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On the New Wave Behaviors of the Gilson-Pickering Equation

Karmina K. Ali[1,2]*, *Hemen Dutta*[3], *Resat Yilmazer*[2] and *Samad Noeiaghdam*[4,5]

[1] Department of Mathematics, Faculty of Science, University of Zakho, Zakho, Iraq, [2] Department of Mathematics, Faculty of Science, Firat University, Elazig, Turkey, [3] Department of Mathematics, Gauhati University, Guwahati, India, [4] Baikal School of BRICS, Irkutsk National Research Technical University, Irkutsk, Russia, [5] South Ural State University, Chelyabinsk, Russia

In this article, we study the fully non-linear third-order partial differential equation, namely the Gilson-Pickering equation. The $(1/G')$-expansion method, and the generalized exponential rational function method are used to construct various exact solitary wave solutions for a given equation. These methods are based on a homogeneous balance technique that provides an order for the estimation of a polynomial-type solution. In order to convert the governing equation into a nonlinear ordinary differential equation, a traveling wave transformation has been implemented. As a result, we have constructed a variety of solitary wave solutions, such as singular solutions, compound singular solutions, complex solutions, and topological and non-topological solutions. Besides, the 2D, 3D, and contour surfaces are plotted for all obtained solutions by choosing appropriate parameter values.

Keywords: the Gilson-Pickering equation, the $(1/G')$-expansion method, the generalized exponential rational function method, analytic methods, exact solutions

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***Correspondence:**
Karmina K. Ali
karmina.ali@uoz.edu.krd

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1. INTRODUCTION

Nonlinear partial differential equations (NLPDEs) are used to represent a variety of nonlinear physical phenomena in different areas of applied sciences like fluid dynamics, plasma physics, optical fibers, and biology. Among the most profitable strategies for examining such nonlinear physical phenomena is to seek for the exact solutions of NLPDEs [1–5]. In recent years, a variety of effective methods have been implemented to investigate the exact solutions of nonlinear partial differential equations, such as Hirota's bilinear method [6], the Adomian decomposition method [7], the $\exp(-\Phi(\xi))$-expansion method [8], the sine-Gordon expansion method [9], the Bernoulli sub-equation method [10, 11], the shooting method with the fourth-order Runge-Kutta scheme [12, 13], the generalized exponential rational function method [14–18], the modified exponential function method [19], the modified auxiliary expansion method [20], the homotopy perturbation Sumudu transform method [21], the homotopy perturbation transform method [22, 23], and the fractional homotopy analysis transform method [24].

The third-order nonlinear partial differential equation (NLPDE) was introduced in [25] by Gilson and Pickering as

$$u_t - \epsilon u_{xxt} + 2ku_x - uu_{xxx} - \alpha uu_x - \beta u_x u_{xx} = 0, \tag{1}$$

where $\varepsilon, \alpha, \kappa$, and β are non-zero real numbers. Recently, the Gilson-Pickering equation has been investigated using a variety of methods, such as the (G'/G)-expansion method [26], the anstaz method [27], the (G'/G)-expansion method to tanh, the coth, cot, and the logical forms under certain conditions [28], the Bernoulli sub-equation model [29], a not a knot meshless method [30], and the symmetry method [31].

The core of this paper is to investigate the Gilson-Pickering equation using the $(1/G')$-expansion method and the generalized exponential rational function method (GERF).

2. APPLICATIONS OF THE GILSON PICKERING EQUATION

This section presents specific instances of the Gilson Pickering equation and their applications. When $\varepsilon = 1, \alpha = -3$, and $\beta = 2$, Equation (1) gives the Fuchssteiner-Fokas-Camassa-Holm equation, which is a completely integrable nonlinear partial differential equation that arises at different levels of approximation in shallow water theory [32, 33]. When $\varepsilon = 0, \alpha = 1, \kappa = 0$, and $\beta = 3$, Equation (1) reduces to the Rosenau-Hyman equation (RH), which arises in the study of the influence of nonlinear dispersion on the structure of patterns in liquid drops [34]. When $\varepsilon = 1, \alpha = -1, \kappa = 0.5$, and $\beta = 3$, Equation (1) gives the Fronberg-Whitham (FW), which was developed to analyze the qualitative characteristics of wave breakage and admits a wave of the highest height [35–37].

