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Finite-Time Neural Network Backstepping Control of an Uncertain Fractional-Order Duffing System With Input Saturation

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In this paper, neural network (NN) control of the fractional-order Duffing system (FODS) by using a backstepping method within finite time in the presence of input saturation has been investigated. A fractional-order filter with an order lying on the interval (1,2) was used to estimate the virtual input together with its fractional derivative, and this showed that the estimation error tends to a small region in some finite time. Fractional-order law is designed for the parameter of the NN, and an adaptive NN controller was given. The proposed method drives the tracking error, tending to an arbitrary small region within a finite time. The simulation results verify the validity of the proposed method.

Keywords: finite time control, fractional-order system, fractional filter, adaptive neural network control, chaos control

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1. INTRODUCTION

It is a well-known fact that classical differential operators are local operators and cannot describe some complex properties. For example, Brownian motion, viscoelastic materials, anomalous diffusion, and irregular fluctuations in the turbulent velocity field have memory problems. Fractional-order differential operators are non-local and can well-characterize memory, genetic, and global correlation in the real world. The physical process is an important tool for describing physical processes and complex mechanics [1, 2]. In fact, fractional derivatives exhibit several advantages over integer derivatives: (1) fractional derivatives have a global correlation and can reflect the historical dependence of function development in the system; (2) the fractional derivative model is more consistent with the experimental results when simulating some complex properties, and the effect is better; and (3) when simulating complex mechanics and physical process problems, the expression of fractional-order model is more concise and the meaning is clearer [3, 4]. In view of these three advantages of fractional derivative, scholars have gradually used fractional differential equations to describe some practical problems. In recent decades, fractional calculus and fractional differential equations have developed rapidly and have gradually matured, and they have also been applied in other disciplines, such as quantum mechanics, economics and finance, turbulence, viscoelasticity theory, and superconductivity. A large number of papers on fractional calculus and fractional differential equations and works and so on have appeared [5–10]. The research contents include the theory and application of fractional calculus, the existence and uniqueness of solutions to the Cauchy problem, stability, controllability, the existence and uniqueness of solutions to boundary value problems, analytical solutions, and numerical algorithms. However, some research methods in integer-order differential equations cannot be directly applied to the study of fractional-order differential equations, and new theories and methods need to be sought. There are many research fields for integer-order differential equations, and research fields for fractional-order differential equations are limited, and the solution mapping of fractional-order differential equations does not have a semigroup property. Therefore, there are many difficulties in the study of fractional differential equations.

On the other hand, it is well-known that chaos control is a research hot topic and has some successful applications. With the in-depth study of chaotic systems, people began to try to migrate the synchronization method of integer-order chaotic systems to the synchronization of fractional-order chaotic systems (FOCSs). This natural idea is not easy to implement. For this reason, some people try to use the Laplace transform method and time-frequency domain transformation method. By solving the s^{α} by using the Laplace transform method, finite time control was investigated in Tavazoei and Haeri [5]. Up to now, many control methods have been used to control or synchronize FOCSs, for example, adaptive robust control, adaptive fuzzy control (AFC), adaptive neural network control (ANNC), sliding mode control (SMC), command filtered control (CFC), etc. [11–18]. In Pham et al. [19], a three-dimensional FOCS that had no equilibrium was introduced and investigated, and it was shown that the system shows chaotic phenomenon when the order <2.7. In Zhang et al. [20], the lag projective synchronization of FOCSs with time-varying delays was considered by using a comparison principle of linear fractional equation. In Liu et al. [16], the NN was used to control FOCSs in the presence of input faults. It should be mentioned that above works do not consider the finite time stability is. Up to now, the finite time control of the FOCS has rarely been investigated [21–23].

Inspired with above discussion, we will address the finite time NN control of the fractional-order Deffuing system (FODS) with input saturation. Take some related works, such as Liu et al. [16] and Ha et al. [24, 25], the our work has included several features: (1) a fractional-order filter whose order lies on (1,2), designed to evaluate the immediate controller and its fractional derivative within some finite time. However, a fractional filter was also used in Liu et al. [16] and Ha et al. [25] whose order lies on (0, 1), and, in addition, the finite-time stability cannot be guaranteed; (2) to cancel the estimation error of the filter, a fractional-order compensated signal was proposed. Compared with the compensated signals proposed in Ha et al. [24], our method can obtain a more rapid convergence; and (3) in the FOCS's mode, we have considered the case of input saturation.

2. PRELIMINARIES

2.1. Description of the NN

The NN with three layers is expressed as

$$y_j(s,\mu_j) = \sum_{\eta=1}^{h} \omega_{j\eta}\varphi_{j\eta}\left(\sum_{i=1}^{n} \nu_{\eta i}s_i + \gamma_\eta\right) = \mu_j^T\chi_j(\cdot), \tag{1}$$

where $n, h, and m \in \mathbb{N}_+$ denote the amount of neurons three layers (input, middle, and output), $\mu_j = \begin{bmatrix} \omega_{j1} \\ \vdots \\ \omega_{jh} \end{bmatrix}$, and $\chi_j = \begin{bmatrix} \varphi_{j1}\left(\sum_{i=1}^{n} \nu_{1i}s_i + \gamma_1\right) \\ \vdots \\ \varphi_{jh}\left(\sum_{i=1}^{n} \nu_{hi}s_i + \gamma_h\right) \end{bmatrix}$. ν_{ji} denotes a weight whose value is on the interval $[-1, 1]$. Usually, $\varphi(\cdot)$ can be defined by

$$\varphi(\hbar) = \frac{e^{\hbar} - e^{-\hbar}}{e^{\hbar} + e^{-\hbar}}. \tag{2}$$

