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Helicoidal Surfaces in Galilean Space With Density

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In this paper, we construct helicoidal surfaces in the three dimensional Galilean space G^3. The First and the Second Fundamental Forms for such surfaces will be obtained. Also, mean and Gaussian curvature given by smooth functions will be derived. We consider the Galilean 3−space with a linear density e^{ϕ} and construct a weighted helicoidal surfaces in G^3 by solving a second order non-linear differential equation. Moreover, we discuss the problem of finding explicit parameterization for the helicoidal surfaces in G^3.
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1. INTRODUCTION

Due to its applications in probability and statistics, the study of manifolds with density has increased in the last years after Morgan's published his paper "Manifolds with density" [1]. As a new field in geometry, manifolds with density appear in different ways in mathematics, for example as quotients of Riemannian manifolds or as Gauss space [2].

Helicoidal surface is a natural generalization of rotation surface, of which many excellent works have been done, such as Kenmotsu [3].

For helicoidal surface in R^3, the cases with prescribed mean curvature or Gauss curvature have been studied by Baikoussis and Koufogiorgos [4]. Also, helicoidal surfaces in three dimensional Minkowski space has been considered by Beneki et al. [5]. A kind of helicoidal surface in 3−dimensional Minkowski space was constructed by Ji and Hou [6].

Construction of helicoidal surfaces in Euclidean space with density by solving second-order non-linear ordinary differential equation with weighted minimal helicoidal surface was introduced in Kim et al. [7]. For weighted type integral inequalities, one can see Agarwal et al. [8].

Mean and Gaussian curvature for surfaces are one of the main objects, which have geometers interest for along time. A manifold with density is a Riemannian manifold M^n with a positive function e^{ϕ}, known as density, used to weight volume and hypersurface area [2, 9]. A nice example of manifolds with density is Gauss space, the Euclidean space with Gaussian probability density $(2\pi)^{\frac{-n}{2}}\, e^{\frac{-r^2}{2}}$, which is very useful to probabilists [2].

On a manifold with density e^{ϕ}, the weighted mean curvature of a hypersurface with unit normal N is defined by

$$H_{\phi} = H - \frac{1}{n}\frac{d\phi}{dN} \tag{1}$$

where H is the Riemannian mean curvature of the hypersurface [9]. The weighted mean curvature H_{ϕ} of a surface in E^3 with density e^{ϕ} was introduced by Gromov [10], and it is a natural generalization of the mean curvature H of a surface. The curvature concept is not confined to continuous space, it has been intensively studied in discrete mathematics including networks, for more details one can see Shang [11].

A surface with $H_\phi = 0$ is known as a weighted minimal surface or a $\phi-$minimal surface in E^3 [12]. For more details about manifolds with density and other relative topics, we refer the reader to [1–3, 5–7, 9, 10, 13–16]. In particular, Yoon et al. [17] studied rational surfaces in Pseudo-Galilean space with a log-linear density and investigated $\phi-$minimal rotational surfaces. Also, they classified the weighted minimal helicoidal surfaces in the Euclidean space E^3 [7].

The purpose of this paper is to construct helicoidal surface in Galilean space G^3. Firstly, we choose orthonormal basis as the coordinate frame and define helicoidal surface with density. The first fundamental form ds^2, the second fundamental form II, the Gaussian and Mean curvature of helicoidal surface will be obtained in section 3. Secondly in section 4, we prescribed the parametrization of a weighted mean curvature $H_\phi = H - \frac{1}{2} < N, \nabla\phi >$ and solving the non-linear differential equation.

2. PRELIMINARIES

In this part, we give a brief review of curves and surfaces in the Galilean space G^3. For more details, one can see [12, 14–16, 18].

The Galilean 3−space G^3 can be defined in the three-dimensional real projective space $P_3(R)$ and its *absolute figure* is an ordered triple $\{\rho, f, I\}$, where ρ is the ideal (absolute) plane, f a line in ρ and I is the fixed elliptic involution of the points of f. We introduce homogeneous coordinates in G^3 in such a way that the absolute plane ρ is given by $x_o = 0$, the absolute line f by $x_o = x_1 = 0$ and the elliptic involution by

$$(0:0:x_2:x_3) \rightarrow (0:0:x_3:-x_2) \tag{2}$$

A plane is said to be Euclidean if it contains f, otherwise it is called *isotropic*, i.e., the planes $x = const.$ are Euclidean, in particular the plane ρ. Other planes are isotropic.