3. THE BASIC CONCEPTS OF THE (1/*G′*)-EXPANSION METHOD

In this section, the fundamental steps of the $(1/G')$-expansion method are presented [38, 39]:

Step 1. Let us consider the general form of a two-variable nonlinear partial differential equation (NPDE) as follows:

$$Q(p, p_t, p_x, p_{xx}, ...) = 0, \tag{2}$$

where $p = p(x, t)$, and Q is a partial differential equation.
Step 2. To convert Equation (2) to a nonlinear ordinary differential equation (NODE), we employ the following wave transformation

$$p(x, t) = P(\eta), \ \eta = (x - ht), \tag{3}$$

where h is a scalar. After some procedures, Equation (2) reduces to the following NODE:

$$W(P', P'', P''', ...) = 0, \tag{4}$$

where W is an ordinary differential equation.
Step 3. Assume that Equation (4) has a solution of the form

$$P(\eta) = \sum_{i=0}^{m} a_i \left(\frac{1}{G'}\right)^i, \tag{5}$$

where $a_0, a_1, a_2, ..., a_m$ are scalars to be determined, m is a balance term, and $G = G(\eta)$ satisfies the following second-order linear ODE:

$$G'' + \lambda G' + \mu = 0, \tag{6}$$

where λ *and* μ are scalars.

The solution of Equation (6) is given by

$$G(\eta) = a_0 + a_1 \left(\frac{1}{-\mu/\lambda + be^{-\lambda\eta}}\right). \tag{7}$$

If we convert the algebraic expression given by Equation (7) to a trigonometric function, we can write it as the following:

$$G(\eta) = a_0 + \frac{a_1}{-\frac{\mu}{\lambda} + b\cosh(\lambda\eta) - b\sinh(\lambda\eta)}. \tag{8}$$

Inserting Equation (6) and its necessary derivatives along with Equation (5) into Equation (4) returns the polynomial of $\left(\frac{1}{G'}\right)^i$. Summing the $\left(\frac{1}{G'}\right)^i$ coefficients with the same power and then setting every summation to zero, we get a system of algebraic equations for $a_i, i \geq 0$. Eventually, solving this system simply gives the value of the variables. Putting these values of variables with the value of the balance term m into Equation (4), we can get solutions for Equation (2).

4. THE BASIC CONCEPTS OF THE GERF

In this section, the basic steps of the GERF are presented.

Step1. Let us consider that the general form of a nonlinear partial differential equation is given by:

$$Q(p, p_x, p_t, p_{xx}, ...) = 0, \tag{9}$$

where Q is a partial differential equation.
Suppose that the wave transformation takes the form:

$$p(x, t) = P(\eta), \ \eta = x - ht, \tag{10}$$

where h is a scalar.
Using Equation (10) in Equation (9), we get the nonlinear ordinary differential equation

$$W(P, P', P'', ...) = 0, \tag{11}$$

where W is an ordinary differential equation.
Step 2. Suppose that the solitary wave solutions of Equation (11) are given by:

$$P(\eta) = A_0 + \sum_{K=1}^{m} A_K \varphi(\eta)^K + \sum_{K=1}^{m} B_K \varphi(\eta)^{-K}, \tag{12}$$

where

$$\varphi(\eta) = \frac{r_1 e^{s_1\eta} + r_2 e^{s_2\eta}}{r_3 e^{s_3\eta} + r_4 e^{s_4\eta}}, \tag{13}$$

where r_m, s_m $(1 \leq n \leq 4)$ are real/complex constants, A_0, A_K, B_K are constants to be determined, and m will be determined by the balance principle.
Step 3. Substituting Equation (12) into Equation (11), we get the polynomials that are dependent on Equation (12). By equating the same order terms, we obtain an algebraic system of equations. With the help of computational programs such as Mathematica, Matlab, and Maple, we solve this system and determine the values of A_0, A_K, B_K. Finally one can easily obtain the nontrivial exact solutions of Equation (11).

5. MATHEMATICAL CALCULATION

In this section, the mathematical calculation of the Gilson-Pickering equation is presented.

Consider the Gilson-Pickering equation (Equation 1) stated in section 1. Inserting the wave transformation

$$u = P(\eta),\ \eta = x - ht, \tag{14}$$

into Equation (1), the following NODE can be obtained

$$\left(2k-h\right)P' + \epsilon h P''' - PP' - \beta P'P' - \alpha PP' = 0, \tag{15}$$

where $\epsilon, \beta, \alpha, h$, and k are non-zero real numbers.
Integrating Equation (15) once with respect to η and assuming that the integration constant is zero, we have.

$$\left(2k-h\right)P + \left(\epsilon h - P\right)P'' + \frac{1-\beta}{2}\left(P'\right)^2 - \frac{\alpha}{2}P^2 = 0. \tag{16}$$

6. IMPLEMENTATION OF THE (1/*G*′)-EXPANSION METHOD

In this section, the application of the $(1/G')$-expansion method to the Gilson-Pickering equation is presented.