Then, the NN is given as

$$y = \theta^T\chi(\hbar) \tag{3}$$

with $\theta = \begin{bmatrix} \mu_1^T \\ \mu_2^T \\ \vdots \\ \mu_m^T \end{bmatrix}$, and $\chi(\hbar) = \begin{bmatrix} \chi_1(\hbar) & 0 & \cdots & 0 \\ 0 & \chi_2(\hbar) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \chi_m(\hbar) \end{bmatrix}$.
Suppose that $f(\hbar), \hbar \in \mathbb{R}^n$ is unknown, then it can be approximated by the NN as

$$f(x) = \theta^{*T}\chi(\hbar) + \varepsilon(\hbar), \tag{4}$$

with $\varepsilon(\hbar)$ denoting the optimal approximation error, where

$$\theta^* = \arg\min_{\theta}\left[\sup|\hat{f}(\hbar) - f(\hbar)|\right], \tag{5}$$

with $\hat{f}(\hbar) = \theta^T(t)\chi(\hbar)$.

2.2. Basic Lemmas

The q-th fractional integral for a function $g(t)$ is defined as

$$\mathcal{I}^q g(t) = \frac{1}{\Gamma(q)}\int_0^t \frac{g(\tau)}{(t-\tau)^{1-q}}d\tau, \tag{6}$$

with $\Gamma(\cdot)$ representing Euler's function, and the q-th fractional-order derivative for a function $g(t)$, which has a k-th continuous derivative, is

$$\mathcal{D}^q g(t) = \frac{1}{\Gamma(k-q)}\int_0^t \frac{g^{(k)}(\tau)}{(t-\tau)^{q+1-k}}d\tau, \tag{7}$$

where $k-1 \le q < k$ $(k \in \mathbb{N})$. The following always assumes that $0 < q \le 1$ for convenience. The fractional-order calculus has the following properties.

Lemma 1. *[16] For a smooth function $x(t)$, it holds that*

$$\frac{1}{2}\mathcal{D}^{\alpha}x^2(t) \le x(t)\mathcal{D}^{\alpha}x(t). \tag{8}$$

Lemma 2. *[22] Let $V(\zeta)$ be a smooth function satisfying $\mathcal{D}_t^q V(\zeta) + \alpha_1 V^{\alpha_2}(\zeta) \le 0$, $\zeta \in \Omega_1 \subset \mathbb{R}^n$, $\alpha_1 \in \mathbb{R}^+$, and $0 < \alpha_2 < 1$. Then, one can find $\Omega_2 \subset \mathbb{R}^n$, which holds that $V(\zeta)$ begins within Ω_2 will reach a sufficient small region in some finite time T^*.*

Lemma 3. *[23] Assume $g_1, g_2 > 0$, $0 < g_3 < 1$, and*

$$\mathcal{D}_t^q V(\zeta) + g_1 V(\zeta) + g_2 V^{g_3}(\zeta) \le 0$$

where $\zeta \in \mathbb{R}^n$. Then, the system is finite time stable.

Lemma 4. *[23] Consider*

$$\begin{cases} \mathcal{D}_t^q \pi_1(t) = F(t), \\ \quad F(t) = -g_4 \mathrm{sign}(\pi_1(t) - \zeta(t)) + \pi_2(t) \\ \mathcal{D}_t^q \pi_2(t) = -g_5 \mathrm{sign}(\pi_2(t) - F(t)), \end{cases}$$

where $\zeta(t) \subset \mathbb{R}$, $g_4, g_5 \in \mathbb{R}^+$. *Let*

$$\hat{e}_1 = \pi_1 - \zeta, \quad \hat{e}_2(t) = F - \mathcal{D}_t^q \zeta.$$

Then, $\hat{e}_1$ *and* $\hat{e}_2$ *are finite time stable.*

3. MAIN RESULTS

The integer-order Duffing system is written as

$$\ddot{y}(t) - y(t) + a\dot{y}(t) + y^3(t) = b\cos(\omega t) \tag{9}$$

where a, b are parameters. Denote $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$, $x(t) = [x_1(t), x_2(t)]^T$ and $f(x(t)) = x_1(t) - ax_2(t) - x_1^3(t) + b\cos(\omega t)$. By putting the fractional calculus into system (9) and considering the input saturation, the controlled FODS is written as

$$\begin{cases} \mathcal{D}_t^q x_1(t) = x_2(t), \\ \mathcal{D}_t^q x_2(t) = \mathrm{sat}(u(t)) + d(t) + f(x(t)). \end{cases} \tag{10}$$

in which $\mathrm{sat}: u(t) \rightarrow \mathrm{sat}(u(t))$ is called a saturator. It can be expressed as:

$$\mathrm{sat}(u(t)) = \begin{cases} u_r, & u \geq u_r \\ u(t), & u_l < u(t) < u_r \\ u_l, & u \leq u_l, \end{cases} \tag{11}$$

with $u_r > 0, u_l < 0$. Denoting the term that exceeds the saturation limiter as $\gamma(t)$:

$$\gamma(t) = \begin{cases} u_r - u(t), & u(t) \geq u_r, \\ 0, & u_l < u(t) < u_r, \\ u_l - u(t), & u(t) \leq u_l. \end{cases} \tag{12}$$