The *Galilean distance* between the points $Q_i = (r_i, s_i, t_i)$, $(i = 1, 2)$ is given by

$$d(Q_1, Q_2) = \begin{cases} \mid r_2 - r_1 \mid, & \text{if } r_1 \neq 0 \text{ or } r_2 \neq 0; \\ \sqrt{(s_2 - s_1)^2 + (t_2 - t_1)^2}, & \text{if } r_1 = 0 \text{ and } r_2 = 0. \end{cases} \tag{3}$$

Moreover, the distance in the Euclidean space between Q_1 and Q_2 is given by

$$d(Q_1, Q_2) = \sqrt{(r_2 - r_1)^2 + (s_2 - s_1)^2 + (t_2 - t_1)^2}$$

The *Galilean scalar product* between two vectors $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ is defined by

$$< P, Q >_G = \begin{cases} p_1 q_1, & \text{if } p_1 \neq 0 \text{ or } q_1 \neq 0; \\ p_2 q_2 + p_3 q_3, & \text{if } p_1 = 0 \text{ and } q_1 = 0. \end{cases} \tag{4}$$

For this, the *Galilean norm* of a vector P is $\parallel P \parallel = \sqrt{< P, P >_G}$. Moreover, the *cross product* in the Galilean space is defined by

$$< P \times_G Q > = \left(0, \begin{vmatrix} p_1 & p_3 \\ q_1 & q_3 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}\right)$$

A vector $P = (p_1, p_2, p_3)$ is said to be *isotropic* if $p_1 = 0$, otherwise it is known as *non-isotropic*. The following definitions will be helpful [19].

Definition 1. Let $a = (1, y_2, y_3)$ and $b = (1, z_2, z_3)$ be two unit non-isotropic vectors in general position in G^3. Then an angle θ between a and b is given by

$$\theta = \sqrt{(z_2 - y_2)^2 + (z_3 - y_3)^2}$$

Definition 2. An angle ψ between a unit non-isotropic vector $a = (1, y_2, y_3)$ and an isotropic vector $c = (0, z_2, z_3)$ in G^3 is given by

$$\psi = \frac{y_2 z_2 + y_3 z_3}{\sqrt{z_2^2 + z_3^2}}$$

Definition 3. An angle η between two isotropic vectors $c = (0, y_2, y_3)$ and $d = (0, z_2, z_3)$ parallel to the Euclidean plane in G^3 is equal to the Euclidean angle between them. Namely,

$$cos\eta = \frac{y_2 z_2 + y_3 z_3}{\sqrt{y_2^2 + y_3^2}\sqrt{z_2^2 + z_3^2}}$$

Definition 4. The curve $\alpha(t) = (x(t), y(t), z(t))$ in the Galilean space G^3 is said to be admissible if it has no inflection points $(\dot\alpha(t) \times_G \ddot\alpha(t) \neq 0)$ and no isotropic tangents $(\dot x(t) \neq 0)$.

Let C be an open subset of R^2 and M a surface in G^3 parameterized by

$$r : C \subseteq R^2 \rightarrow G^3,\ r(u, v) = (x(u, v), y(u, v), z(u, v)) \tag{5}$$

In order to specify the partial derivatives we will denote:

$$x_u = \frac{\partial x}{\partial u},\ x_v = \frac{\partial x}{\partial v}\ \ and\ \ x_{uv} = \frac{\partial^2 x}{\partial u \partial v} \tag{6}$$

Then r is satisfied *admissibility criteria* if no where it has Euclidean tangent planes. The first fundamental form is given by

$$ds^2 = (g_1 du + g_2 dv)^2 + \varepsilon(h_{11} du^2 + 2h_{12} du dv + h_{22} dv^2) \tag{7}$$

where $g_1 = x_u = \frac{\partial x}{\partial u}$, $g_2 = x_v = \frac{\partial x}{\partial v}$, and $h_{11} = y_u^2 + z_u^2$, $h_{22} = y_v^2 + z_v^2$, $h_{12} = y_u y_v + z_u z_v$, also

$$\varepsilon = \begin{cases} 0\ , & \text{if the direction } du : dv \text{ is non-isotropic;} \\ 1\ , & \text{if the direction } du : dv \text{ is isotropic.} \end{cases} \tag{8}$$