Applying the balance principle, by taking the nonlinear term P^2 and the highest derivative P'' in Equation (16) gives $m = 2$. With $m = 2$, Equation (5) takes the form

$$P(\eta) = a_0 + a_1\left(\frac{1}{G'}\right) + a_2\left(\frac{1}{G'}\right)^2. \tag{17}$$

Inserting Equation (17) and its necessary derivatives into Equation (16), returns the polynomial of $\left(\frac{1}{G'}\right)^i$. Summing the $\left(\frac{1}{G'}\right)^i$ coefficients with the likely power and then setting every summation to zero, we get a system of algebraic equations. Solving this system simply gives the following families of solutions:

Family 1. When

$$\begin{aligned} a_0 &= -\frac{2\left(h-2k\right)}{\alpha},\\ a_1 &= -\frac{12\sqrt{-\left(h-2k\right)\alpha}\left(-4k+h\left(2+\alpha\epsilon\right)\right)^{3/2}\mu}{\alpha^2\left(-6k+h\left(3+\alpha\epsilon\right)\right)},\\ a_2 &= \frac{12\left(-4k+h\left(2+\alpha\epsilon\right)\right)^2\mu^2}{\alpha^2\left(-6k+h\left(3+\alpha\epsilon\right)\right)},\ \lambda = -\frac{\sqrt{-\left(h-2k\right)\alpha}}{\sqrt{2h-4k+h\alpha\epsilon}},\\ \beta &= -2, \end{aligned} \tag{18}$$

we get

$$\begin{aligned} u_1(x,t) =& \frac{12\left(-4k+h\left(2+\alpha\epsilon\right)\right)^2\mu^2}{\alpha^2\left(-6k+h\left(3+\alpha\epsilon\right)\right)\left(-\frac{L\mu}{M}+C_1\cosh\left(\frac{M\xi}{L}\right)-C_1\sinh\left(\frac{M\xi}{L}\right)\right)^2}\\ &+\frac{12M\left(-4k+h\left(2+\alpha\epsilon\right)\right)^{3/2}\mu}{\alpha^2\left(-6k+h\left(3+\alpha\epsilon\right)\right)\left(-\frac{L\mu}{M}+C_1\cosh\left(\frac{M\xi}{L}\right)-C_1\sinh\left(\frac{M\xi}{L}\right)\right)}\\ &-\frac{2\left(h-2k\right)}{\alpha}, \end{aligned} \tag{19}$$

where $M = \sqrt{\left(-h+2k\right)\alpha}$, $L = \sqrt{2h-4k+h\alpha\epsilon}$.

Family 2. When

$$\begin{aligned} &a_0 = 0, a_1 = \frac{12h^{3/2}\sqrt{h-2k}\epsilon^{3/2}\mu}{2k+h\left(-1+\alpha\epsilon\right)}, a_2 = \frac{12h^2\epsilon^2\mu^2}{2k+h\left(-1+\alpha\epsilon\right)},\\ &\lambda = \frac{\sqrt{h-2k}}{\sqrt{h\sqrt{\epsilon}}}, \beta = -2, \end{aligned} \tag{20}$$

we get

$$\begin{aligned} u_2(x,t) =& \frac{12h^2\epsilon^2\mu^2}{\left(2k+h\left(-1+\alpha\epsilon\right)\right)\left(-\frac{\sqrt{h}\sqrt{\epsilon}\mu}{\sqrt{h-2k}}+C_1\cosh\left(S\right)-C_1\sinh\left(S\right)\right)^2}\\ &+\frac{12h^{3/2}\sqrt{h-2k}\epsilon^{3/2}\mu}{\left(2k+h\left(-1+\alpha\epsilon\right)\right)\left(-\frac{\sqrt{h}\sqrt{\epsilon}\mu}{\sqrt{h-2k}}+C_1\cosh\left(S\right)-C_1\sinh\left(S\right)\right)}, \end{aligned} \tag{21}$$

where $S = \frac{\sqrt{h-2k}\xi}{\sqrt{h}\sqrt{\epsilon}}$.