For the target, let $x_1(t)$ track a known smooth signal $x_d(t) \in \mathbb{R}$ in finite time. In this paper, we have used the backstepping method. Define $e_1(t) = x_1(t) - x_d(t)$, and let us construct a virtual input $\alpha(t)$, giving us

$$\begin{aligned} \mathcal{D}_t^q e_1(t) &= x_2(t) - \mathcal{D}_t^q x_d(t) \\ &= \alpha(t) + \hat{\alpha}(t) - \alpha(t) + x_2(t) - \hat{\alpha}(t) - \mathcal{D}_t^q x_d(t) \\ &= \alpha(t) + \hat{\alpha}(t) - \alpha(t) + e_2(t) - \mathcal{D}_t^q x_d(t) \end{aligned} \tag{13}$$

with $e_2(t) = x_2(t) - \hat{\alpha}(t)$, and $\hat{\alpha}(t)$ being $\alpha(t)$'s estimation. Noting the estimation error is hard to be canceled, we have designed a compensated signal to solve this problem. Let

$$\mathcal{D}_t^q \beta_1(t) = -k_1\beta_1(t) + \hat{\alpha}(t) - \alpha(t) + \beta_2(t) - c_1 \mathrm{sign}(\beta_1(t)), \tag{14}$$

where $\beta_2(t)$ is given later, $k_1, c_1 > 0$, and $\beta(0) = 0$. Using Lemma 4, we can estimate $\alpha(t)$ and $\mathcal{D}_t^q \alpha(t)$ as

$$\begin{cases} \mathcal{D}_t^q \pi_1(t) = F(t), \\ \quad F(t) = -b_1 \mathrm{sign}(\pi_1(t) - \alpha(t)) + \pi_2(t) \\ \mathcal{D}_t^q \pi_2(t) = -b_2 \mathrm{sign}(\pi_2(t) - F(t)). \end{cases} \tag{15}$$

Thus, (15) and Lemma 4 imply that $\hat{\alpha}(t) = \pi_1(t)$ and $\mathcal{D}_t^q \hat{\alpha}(t) = F(t)$ within finite time. Let

$$\begin{cases} \tilde{e}_1(t) = e_1(t) - \beta_1(t), \\ \tilde{e}_2(t) = e_2(t) - \beta_2(t), \end{cases} \tag{16}$$

where $e_2(t) = x_2(t) - \hat{\alpha}(t)$. Then the victual signal is designed as

$$\alpha(t) = -k_1 e_1(t) + \mathcal{D}_t^q x_d(t) - a_1 \tilde{e}_1^{\nu}(t), \tag{17}$$

with $k_1 \in \mathbb{R}^+$, $\nu \in (0,1)$. Define $V_1 = \frac{1}{2}\tilde{e}_1^2(t)$, according to Lemma 1, and its fractional-order derivative is

$$\begin{aligned} \mathcal{D}_t^q V_1 \leq\ & \tilde{e}_1(t)\mathcal{D}_t^q \tilde{e}_1(t) \\ =\ & \tilde{e}_1(t)\Big[\alpha(t) + \hat{\alpha(t)} - \alpha(t) + e_2(t) \\ & - \mathcal{D}_t^q x_d(t) + k_1\beta_1(t) - \hat{\alpha}(t) + \alpha(t) - \beta_2(t) \\ & + c_1 \mathrm{sign}(\beta_1(t))\Big] \\ =\ & \tilde{e}_1(t)\Big[- k_1\tilde{e}_1(t) - a_1\tilde{e}_1^{\nu}(t) + e_2(t) + k_1\beta_1(t) - \beta_2(t) \\ & + c_1 \mathrm{sign}(\beta_1(t))\Big] \\ =\ & -k_1\tilde{e}_1^2(t) - a_1\tilde{e}_1(t)\tilde{e}_1^{\nu}(t) + \tilde{e}_1(t)e_2(t) - \tilde{e}_1(t)\tilde{e}_2(t) \\ & + c_1\tilde{e}_1(t)\mathrm{sign}(\beta_1(t)) \\ =\ & -k_1\tilde{e}_1^2(t) - a_1\tilde{e}_1(t)\tilde{e}_1^{\nu}(t) + \tilde{e}_1(t)\tilde{e}_2(t) \\ & + c_1\tilde{e}_1(t)\mathrm{sign}(\beta_1(t)). \end{aligned} \tag{18}$$

It follows from (10), (11), (12), and (16) that

$$\begin{aligned} \mathcal{D}_t^q \tilde{e}_2(t) &= \mathrm{sat}(u(t)) + d(t) + f(x(t)) - \mathcal{D}_t^q \hat{\alpha}(t) - \mathcal{D}_t^q \beta_2 \\ &= u(t) + \gamma(t) + d(t) + f(x(t)) - \mathcal{D}_t^q \hat{\alpha}(t) - \mathcal{D}_t^q \beta_2 \\ &= u(t) + \Theta(t) - \mathcal{D}_t^q \hat{\alpha}(t) - \mathcal{D}_t^q \beta_2 \end{aligned} \tag{19}$$

with $\Theta(t) = \gamma(t) + d(t) + f(x(t))$, $\mathcal{D}_t^q \hat{\alpha}(t)$ being driven from (15). The unknown function $\Theta(x)$ in (19) can be approximated by the NN as