Now, consider the function

$$\omega = \parallel r_u \times r_v \parallel = \sqrt{(x_u z_v - x_v z_u)^2 + (x_v y_u - x_u y_v)^2} \tag{9}$$

hence the isotropic unit normal vector field N of the surface $r = r(u, v)$ is given by

$$N = \frac{r_u \times r_v}{\parallel r_u \times r_v \parallel} = \frac{1}{\omega}(0,\ x_u z_v - x_v z_u,\ x_v y_u - x_u y_v) \tag{10}$$

The second fundamental form is obtained by

$$II = L_{11}du^2 + 2L_{12}dudv + L_{22}dv^2 \tag{11}$$

such that

$$\begin{aligned} L_{ij} &= \frac{1}{g_1}(g_1(0, y_{ij}, z_{ij}) - (g_i)_j(0, y_u, z_u)) \cdot N \\ &= \frac{1}{g_2}(g_2(0, y_{ij}, z_{ij}) - (g_i)_j(0, y_v, z_v)) \cdot N \end{aligned}$$

where $i, j = u, v$.

Note that the dot "·" denotes the Euclidean scalar product. Therefore, the *Gaussian* and *mean curvature* are given by.

$$K = \frac{L_{11}L_{22} - L_{12}^2}{\omega^2} \ \ and \ H = \frac{g_2^2L_{11} - 2g_1g_2L_{12} + g_1^2L_{22}}{2\omega^2} \tag{12}$$

3. HELICOIDAL SURFACES IN THE GALILEAN SPACE G^3

We will take a regular plane curve $\alpha(u_1) = (g(u_1), 0, f(u_1))$ with $g(u_1) > 0$ in the $xz-$ plane which is defined on an open interval $I \subset R$. A surface Γ^2 in the Galilean space G^3 is defined by

$$\chi(u_1, u_2) = (g(u_1)cos(u_2), g(u_1)sin(u_2), f(u_1) + bu_2) \tag{13}$$

is said to be helicoidal surface with axis oz, a pitch b and the profile curve α.

Without loss of generality, we assume that $\alpha(u_1) = (u_1, 0, f(u_1))$ is the profile curve in the $xz-$ plane defined on an open interval I of positive real numbers ($I \subset R^+$). So, the helicoidal surface Γ^2 in G^3 is given by

$$\chi(u_1, u_2) = (u_1cos(u_2), u_1sin(u_2), f(u_1) + bu_2) \tag{14}$$

where $f(u_1)$ is a differentiable function defined on I.

Theorem 5. *Let Γ^2 be helicoidal surface in G^3 defined by*

$$\chi(u_1, u_2) = (u_1cos(u_2), u_1sin(u_2), f(u_1) + bu_2) \tag{15}$$

where $f(u_1)$ is a differentiable function defined on I. Then the unit normal vector field N of the surface Γ^2 is given by

$$N = \frac{1}{\omega}(0, u_1 f'(u_1)\ sin(u_2) + b\ cos(u_2), -u_1) \tag{16}$$

The first and the second fundamental forms of the surface Γ^2 in G^3 are given respectively by

$$ds^2 = cos^2(u_2)du_1^2 - 2u_1sin(u_2)cos(u_2)du_1du_2 + u_1^2sin^2(u_2)du_2^2 \tag{17}$$

and

$$II = \frac{1}{\omega}(-u_1 f''(u_1)du_1^2 + 2b\ du_1\ du_2 - u_1^2 f'(u_1)du_2^2). \tag{18}$$