Family 3. When

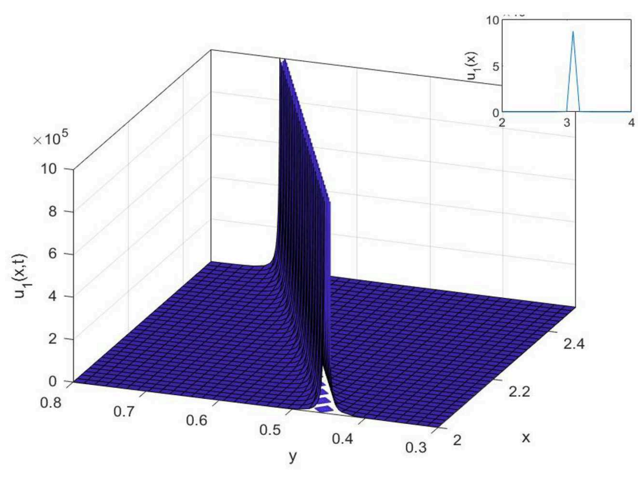

FIGURE 1 | The 3D, 2D, and contour surfaces of Equation (19) when $h = 2$, $k = 2.5$, $\alpha = 2.6$, $\mu = 0.2$, $\epsilon = 3.5$, and $C_1 = 0.6$.

$$a_0 = \frac{4k\epsilon\lambda^2}{\alpha + (2+\alpha\epsilon)\lambda^2},\ a_2 = \frac{\left(\alpha - \lambda^2\right)\left(\alpha + (2+\alpha\epsilon)\lambda^2\right)a_1^2}{24k\alpha\epsilon\lambda^2},$$
$$\mu = \frac{\left(\alpha - \lambda^2\right)\left(\alpha + (2+\alpha\epsilon)\lambda^2\right)a_1}{24k\alpha\epsilon\lambda}, \beta = -2,$$
$$h = \frac{2k\left(\alpha + 2\lambda^2\right)}{\alpha + (2+\alpha\epsilon)\lambda^2}, \tag{22}$$

gives

$$u_3(x,t) = \frac{a_1}{C_1\cosh(\lambda\xi) - C_1\sinh(\lambda\xi) - \frac{(\alpha-\lambda^2)(\alpha+(2+\alpha\epsilon)\lambda^2)a_1}{24k\alpha\epsilon\lambda^2}}$$
$$+ \frac{\left(\alpha-\lambda^2\right)\left(\alpha+(2+\alpha\epsilon)\lambda^2\right)a_1^2}{24k\alpha\epsilon\lambda^2\left(C_1\cosh(\lambda\xi) - C_1\sinh(\lambda\xi) - \frac{(\alpha-\lambda^2)(\alpha+(2+\alpha\epsilon)\lambda^2)a_1}{24k\alpha\epsilon\lambda^2}\right)^2}$$
$$+ \frac{4k\epsilon\lambda^2}{\alpha + (2+\alpha\epsilon)\lambda^2}. \tag{23}$$

Family 4. When

$$a_0 = \frac{4k\epsilon}{1+\alpha\epsilon},\ a_2 = 0,\ \beta = -3,\ \mu = \frac{i\sqrt{\alpha}(1+\alpha\epsilon)a_1}{4k\epsilon},$$
$$h = \frac{2k}{1+\alpha\epsilon},\ \lambda = i\sqrt{\alpha}, \tag{24}$$

we get

$$u_4(x,t) = \frac{4k\epsilon}{1+\alpha\epsilon} + \frac{a_1}{C_1\cos\left(\sqrt{\alpha}\xi\right) - iC_1\sin\left(\sqrt{\alpha}\xi\right) - \frac{(1+\alpha\epsilon)a_1}{4k\epsilon}}. \tag{25}$$

Family 5. When

$$a_0 = \frac{i\sqrt{\alpha}a_1}{\mu},\ a_2 = 0,\ \beta = -3,\ h = \frac{i\sqrt{\alpha}a_1}{2\epsilon\mu},$$
$$k = \frac{i\sqrt{\alpha}(1+\alpha\epsilon)a_1}{4\epsilon\mu},\ \lambda = i\sqrt{\alpha}, \tag{26}$$

we get

$$u_5(x,t) = \frac{i\sqrt{\alpha}a_1}{\mu} + \frac{a_1}{\frac{i\mu}{\sqrt{\alpha}} + C_1\cos\left(\sqrt{\alpha}\xi\right) - iC_1\sin\left(\sqrt{\alpha}\xi\right)}. \tag{27}$$