$$\hat{\Theta}(t) = \theta^T(t)\chi(x(t)). \tag{20}$$

Let the optimal parameter of NN be $\theta^* = \arg\min_{\theta(t)} \left[\sup_{x(t)} \left|\hat{\Theta}(t) - \Theta(t)\right|\right]$. Define $\tilde{\theta}(t) = \theta(t) - \theta^*$, and $\epsilon(t) = \hat{\Theta}(t) - \Theta(t)$. In fact, according to universal approximation theorem of the NN, we know that, for any continuous non-linear function defined on a compact set, there is a NN in order for the optimal to be as small as possible [16, 26, 27]. Thus, it is possible for us to assume the optimal estimation error is bounded, i.e., $|\epsilon| \leq \bar{\epsilon}$, where $\bar{\epsilon} \in \mathbb{R}^+$ is a constant. We then have

$$\hat{\Theta}(t) - \Theta(t) = \theta(t)^T \chi(x(t)) - \theta(t)^{*T} \chi(x(t)) + \theta^{*T} \chi(x(t)) - \Theta(t) = \tilde{\theta}^T(t)\chi(x(t)) - \epsilon_i(t). \tag{21}$$

To meet the control objective, we can design the compensated signal as

$$\mathcal{D}_t^q \beta_2(t) = -k_2\beta_2(t) - \beta_1(t) - c_2\text{sign}(\beta_2(t)) \tag{22}$$

with $k_2, c_2 > 0$. Then, let us construct the final input as

$$u(t) = -k_2 e_2(t) + \mathcal{D}_t^q \hat{\alpha}(t) - \hat{\theta}^T(t)\chi(x(t)) - \sigma\,\text{sign}(\tilde{e}_2(t)) - a_2\tilde{e}_2^\nu(t) \tag{23}$$

where $\sigma, a_2 > 0$, and $\sigma \geq \bar{\epsilon}$ can be satisfied. It follows from (22) and (23) into (19) that

$$\begin{aligned}
\mathcal{D}_t^q \tilde{e}_2(t) &= -k_2 e_2(t) - \hat{\theta}^T(t)\chi(x(t)) - \sigma\,\text{sign}(\tilde{e}_2(t)) \\
&\quad - a_2\tilde{e}_2^\nu(t) + \Theta(t) - \mathcal{D}_t^q \beta_2(t) \\
&= -k_2 e_2(t) - \tilde{\theta}^T(t)\chi(x(t)) + \epsilon(t) - \sigma\,\text{sign}(\tilde{e}_2(t)) \\
&\quad - a_2\tilde{e}_2^\nu(t) + k_2\beta_2(t) \\
&\quad - \tilde{\beta}_1(t) + c_2\text{sign}(\beta_2(t)) \\
&= -k_2\tilde{e}_2(t) - \tilde{\theta}^T(t)\chi(x(t)) + \epsilon(t) - \sigma\,\text{sign}(\tilde{e}_2(t)) \\
&\quad - a_2\tilde{e}_2^\nu(t) - \tilde{\beta}_1(t) + c_2\text{sign}(\beta_2(t)).
\end{aligned} \tag{24}$$

Then, (24) implies

$$\begin{aligned}
\tilde{e}_2(t)\mathcal{D}_t^q \tilde{e}_2(t) &= -k_2\tilde{e}_2^2(t) - \tilde{e}_2(t)\tilde{\theta}^T(t)\chi(x(t)) + \tilde{e}_2(t)\epsilon(t) \\
&\quad - \sigma\tilde{e}_2(t)\text{sign}(\tilde{e}_2(t)) - a_2\tilde{e}_2(t)\tilde{e}_2^\nu(t) \\
&\quad - \tilde{e}_2(t)\tilde{e}_1(t) + c_2\tilde{e}_2(t)\text{sign}(\beta_2(t)) \\
&\leq -k_2\tilde{e}_2^2(t) - \tilde{e}_2\tilde{\theta}^T(t)\chi(x)(t) + |\tilde{e}_2(t)|\bar{\epsilon} \\
&\quad - \sigma|\tilde{e}_2(t)| - a_2\tilde{e}_2(t)\tilde{e}_2^\nu(t) - \tilde{e}_2(t)\tilde{e}_1(t) \\
&\quad + c_2\tilde{e}_2(t)\text{sign}(\beta_2(t)) \\
&\leq -k_2\tilde{e}_2^2(t) - e_2(t)\tilde{\theta}^T(t)\chi(x(t)) - a_2\tilde{e}_2(t)\tilde{e}_2^\nu(t) \\
&\quad - \tilde{e}_2(t)\tilde{e}_1(t) + c_2\tilde{e}_2(t)\text{sign}(\beta_2(t)).
\end{aligned} \tag{25}$$