Proof: Let Γ^2 be helicoidal surface in G^3 defined by

$$\chi(u_1, u_2) = (u_1cos(u_2), u_1sin(u_2), f(u_1) + bu_2) \tag{19}$$

Then the unit normal vector field N of the surface Γ^2 is an isotropic vector field defined by

$$\begin{aligned} N &= \frac{1}{\omega}(0, x_{u_1}z_{u_2} - x_{u_2}z_{u_1}, x_{u_2}y_{u_1} - x_{u_1}y_{u_2}) \\ &= \frac{1}{\omega}(0, u_1 f'(u_1)\ sin(u_2) + b\ cos(u_2), -u_1) \end{aligned}$$

where the positive function ω is given by

$$\omega = \sqrt{(b\ cos(u_2) + u_1 f'(u_1)\ sin(u_2)^2 + u_1^2} \tag{20}$$

Here the partial derivatives of the functions $x, y,\ and\ z$ with respect to $u_i \ \ (i = 1, 2)$ are denoted by $x_{u_i},\ y_{u_i},\ and\ z_{u_i}$, respectively. On the other hand, let us define $g_i = x_{u_i}$, $h_{ij} = y_{u_i}y_{u_j} + z_{u_i}z_{u_j}$, $i, j = 1, 2$. So, the first fundamental form of the surface Γ^2 in G^3 is given by

$$ds^2 = ds_1^2 + \varepsilon ds_2^2 \tag{21}$$

where

$$\begin{aligned} ds_1^2 &= (g_1du_1 + g_2du_2)^2 \\ &= (cos(u_2)\ du_1 - u_1\ sin(u_2)\ du_2)^2 \end{aligned}$$

and

$$\begin{aligned} ds_2^2 &= h_{11}du_1^2 + 2h_{12}du_1du_2 + h_{22}du_2^2 \\ &= (sin^2(u_2) + f'^2(u_1))\ du_1^2 + 2(u_1sin(u_2)cos(u_2) \\ &+ bf'(u_1))du_1\ du_2 + (u_1cos^2(u_2) + b^2)\ du_2^2 \end{aligned}$$

Then

$$ds^2 = cos^2(u_2)du_1^2 - 2u_1sin(u_2)cos(u_2)du_1du_2 + u_1^2sin^2(u_2)du_2^2 \tag{22}$$

In the sequel, the second fundamental form II of Γ^2 is given by

$$II = \frac{1}{\omega}(-u_1 f''(u_1)du_1^2 + 2b\ du_1\ du_2 - u_1^2 f'(u_1)du_2^2) \tag{23}$$

where $L_{11} = \frac{-u_1}{\omega}f''(u_1)$, $L_{22} = \frac{-u_1^2}{\omega}f'(u_1)$ and $L_{12} = \frac{b}{\omega}$. ■

Corollary 6. *The Gaussian curvature K of the surface $\Gamma^2$is obtained by*

$$K = \frac{1}{\omega^4}(u_1^3f'(u_1)f''(u_1) - b^2) \tag{24}$$

Moreover, the mean curvature of the surface Γ^2 is given by

$$\begin{aligned} H &= \frac{1}{2\omega^3}(-u_1^3 f''(u_1)sin^2(u_2) + 2bu_1\ sin(u_2)\ cos(u_2) \\ &- u_1^2 f'(u_1)\ cos^2(u_2)) \end{aligned} \tag{25}$$

Proof: Since the Gaussian curvature K is given by $K = \frac{L_{11}L_{22}-L_{12}^2}{\omega^2}$, then

$$K = \frac{1}{\omega^2}(\frac{u_1 f''(u_1)}{\omega} \times \frac{u_1^2 f'(u_1)}{\omega} - \frac{b^2}{\omega^2}) = \frac{1}{\omega^4}(u_1^3 f''(u_1) f'(u_1) - b^2)$$

The mean curvature of the surface is obtain from

$$H = \frac{g_2^2 \, L_{11} - 2\, g_1 g_2 \, L_{12} + g_1^2 \, L_{22}}{2\, \omega^2}$$

By substituting, we get

$$H = \frac{1}{2\,\omega^2}[(-u_1 \, sin(u_2))^2 (\frac{-u_1 f''(u_1)}{\omega}) - 2\, cos(u_2)(-u_1 \, sin(u_2))(\frac{b}{\omega}) + cos^2(u_2)\, (\frac{-u_1^2 f'(u_1)}{\omega})] = \frac{1}{2\,\omega^3}[-u_1^3 f''(u_1)\, sin^2(u_2) + 2bu_1 \, sin(u_2)\, cos(u_2) - u_1^2 f'(u_1)\, cos^2(u_2)] \quad \blacksquare$$