Family 6. When

$$a_0 = \frac{12h\epsilon\mu + 3\lambda a_1 - \sqrt{-96h\epsilon\lambda\mu a_1 + 9\left(4h\epsilon\mu + \lambda a_1\right)^2}}{24\mu},\ a_2 = \frac{\mu a_1}{\lambda},$$
$$\alpha = \frac{\lambda\left(12h\epsilon\mu - \lambda a_1 + \sqrt{3}\sqrt{48h^2\epsilon^2\mu^2 + \lambda a_1\left(-8h\epsilon\mu + 3\lambda a_1\right)}\right)}{2a_1},$$
$$k = \frac{24h\mu + 12h\epsilon\lambda^2\mu - 3\lambda^3 a_1 + \lambda^2\sqrt{-96h\epsilon\lambda\mu a_1 + 9\left(4h\epsilon\mu + \lambda a_1\right)^2}}{48\mu},$$
$$\beta = -2, \tag{28}$$

we have

$$u_6(x,t) = \frac{\mu a_1}{\lambda\left(-\frac{\mu}{\lambda} + C_1\cosh(\lambda\xi) - C_1\sinh(\lambda\xi)\right)^2}$$
$$+ \frac{a_1}{-\frac{\mu}{\lambda} + C_1\cosh(\lambda\xi) - C_1\sinh(\lambda\xi)}$$
$$+ \frac{12h\epsilon\mu + 3\lambda a_1 - \sqrt{-96h\epsilon\lambda\mu a_1 + 9\left(4h\epsilon\mu + \lambda a_1\right)^2}}{24\mu}. \tag{29}$$

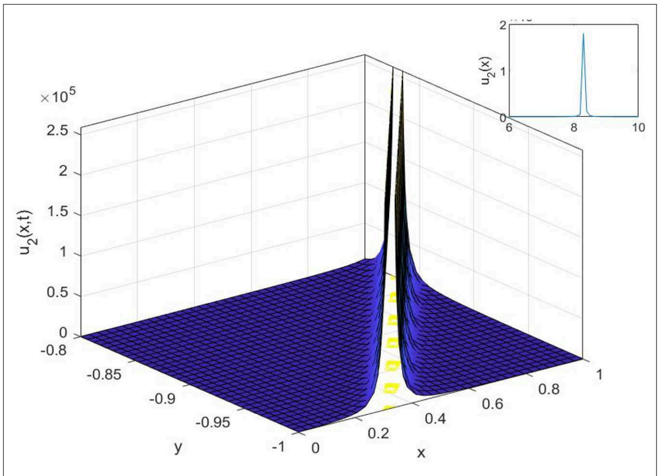

FIGURE 2 | The 3D, 2D, and contour surfaces of Equation (21) using $h = 4$, $k = 0.5$, $\alpha = 2.6$, $\mu = 0.2$, $\epsilon = 4$, and $C_1 = 3$.

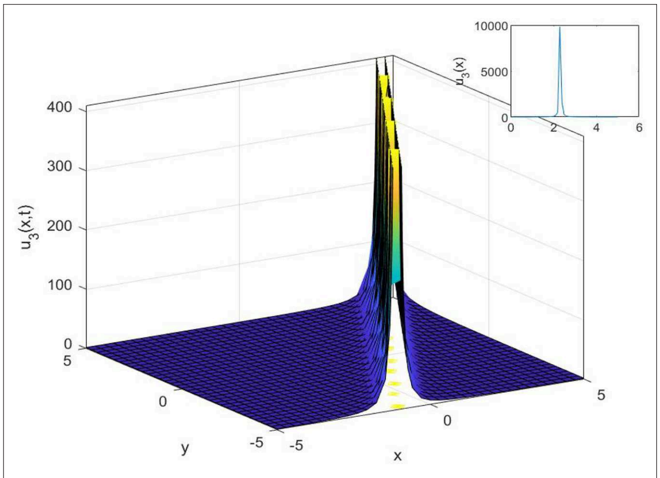

FIGURE 3 | The 3D, 2D, and contour surfaces of Equation (23) using $k = 2$, $\alpha = 5$, $\lambda = 1.2$, $\epsilon = 6.6$, $C_1 = 2$, and $a_1 = 2.8$.

7. IMPLEMENTATION OF THE GERF METHOD

In this section, the application of the GERF method to the Gilson-Pickering equation is presented.

Applying the balance principle, by taking the nonlinear term P^2 and the highest derivative P'' in Equation (16) gives $m = 2$. With $m = 2$, Equation (12) takes the form

$$P(\eta) = A_0 + A_1\varphi(\eta) + \frac{B_1}{\varphi(\eta)} + A_2\varphi(\eta)^2 + \frac{B_2}{\varphi(\eta)^2}, \quad (30)$$

where $\varphi(\eta)$ is given by Equation (13). Following the methodology described above in section 4, we obtain the following nontrivial solutions of Equation (1):

Family 1. When $r_i = \{-2, -1, 1, 1\}$, $s_i = \{0, 1, 0, 1\}$, we get

$$\varphi(\eta) = \frac{-2 - e^{\eta}}{1 + e^{\eta}}, \quad (31)$$

Case 1.