Define

$$V_2(t) = V_1(t) + \frac{1}{2}\tilde{e}_2^2(t). \tag{26}$$

According to (18), (25), and (26), we have

$$\begin{aligned}
\mathcal{D}_t^q V_2(t) &= -k_1\tilde{e}_1^2(t) - a_1\tilde{e}_1(t)\tilde{e}_1^\nu(t) + c_1\tilde{e}_1(t)\text{sign}(\beta_1(t)) \\
&\quad - k_2\tilde{e}_2^2(t) - \tilde{e}_2\tilde{\theta}^T\chi(x(t)) \\
&\quad - a_2\tilde{e}_2(t)\tilde{e}_2^\nu(t) + c_2\tilde{e}_2(t)\text{sign}(\beta_2(t)) = -\sum_{j=1}^{2} k_j\tilde{e}_j^2(t) \\
&\quad - \sum_{j=1}^{2} a_j\tilde{e}_j(t)\tilde{e}_j^\nu(t) + \sum_{j=1}^{2} c_j\tilde{e}_j(t)\text{sign}(\beta_j(t)) \\
&\quad - \tilde{e}_2\tilde{\theta}^T(t)\chi(x(t)) \\
&= \sum_{j=1}^{2}\Big[-k_j\tilde{e}_j^2(t) - a_j\tilde{e}_j^{\nu+1}(t) + c_j\tilde{e}_j(t)\text{sign}(\beta_j(t))\Big] \\
&\quad - \tilde{e}_2(t)\tilde{\theta}^T(t)\chi(x(t)).
\end{aligned} \tag{27}$$

The fractional-order adaptation law is

$$\mathcal{D}_t^q \theta(t) = \kappa_1\tilde{e}_2(t)\chi(x(t)) - \kappa_1\kappa_2\theta(t) \tag{28}$$

with $\kappa_1, \kappa_2 > 0$.

The following theorem provides a conclusion for the discussion.

Theorem 1. *Let the immediate controller be (17) with the fractional filter (15). Let the compensated signal be (14) and (22). Then, the NN controller (23) with adaptation law (28) drive $e_1(t)$ to be arbitrary small in finite time.*

Proof. Let

$$V(t) = V_2(t) + \frac{1}{2\kappa_1}\tilde{\theta}^T(t)\tilde{\theta}(t). \tag{29}$$

Then, based on (27), (28), and (29), we obtain

$$\begin{aligned}
\mathcal{D}_t^q V(t) &\leq \textstyle\sum_{j=1}^{2}\Big[-k_j\tilde{e}_j^2(t) - a_j\tilde{e}_j^{\nu+1}(t) + c_j\tilde{e}_j(t)\text{sign}(\beta_j(t))\Big] \\
&\quad - e_2\tilde{\theta}^T(t)\chi(x(t)) + \tfrac{1}{\kappa_1}\tilde{\theta}^T(t)\mathcal{D}_t^q\theta(t) \\
&= \textstyle\sum_{j=1}^{2}\Big[-k_j\tilde{e}_j^2(t) - a_j\tilde{e}_j^{\nu+1}(t) + c_j\tilde{e}_j(t)\text{sign}(\beta_j(t))\Big] \\
&\quad - \kappa_2\tilde{\theta}^T(t)\theta(t) \\
&\leq \textstyle\sum_{j=1}^{2}\Big[-\tfrac{2k_j-c_j}{2}\tilde{e}_j^2(t) - a_j\tilde{e}_j^{\nu+1}(t)\Big] - \kappa_2\tilde{\theta}^T(t)\theta(t) \\
&\quad + \textstyle\sum_{j=1}^{2}\tfrac{c_j}{2} \\
&= \textstyle\sum_{j=1}^{2}\Big[-\tfrac{2k_j-c_j}{2}\tilde{e}_j^2(t) - a_j\tilde{e}_j^{\nu+1}(t)\Big] - \kappa_2\tilde{\theta}^T(t)(\tilde{\theta}(t) \\
&\quad + \theta^*(t)) + \textstyle\sum_{j=1}^{2}\tfrac{c_j}{2} \\
&\leq \textstyle\sum_{j=1}^{2}\Big[-\tfrac{2k_j-c_j}{2}\tilde{e}_j^2(t) - a_j\tilde{e}_j^{\nu+1}(t)\Big] - \tfrac{3\kappa_2}{4}\tilde{\theta}^T(t)\tilde{\theta}(t) \\
&\quad + \textstyle\sum_{j=1}^{2}\tfrac{c_j}{2} + \kappa_2\theta^{*T}\theta^*(t).
\end{aligned} \tag{30}$$

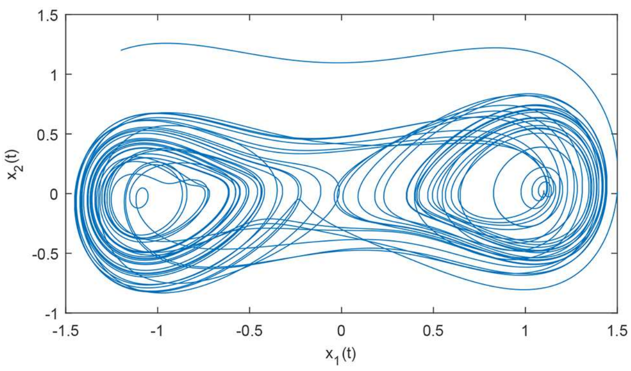

FIGURE 1 | Chaotic phenomenon of uncontrolled FODS (10).