4. WEIGHTED HELICOIDAL SURFACES IN G^3

Let Γ^2 be a helicoidal surface in G^3 defined by

$$\chi(u_1, u_2) = (u_1 cos(u_2), u_1 sin(u_2), f(u_1) + bu_2) \tag{26}$$

where $f(u_1)$ is a differentiable function defined on I. Suppose that Γ^2 is the surface in G^3 with a linear density e^{ϕ}, where $\phi = \alpha x + \beta y + \gamma z, \alpha, \beta, \gamma$ not all zero.

In this case, the weighted mean curvature H_ϕ of Γ^2 can be expressed as

$$H_\phi = H - \frac{1}{2} < N, \nabla\phi >_{G^3} \tag{27}$$

where $\nabla\phi$ is the gradient of ϕ. If Γ^2 is the weighted minimal surface, then

$$H = \frac{1}{2} < N, \nabla\phi >_{G^3} \tag{28}$$

Theorem 7. *Let Γ^2 be weighted minimal helicoidal surface in G^3 defined by*

$$\chi(u_1, u_2) = (u_1 cos(u_2), u_1 sin(u_2), f(u_1) + bu_2) \tag{29}$$

with a linear density e^{ϕ}, then $f(u_1)$ will be one of the following

1. $f(u_1) = \frac{2b\, cot(u_2)}{cot^2(u_2)-1}\, \frac{1}{u_1} \, + \, c_1\, u_1^{-cot^2(u_2)}$
2. $f(u_1) = \frac{1}{2B}\, [\, A\, ln(u_1) - 2\, ln(\frac{r(u_1)}{p(u_1)}) + ln(BD) + ln(u_1)]$
3. *$f(u_1)$ is the solution of the differential equation* $f''(u_1) + [\frac{A}{u_1} + \frac{B}{u_1^2} + C] f'(u_1) + \frac{D}{u_1} f'^2(u_1) + E f'^3(u_1) + [\frac{F}{u_1} + \frac{G}{u_1^2} + \frac{H}{u_1^3}] = 0$
4. *$f(u_1)$ is the solution of the differential equation* $f''(u_1) + (B + \frac{A}{u_1} + \frac{C}{u_1^2}) f'(u_1) + (E + \frac{D}{u_1}) f'^2(u_1) + F f'^3(u_1) + (J + \frac{H}{u_1} + \frac{G}{u_1^2} + \frac{I}{u_1^3}) = 0$

Proof: Let Γ^2 be a helicoidal surface in G^3 defined by

$$\chi(u_1, u_2) = (u_1 cos(u_2), u_1 sin(u_2), f(u_1) + bu_2) \tag{30}$$

where $f(u_1)$ is a differentiable function defined on I. By substituting in equation (27) we obtain

$$- u_1^3 f''(u_1) sin^2(u_2) + 2bu_1 \, sin(u_2)\, cos(u_2) - u_1^2 f'(u_1)\, cos^2(u_2) = ((b\, cos(u_2) + u_1 f'(u_1)\, sin(u_2))^2 + u_1^2) < (0, u_1 f'(u_1)\, sin(u_2) + b\, cos(u_2), -u_1), (\alpha, \beta, \gamma) >$$

Now, we can distinguish two cases according to the value of α.
Case 1. If $\alpha \neq 0$
In this case the vector (α, β, γ) is non-isotropic, with some simple calculation we can obtain the following differential equation

$$f''(u_1) + \frac{1}{u_1}\, cot^2(u_2) f'(u_1) = \frac{2b}{u_1^2}\, cot(u_2) \tag{31}$$

To solve this equation, we make reduction of the order as: Let $f'(u_1) = y(u_1)$ which gives $f''(u_1) = y'(u_1)$, substitutes into equation (31) we obtain the differential equation

$$y'(u_1) + \frac{1}{u_1}\, cot^2(u_2)\, y(u_1) = \frac{2b}{u_1^2}\, cot(u_2) \tag{32}$$