$$A_0 = \frac{A_1(-1 + 13\alpha)}{18\alpha},\ B_1 = 0,\ A_2 = \frac{A_1}{3},\ B_2 = 0,\ \beta = -2,$$
$$h = \frac{A_1\left(-2 + \alpha + \alpha^2\right)}{36\alpha\epsilon},\ k = \frac{A_1(-1 + \alpha)(2 + \alpha + \alpha\epsilon)}{72\alpha\epsilon}, \quad (32)$$

we get

$$u_7(x, t) = \frac{A_1\left(-2 - e^{x - \frac{A_1 t\left(-2+\alpha+\alpha^2\right)}{36\alpha\epsilon}}\right)^2}{3\left(1 + e^{x - \frac{A_1 t\left(-2+\alpha+\alpha^2\right)}{36\alpha\epsilon}}\right)^2} + \frac{A_1\left(-2 - e^{x - \frac{A_1 t\left(-2+\alpha+\alpha^2\right)}{36\alpha\epsilon}}\right)}{1 + e^{x - \frac{A_1 t\left(-2+\alpha+\alpha^2\right)}{36\alpha\epsilon}}} + \frac{A_1(-1 + 13\alpha)}{18\alpha}, \quad (33)$$

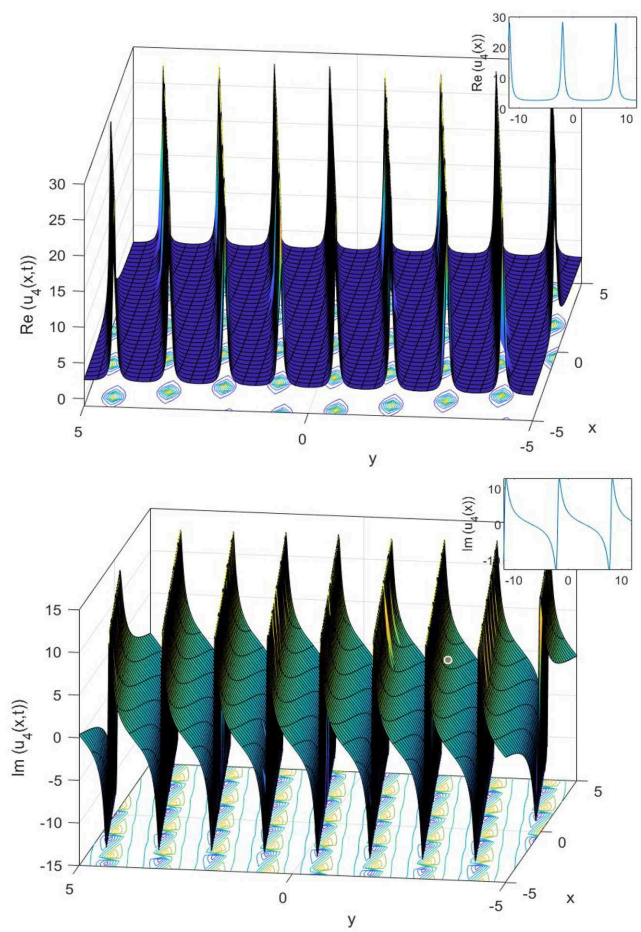

FIGURE 4 | The 3D, 2D, and contour surfaces of Equation (25), using $k = 4.5$, $\alpha = 0.4$, $\epsilon = 0.3$, $C_1 = 0.2$, and $a_1 = 0.8$.