Then, (30) implies

$$\begin{aligned}\mathcal{D}_t^q V(t) \leq \sum_{j=1}^{2}\left[-\frac{2k_j - c_j}{2}\tilde{e}_j^2(t) - a_j\tilde{e}_j^{\nu+1}(t)\right] \\ - \left(\frac{\kappa_2}{2}\tilde{\theta}^T(t)\tilde{\theta}(t)\right)^{\frac{1}{2}(\nu+1)} - \frac{3\kappa_2}{4}\tilde{\theta}^T(t)\tilde{\theta}(t) \\ + \kappa_2\theta^{*T}\theta_i^* + \sum_{j=1}^{2}\frac{c_j}{2} + \left(\frac{\kappa_2}{2}\tilde{\theta}^T(t)\tilde{\theta}(t)\right)^{\frac{1}{2}(\nu+1)}.\end{aligned} \tag{31}$$

If $\left(\frac{\kappa_{2i}}{2}\tilde{\theta}_i^T\tilde{\theta}_i\right)^{\frac{1}{2}(\nu+1)} \geq 1$, it is easy to know that

$$\begin{aligned}\left(\frac{\kappa_{2i}}{2}\tilde{\theta}^T(t)\tilde{\theta}(t)\right)^{\frac{1}{2}(\nu+1)} - \frac{\kappa_2}{2}\tilde{\theta}^T(t)\tilde{\theta}(t) + \kappa_2\theta^{*T}\theta^* \\ \leq \frac{\kappa_2}{2}\tilde{\theta}^T(t)\tilde{\theta}(t) - \frac{\kappa_2}{2}\tilde{\theta}^T(t)\tilde{\theta}(t) + \kappa_2\theta^{*T}\theta^* \\ = \kappa_2\theta^{*T}\theta^*.\end{aligned} \tag{32}$$

On the contrary, if $\left(\frac{\kappa_2}{2}\tilde{\theta}^T(t)\tilde{\theta}(t)\right)^{\frac{1}{2}(\nu+1)} < 1$, one has

$$\begin{aligned}\left(\frac{\kappa_2}{2}\tilde{\theta}^T\tilde{\theta}\right)^{\frac{1}{2}(\nu+1)} - \frac{\kappa_2}{2}\tilde{\theta}^T\tilde{\theta} + \kappa_2\theta^{*T}\theta^* \\ < 1 - \frac{\kappa_2}{2}\tilde{\theta}^T(t)\tilde{\theta}(t) + \kappa_2\theta^{*T}\theta^* \leq 1 + \kappa_2\theta^{*T}\theta^*.\end{aligned} \tag{33}$$

Thus, it follows from (32) and (33) that

$$\left(\frac{\kappa_2}{2}\tilde{\theta}^T(t)\tilde{\theta}(t)\right)^{\frac{1}{2}(\nu+1)} - \frac{\kappa_2}{2}\tilde{\theta}^T(t)\tilde{\theta}(t) + \kappa_2\theta^{*T}\theta^* \leq 1 + \kappa_2\theta^{*T}\theta^*. \tag{34}$$

Substituting (34) into (31) yields

$$\begin{aligned}\mathcal{D}_t^q V(t) \leq \sum_{j=1}^{2}\left[-\frac{2k_j - c_j}{2}\tilde{e}_j^2(t) - a_j\tilde{e}_j^{\nu+1}(t)\right] \\ - \left(\frac{\kappa_2}{2}\tilde{\theta}^T(t)\tilde{\theta}(t)\right)^{\frac{1}{2}(\nu+1)} - \frac{\kappa_2}{4}\tilde{\theta}^T(t)\tilde{\theta}(t) \\ + \left[1 + \kappa_2\theta^{*T}\theta^*\right] + \sum_{j=1}^{2}\frac{c_j}{2} \\ \leq -\varsigma_1 V - \varsigma_2 V^{\frac{\nu+1}{2}} + \varsigma_3\end{aligned} \tag{35}$$

with $\varsigma_1 = \min\left\{\frac{2k_1-c_1}{2}, \frac{2k_2-c_2}{2}, \frac{\kappa_{min}}{2}\right\}$, $\varsigma_2 = \min\left\{2^{\frac{\nu+1}{2}}a_1, 2^{\frac{\nu+1}{2}}a_2, \kappa_{min}^{\frac{\nu+1}{2}}\right\}$, and $\varsigma_3 = 1 + \kappa_2\theta^{*T}\theta^* + \sum_{j=1}^{2}\frac{c_j}{2}$, and $\kappa_{min} = \min\{\kappa_1, \kappa_2\}$. As a result, (35) can be arranged as

$$\mathcal{D}_t^q V(t) \leq -\left(\varsigma_1 - \frac{\varsigma_3}{2V(t)}\right)V(t) - \left(\varsigma_2 - \frac{\varsigma_3}{2V^{\frac{\nu+1}{2}}(t)}\right)V^{\frac{\nu+1}{2}}(t). \tag{36}$$

According to (36) and Lemma 3, when $k > \frac{1}{2}c$, $e_1(t)$ will tend to the region

$$|e_1(t)| \leq \max\left\{\sqrt{\frac{\varsigma_3}{\varsigma_1}}, \sqrt{2\left(\frac{\varsigma_3}{2\varsigma_2}\right)^{\frac{\nu+1}{2}}}\right\}$$

in some finite time. Since $e_1(t) = \tilde{e}_1(t) + \beta_1(t), e_2(t) = \tilde{e}_2(t) + \beta_2(t)$, if $\beta_1(t)$ and $\beta_2(t)$ are bounded, then all signals are bounded. Let $V_3(t) = \frac{1}{2}\beta_1^2(t) + \frac{1}{2}\beta_2^2(t)$. Then, (14) and (22) imply

