{1}{2}Q_{n'_i}$ is a convex subgraph of $\frac{1}{2}Q_{n_i}$ $(1 \leq i \leq t)$ and $K_{m'_i\times 2}$ is an isometric subgraph of $K_{m_i\times 2}$ $(1 \leq i \leq s)$.

As $\frac{1}{2}Q_n$ is l_1-rigid, the collection $\mathcal{C}(\frac{1}{2}Q_{n'_i})$ can expand a collection $\mathcal{C}(\frac{1}{2}Q_{n_i})$ for $1 \leq i \leq t$.

By Theorem 3.3, every collection $\mathcal{C}(K_{m'_i\times 2})$ can expand a collection $\mathcal{C}(K_{m_i\times 2})$ $(1 \leq i \leq s)$.

Without loss of generality, suppose that every collection of $\mathcal{C}(K_{m_j\times 2})$ $(1 \leq j \leq s)$ and $\mathcal{C}(\frac{1}{2}Q_{n_k})$ $(1 \leq k \leq t)$ cuts the edges of the corresponding factors $K_{m_j\times 2}$ and $\frac{1}{2}Q_{n_k}$ exactly $\lambda_1, \lambda_2, ..., \lambda_{s+t}$ times, respectively. Take the least common multiple $\lambda = [\lambda_1, \lambda_2, ..., \lambda_{s+t}]$. By Lemma 2.2, we have a list of collections $\mathcal{C}'(K_{m_j\times 2})$ $(1 \leq j \leq s)$ and $\mathcal{C}'(\frac{1}{2}Q_{n_k})$ $(1 \leq k \leq t)$ such that every edge of factors $K_{m_j\times 2}$ and $\frac{1}{2}Q_{n_k}$ is cut by exactly λ cuts.

By Theorem 3.6, each convex cut $\{A_{j_i}, B_{j_i}\}$ of $\mathcal{C}'(K_{m_j\times 2})$ $(1 \leq j \leq s)$ can expand a convex cut $\{A, B\}$ of G such that $\{p_j(A), p_j(B)\} = \{A_{j_i}, B_{j_i}\}$. This is similar to any convex cut $\{A_{k_i}, B_{k_i}\}$ of $\mathcal{C}'(\frac{1}{2}Q_{n_k})$ $(1 \leq k \leq t)$.

All such $\{A, B\}$ expanded by $\{A_{j_i}, B_{j_i}\}$ of $\mathcal{C}'(K_{m_j\times 2})$ $(1 \leq j \leq s)$ and $\{A_{k_i}, B_{k_i}\}$ of $\mathcal{C}'(\frac{1}{2}Q_{n_k})$ $(1 \leq k \leq t)$ form a collection $\mathcal{C}(G)$ and every edge of G is cut by exactly λ cuts of $\mathcal{C}(G)$. This completes the proof. □

4. CONCLUSION

In this study, we investigated the l_1-embeddability of the gate-sum graph of two l_1-graphs. We have shown that the gate-sum graph of two l_1-graphs G_1 and G_2 is still an l_1-graph.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary materials, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

GW contributed the conception of gate-sum of the study. GW and CL contributed to the convex cuts of the gate-sum of two l_1-graphs. CL and FW organized the literature. FW performed the design of figures. All authors contributed to manuscript revision and read and approved the submitted version.

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REFERENCES

1. Graham RL, Pollak HO. On the addressing problem for loop switching. *Bell Syst Tech J.* (1972) **50**:2495–519. doi: 10.1002/j.1538-7305.1971.tb02618.x
2. Blake I, Gilchrist J. Addresses for graphs. *IEEE Trans Inf Theory.* (1973) **19**:683–8. doi: 10.1109/TIT.1973.1055087
3. Deza M, Laurent M. *Geometry of Cuts and Metrics.* Berlin: Springer-Verlag (1997).
4. Diestel R. *Graph Theory*. Heidelberg: Springer-Verlag (2006).
5. Bandelt H, Chepoi V. Decomposition and l_1-embedding of weakly median graphs. *Eur J Combin.* (2000) **21**:701–14. doi: 10.1006/eujc.1999.0377
6. Hammack R, Imrich W, Klavžar S. *Handbook of Product Graphs*. Bosa Roca: CRC Press (2011).
7. Bondy J, Murty U. *Graph Theory.* New York, NY: Springer-Verlag (2008).
8. Assouad P, Deza M. Espaces metriques plongeables dans un hypercube: aspects combinatories. *Ann Discret Math.* (1980) **8**:197–210. doi: 10.1016/S0167-5060(08)70874-4
9. Shpectorov S. On scale embeddings of graphs into hypercubes. *Eur J Combin*. (1993) **14**:117–30.
10. Deza M, Grishukhin V. Hypermetric graphs. *Q J Math.* (1993) **44**:393–433. doi: 10.1093/qmath/44.4.399
11. Deza M, Laurent M. l_1-rigid graphs. *J Algebr Combin*. (1994) **3**:153–75.
12. Deza M, Tuma J. A note on l_1-rigid planar graphs. *Eur J Combin*. (1996) **17**:157–60. doi: 10.1006/eujc.1996.0014
13. Chepoi V, Deza M, Grishukhin V. Clin d'oeil on l_1-embeddable planar graphs. *Discret Appl Math*. (1997) **80**:3–19. doi: 10.1016/S0166-218X(97)00066-8
14. Wang G, Zhang H. l_1-embeddability under the edge-gluing operation on graphs. *Discret Math*. (2013) **313**:2115–8. doi: 10.1016/j.disc.2013.04.032
15. Deza M, Grishukhin V, Shtogrin M. *Scale-Isometric Polytopal Graphs in Hypercubes and Cubic Lattices*. London: Imperical College Press (2004).
16. Biggs N. *Algebraic Graph Theory*. Cambridge, UK: Cambridge University Press (1993).
17. Godsil C, Royle G. *Algebraic Graph Theory*. New York, NY: Springer-Verlag (2001).

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