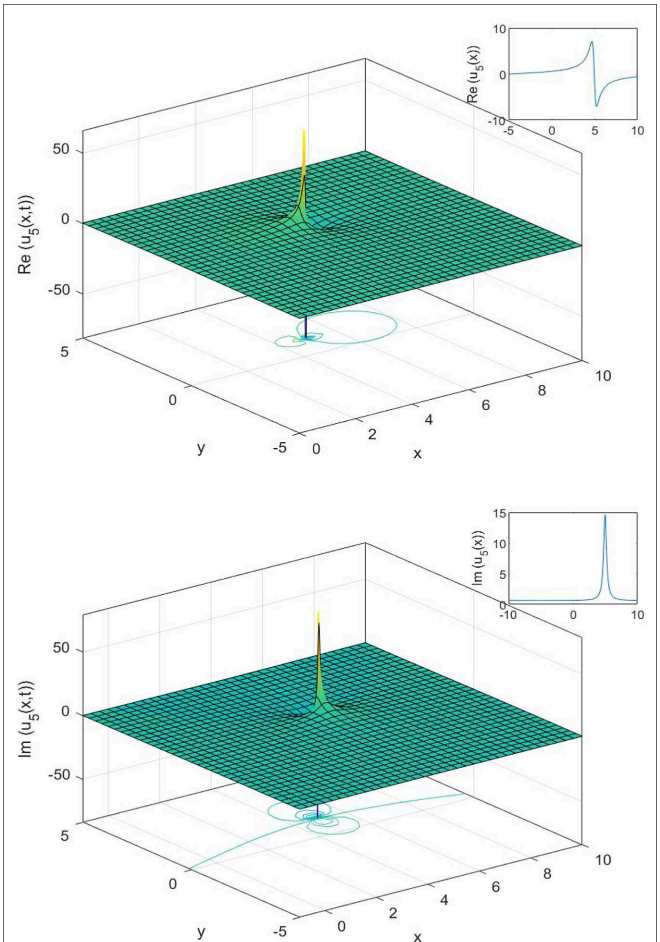

FIGURE 5 | The 3D, 2D, and contour surfaces of Equation (27), using $\mu = 0.4$, $\alpha = 0.1$, $\epsilon = 0.5$, $C_1 = 2$, and $a_1 = 1.5$.

Case 2. When

$$A_0 = -\frac{2\left(h-2k\right)\left(-1+13\alpha\right)}{\left(-1+\alpha\right)\alpha},\ A_1 = 0,\ B_1 = -\frac{72\left(h-2k\right)}{-1+\alpha},$$
$$A_2 = 0,\ B_2 = -\frac{48\left(h-2k\right)}{-1+\alpha},\ \epsilon - \frac{\left(h-2k\right)\left(2+\alpha\right)}{h\alpha},\ \beta = -2, \tag{34}$$

we get

$$u_8\left(x,t\right) = -\frac{72\left(1+e^{-ht+x}\right)\left(h-2k\right)}{\left(-2-e^{-ht+x}\right)\left(-1+\alpha\right)} - \frac{48\left(1+e^{-ht+x}\right)^2\left(h-2k\right)}{\left(-2-e^{-ht+x}\right)^2\left(-1+\alpha\right)} - \frac{2\left(h-2k\right)\left(-1+13\alpha\right)}{\left(-1+\alpha\right)\alpha}. \tag{35}$$

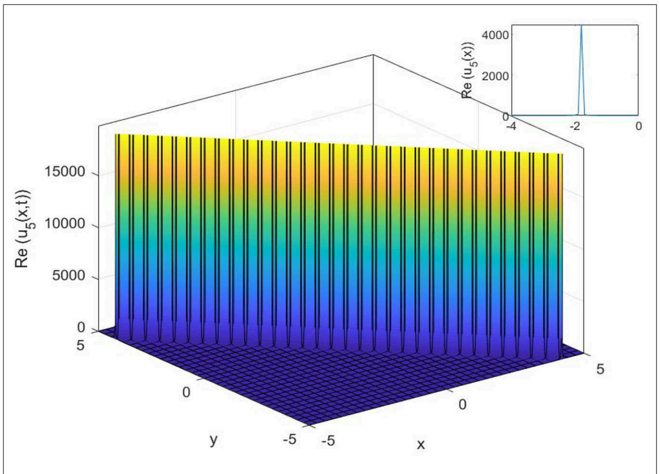

FIGURE 6 | The 3D, 2D, and contour surfaces of Equation (29) using $\mu = 1.5$, $\alpha = 0.4$, $\epsilon = 0.1$, $C_1 = 2$, $a_1 = 0.4$, $h = -1$, and $\lambda = 0.5$.

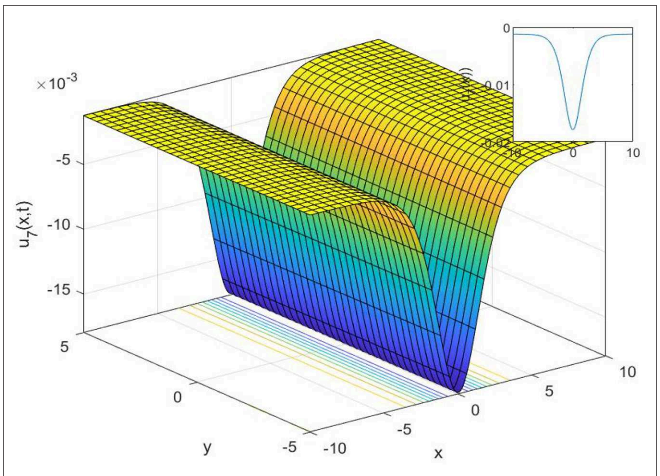

FIGURE 7 | The 3D, 2D, and contour surfaces of Equation (33) using $A_1 = 0.2$, $\alpha = 0.9$, and $\epsilon = 0.6$.