$$\begin{aligned}\mathcal{D}_t^q V_3 \leq \beta_1(t)\mathcal{D}_t^q\beta_1(t) + \beta_2(t)\mathcal{D}_t^q\beta_2(t) \\ = -k_1\beta_1^2(t) + \beta_1(t)(\hat{\alpha(t)} - \alpha(t)) + \beta_1(t)\beta_2(t) \\ - c_1\beta_1(t)\mathrm{sign}(\beta_1(t)) - k_2\beta_2^2(t) \\ - \beta_2(t)\beta_1(t) - c_2\beta_2(t)\mathrm{sign}(\beta_2(t)) \\ \leq -\sum_{j=1}^{2}k_j\beta_j^2(t) - \sum_{j=1}^{2}c_j\beta_j\mathrm{sign}(\beta_j(t)) + \beta_1(t)(\hat{\alpha}(t) \\ - \alpha(t)) \leq -\sum_{j=1}^{2}k_j\beta_j^2(t) - \sum_{j=1}^{2}c_j|\beta_j(t)| + \sum_{j=1}^{2}\rho_j\beta_j\tilde{\alpha}(t),\end{aligned} \tag{37}$$

where $\tilde{\alpha}(t) = \hat{\alpha}(t) - \alpha(t)$. Then, it follows from (15) and Lemma 4 that $\tilde{\alpha}(t)$ is bounded in finite time. As a result, we have

$$\begin{aligned}\mathcal{D}_t^q V_3(t) \leq -\sum_{j=1}^{2}k_j\beta_j^2(t) - \sum_{j=1}^{2}c_j|\beta_j(t)| + \sum_{j=1}^{2}\rho_j\delta|\beta_j(t)| \\ \leq \underline{k}V_3(t) - \underline{c}\sqrt{V_3(t)} + \bar{\rho}\bar{\delta}\sqrt{V_3(t)} \\ = \underline{k}V_3(t) - (\underline{c} - \bar{\rho}\bar{\delta})\sqrt{V_3(t)},\end{aligned} \tag{38}$$

where $\underline{k} = 2\min\{k_1, k_2\}$, $\underline{c} = \min\{c_1, c_2\}$, $\bar{\rho} = \max\{\rho_1, \rho_2\}$ and $\bar{\delta} = \max\{\delta_1, \delta_2\}$. Thus, (38) and Lemma 3 imply that $\beta_1(t)$ and $\beta_2(t)$ are finite time bounded. This concludes our proof. ■

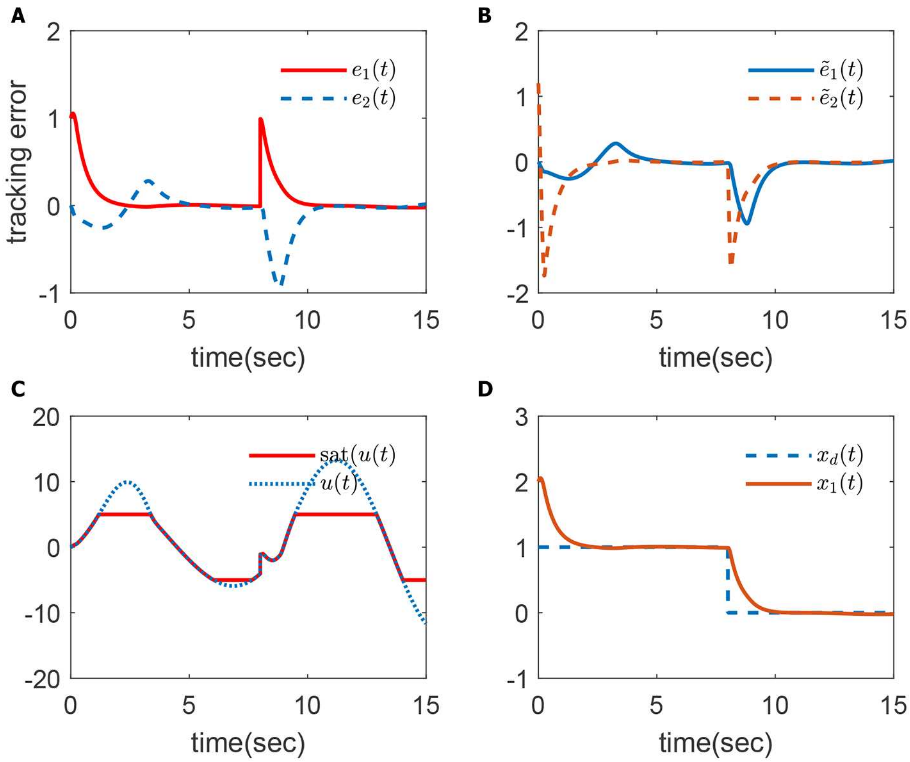

FIGURE 2 | Simulation1 results in **(A)** $e_1(t)$ and $e_2(t)$; **(B)** $\tilde{e}_1(t)$ and $\tilde{e}_2(t)$; **(C)** $u(t)$ and sat($u(t)$); **(D)** $x_d(t)$ and $x_1(t)$.