Integrating factor $IF = u_1^{cot^2(u_2)}$ and hence the solution is given by

$$y(u_1) = \frac{2b\, cot(u_2)}{cot^2(u_2) - 1}\, \frac{1}{u_1} \, + \, c_1\, u_1^{-cot^2(u_2)} \tag{33}$$

i.e.,

$$f'(u_1) = \frac{2b\, cot(u_2)}{cot^2(u_2) - 1}\, \frac{1}{u_1} \, + \, c_1\, u_1^{-cot^2(u_2)} \tag{34}$$

which gives

$$f(u_1) = \frac{2b\, cot(u_2)}{cot^2(u_2) - 1}\, ln(u_1) \, + \, \frac{c_1}{1 - cot^2(u_2)}\, u_1^{1-cot^2(u_2)} \, + \, c_2 \tag{35}$$

Therefore, Γ^2 is determined by

$$\chi(u_1, u_2) = (u_1 cos(u_2), u_1 sin(u_2), \frac{2bcot(u_2)}{cot^2(u_2) - 1} ln(u_1) + \frac{c_1}{1 - cot^2(u_2)} u_1^{1-cot^2(u_2)} + bu_2 + c_2) \tag{36}$$

where

$$z(u_1, u_2) = \frac{2b\, cot(u_2)}{cot^2(u_2) - 1}\, ln(u_1) \, + \, \frac{c_1}{1 - cot^2(u_2)}\, u_1^{1-cot^2(u_2)} + b\, u_2 \, + \, c_2 \tag{37}$$

c_1, c_2 are constants.

Case 2. If $\alpha = 0$

In this case the vector $(0, \beta, \gamma)$ is an isotropic and as before, we obtain

$$\frac{1}{\omega^2}(-u_1^3 f''(u_1) sin^2(u_2) + 2b\, u_1\, sin(u_2)\, cos(u_2) - u_1^2 f'(u_1)\, cos^2(u_2)) = \beta u_1 f'(u_1)\, sin(u_2) + \beta b\, cos(u_2) - \gamma\, u_1$$

Case 2.1. If $\beta = 0$, therefore

$$\frac{1}{\omega^2}(-u_1^3 f''(u_1) sin^2(u_2) + 2b\, u_1\, sin(u_2)\, cos(u_2) - u_1^2 f'(u_1)\, cos^2(u_2)) = -\gamma u_1 \quad (38)$$

which gives the following differential equation

$$f''(u_1) + A \frac{f'(u_1)}{u_1} - B f'^2(u_1) = \frac{C}{u_1^2} + D \quad (39)$$

where $A = cot^2(u_2) - 2b\,\gamma\, cot(u_2)$, $B = \gamma$, $C = \gamma\, b^2\, cot^2(u_2) + 2b\, cot(u_2)$, $D = \frac{\gamma}{sin^2(u_2)}$.

By using Bessel's functions of first and second order, a simple computations gives that the solution of Equation (39) can be written in the form

$$f(u_1) = \frac{1}{2B}\, [\, A\, ln(u_1) - 2\, ln(\frac{r(u_1)}{p(u_1)}) + ln(BD) + ln(u_1)] \quad (40)$$

such that

$$r(u_1) = B\, [c_2\, Y_n(\sqrt{BD}\, u_1) - c_1\, J_n(\sqrt{BD}\, u_1)] \quad (41)$$

$$p(u_1) = [J_{n+1}(\sqrt{BD}\, u_1)\, Y_n(\sqrt{BD}\, u_1) - J_n(\sqrt{BD}\, u_1)\, Y_{n+1}(\sqrt{BD}\, u_1)] \quad (42)$$

and

$$n = \frac{1}{2}\sqrt{A^2 - 4BC - 2A + 1} \quad (43)$$

with an arbitrary constants c_1, c_2. Therefore, in this case, the surface Γ^2 is given by

$$\chi(u_1, u_2) = (u_1\, cos(u_2),\ u_1\, sin(u_2),\ \frac{1}{2B}[Aln(u_1) - 2ln(\frac{r(u_1)}{p(u_1)}) + ln(BD) + ln(u_1)] + bu_2) \quad (44)$$