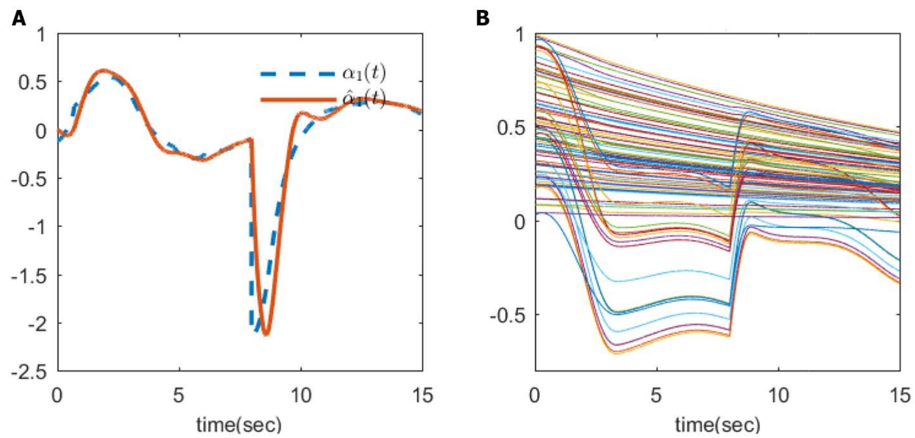

FIGURE 3 | Virtual input and NN parameters in **(A)** virtual inputs $\alpha_1(t)$ and $\alpha_2(t)$; **(B)** NN parameters.

Remark 1. *In this paper, the finite-control of fractional-order Duffing system was considered. It can be seen from the system model (10) that the non-linear function $f(x)$ is Lipschitz continuous. In addition, under the proposed controller (23), for any initial condition, the solution to the fractional-order Duffing system exists and is unique. In addition, from Theorem 1, it is obvious that all the signals in the closed loop system keep bounded. Thus, the solution of the controlled system (10) is stable.*

Remark 2. *In should be emphasized that the proposed fractional-order finite-time filter has very convergence ability compared with some related works, such as Liu et al. [16] and Ha et al. [25], where only the following class of lower filter (the order of the filter lying on (0,1)) is used:*

$$\mathcal{D}_t^q z(t) = \frac{1}{k}(z(t) - \alpha(t)), \tag{39}$$

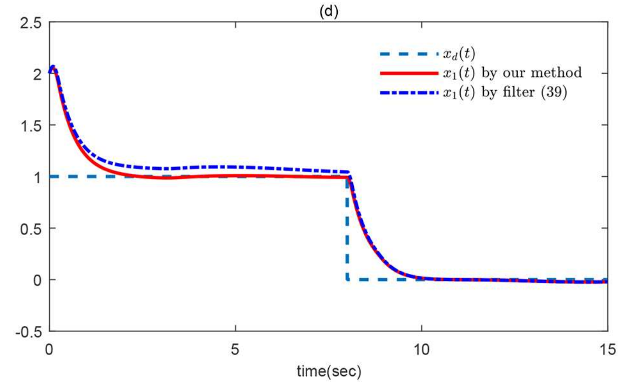

FIGURE 4 | Comparison results.

where $k > 0$. The fractional-order filter (39) can also guarantee that the approximation errors of the virtual input and its fractional converge to a small region of zero; however, the finite-time convergence cannot be guaranteed. In addition, to drive the approximation smaller, larger design parameter k should be used, which usually results in the signal $z(t)$ being too big. However, the proposed filter (15) has no such problems. To show the effectiveness of the proposed high order filter, some comparisons have been given in the following section.

4. SIMULATION RESULTS

In system (10), let parameters $a = 0.15, b = 0.23$, and the initial conditions be $x_1(0) = -1.2, x_2(0) = 1.2$. When $d(t) = u(t) = 0$, under above parameters and initial conditions, system (10) exhibits chaotic phenomenon, as shown in **Figure 1**.

In the simulation, let $x_1(0) = 2, x_2(0) = 0$, and let the reference signal be

$$x_d(t) = \begin{cases} 1, & t \leq 8, \\ 0, & t < 8. \end{cases} \tag{40}$$

The design parameters are $k_1 = k_2 = 0.9$; $a_1 = a_2 = 1, c_1 = c_2 = 1, \kappa_1 = \kappa_2 = 1, \nu = 0.70, b_1 = b_2 = 1$. The NN uses $x_1(t), x_2(t)$ as input variables with $\theta(0) = 0 \in \mathbb{R}^{81}$. The saturation parameters are $u_l = -5, u_r = 5$.

Then, the simulation results can be seen in **Figures 2–4**. The tracking errors $e_1(t)$ and $e_2(t)$ are given in **Figure 2A**, and we can see that the tracking error converges quickly. The compensated tracking errors are given in **Figure 2B**, where the proposed filter has very good approximation ability. The control input is given in **Figure 2C**. The tracking performance is in **Figure 2D**. The virtual input and its approximation is given in **Figure 3A**, and the NN parameters are shown in **Figure 3B**.

To show the rapid convergence speed of the proposed high-order filter, some comparative simulation results will be given here. Noting that in Liu et al. [16] and Ha et al. [25], the lower filter (39) was used. The simulation results under our filter (15) and (39) are given in **Figure 4**; in order to make a fair comparison, the design parameters in (39) are taken as 0.9 just the same as the value we took above. Obviously, compared with the lower filter (39), our method can guarantee a quicker convergence speed.

5. CONCLUSIONS

This paper addressed the finite time control of an unknown disturbed FODS in the presence of input saturation. By using the backstepping technique, a high order fractional filter with the order lying on (1,2) is proposed, and thus, the virtual input and its fractional derivative can be approximated. It is proven that the filter's approximation error can be enough small and can converge to the small region in some finite time. Then, an adaptive NN controller is given. The stability is proven strictly. In addition, the robustness of the proposed method is shown in simulation results. Our future research directions including how to design sliding mode surface for FODS and how to construct a high-order filter.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

HL and XZ contributed the conception of the study. HL wrote the study and organized the literature. XZ wrote the simulation programs. All authors contributed to the manuscript revision, read, and approved the submitted version.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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