Case 2.2. If $\gamma = 0$, then

$$-u_1^3 f''(u_1) sin^2(u_2) + 2bu_1 sin(u_2) cos(u_2) - u_1^2 f'(u_1) cos^2(u_2) = \omega^2(\beta u_1 f'(u_1) sin(u_2) + \beta b cos(u_2))$$

a simple computations gives the next differential equation

$$f''(u_1) + [\frac{A}{u_1} + \frac{B}{u_1^2} + C] f'(u_1) + \frac{D}{u_1} f'^2(u_1) + E f'^3(u_1) + [\frac{F}{u_1} + \frac{G}{u_1^2} + \frac{H}{u_1^3}] = 0 \quad (45)$$

where $A = cot^2(u_2)$, $B = 3\,\beta b^2\, cos(u_2)\, cot(u_2)$, $C = \frac{\beta}{sin(u_2)}$, $D = 3\,\beta b\, cos(u_2)$, $E = \beta sin(u_2)$, $F = \frac{\beta b\, cot(u_2)}{sin(u_2)}$, $G = -2\, b\, cot(u_2)$ and $H = \beta\, b^3\, cos(u_2)\, cot^2(u_2)$.

Case 2.3. If $\gamma\beta \neq 0$, therefore

$$-u_1^3 f''(u_1) sin^2(u_2) + 2bu_1 sin(u_2) cos(u_2) - u_1^2 f'(u_1) cos^2(u_2) = \omega^2(\beta u_1 f'(u_1) sin(u_2) + \beta b cos(u_2) - \gamma u_1)$$

which gives the following differential equation

$$f''(u_1) + (B + \frac{A}{u_1} + \frac{C}{u_1^2}) f'(u_1) + (E + \frac{D}{u_1}) f'^2(u_1) + F f'^3(u_1) + (J + \frac{H}{u_1} + \frac{G}{u_1^2} + \frac{I}{u_1^3}) = 0 \quad (46)$$

where $A = cot^2(u_2) - 2\,\gamma b\, cot(u_2)$, $B = \frac{\beta}{sin(u_2)}$, $C = \frac{3\,\beta b^2\, cos^2(u_2)}{sin(u_2)}$, $D = 3\,\beta\, b\, cos(u_2)$, $E = -\gamma$, $F = \beta sin(u_2)$, $G = -(2\, b\, cot(u_2) + \gamma b^2\, cot^2(u_2))$, $H = \frac{\beta b\, cos(u_2)}{sin^2(u_2)}$, $I = \frac{\beta b^3\, cos^3(u_2)}{sin^2(u_2)}$, and $J = -\frac{\gamma}{sin^2(u_2)}$. ■

5. CONCLUSION AND FURTHER RESEARCH

In this work, we constructed helicoidal surfaces in the Galilean 3−space and studied the First and the Second Fundamental Forms. Moreover, we calculated mean and Gaussian curvature for such surfaces. Also, we considered the Galilean 3−space with a linear density e^{ϕ}, $\phi = \alpha x + \beta y + \gamma z$ such that α, β, γ not all zero and constructed a weighted helicoidal surface by solving a second order non-linear differential equation. Moreover, we discussed an explicit parametrization for the helicoidal surfaces in G^3.

Analogously to how a Minkowski 3−space relates to a Euclidean 3−space, one has the notion of Pseudo-Galilean 3−space G_1^3. As known, G_1^3 is similar to G^3, but the Pseudo-Galilean scalar product of two vectors $r = (r_1, r_2, r_3)$ and $s = (s_1, s_2, s_3)$ is defined by

$$< r, s > = \begin{cases} r_1 s_1, & \text{if } r_1 \neq 0 \text{ or } s_1 \neq 0; \\ r_2 s_2 - r_3 s_3, & \text{if } r_1 = s_1 = 0. \end{cases}$$

Therefore, there exist four types of isotropic vectors $r = (0, r_2, r_3)$ in G_1^3: spacelike vectors (if $r_2^2 - r_3^2 > 0$), timelike vectors (if $r_2^2 - r_3^2 < 0$) and two types of lightlike vectors (if $r_2 = \pm r_3$) [15]. Thus, one can define different types of Helicoidal surfaces in G_1^3.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.