Family 2. When $r_i = \{-2-i, 2-i, -1, 1\}, s_i = \{i, -i, i, -i\}$ we get

$$\varphi\left(\eta\right) = \frac{\cos\left(\eta\right)+2\sin\left(\eta\right)}{\sin\left(\eta\right)}, \tag{36}$$

Case 1. When

$$A_0 = \frac{B_1\left(8-13\alpha\right)}{60\alpha},\ A_1 = 0,\ A_2 = 0,\ B_2 = -\frac{5B_1}{4},\ \beta = -2,$$
$$h = -\frac{B_1\left(-8+\alpha\right)\left(4+\alpha\right)}{240\alpha\epsilon},\ k = \frac{B_1\left(4+\alpha\right)\left(8+\alpha\left(-1+4\epsilon\right)\right)}{480\alpha\epsilon}, \tag{37}$$

we get

$$u_9\left(x,t\right) = \frac{B_1\left(8-13\alpha\right)}{60\alpha} - \frac{5B_1\sin\left(D\right)^2}{4\left(\cos\left(D\right)+2\sin\left(D\right)\right)^2} + \frac{B_1\sin\left(D\right)}{\cos\left(D\right)+2\sin\left(D\right)}, \tag{38}$$

where $D = x + \frac{B_1 t(-8+\alpha)(4+\alpha)}{240\alpha\epsilon}$.

Case 2.

$$A_0 = \frac{A_1\left(8-13\alpha\right)}{12\alpha},\ B_1 = 0,\ A_2 = -\frac{A_1}{4},\ B_2 = 0,\ \beta = -2,$$
$$\epsilon - \frac{A_1\left(-8+\alpha\right)\left(4+\alpha\right)}{48h\alpha},\ k = \frac{1}{24}\left(12h + A_1\left(4+\alpha\right)\right), \tag{39}$$

we get

$$u_{10}\left(x,t\right) = \frac{A1\left(8-13\alpha\right)}{12\alpha} - A_1\csc\left(ht-x\right)\left(\cos\left(ht-x\right) - 2\sin\left(ht-x\right)\right) - \frac{1}{4}A_1\csc\left(ht-x\right)^2\left(\cos\left(ht-x\right) - 2\sin\left(ht-x\right)\right)^2. \tag{40}$$

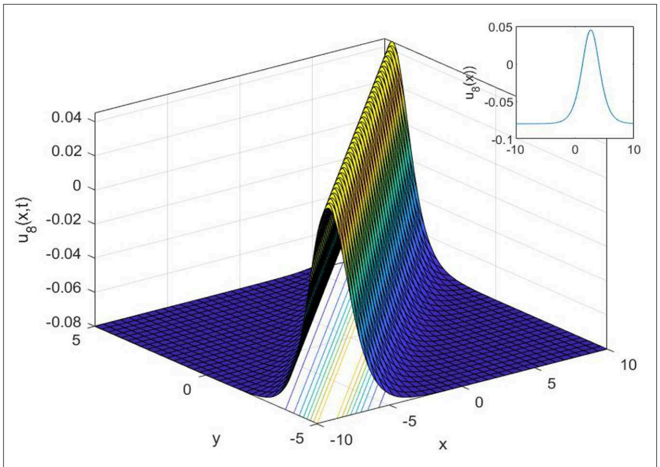

FIGURE 8 | The 3D, 2D, and contour surfaces of Equation (35) using $k = 0.5$, $\alpha = 25$, and $h = 2$.

Family 3. When $r_i = \{2, 0, 1, 1\}, s_i = \{-1, 0, 1, -1\}$

$$\varphi(\eta) = \frac{\left(\cosh(\eta) - \sinh(\eta)\right)}{\cosh(\eta)}, \tag{41}$$

Case 1. When

$$A_0 = -\frac{A_1(-4+\alpha)}{3\alpha},\ B_1 = 0,\ A_2 = -\frac{A_1}{2},\ B_2 = 0,\ \beta = -2,$$
$$h = -\frac{A_1(-4+\alpha)(8+\alpha)}{24\alpha\epsilon},\ k = -\frac{A_1(-4+\alpha)(8+\alpha+4\alpha\epsilon)}{48\alpha\epsilon}, \tag{42}$$

we have

$$u_{11}(x,t) = A_1 Sech(D)\left(\cosh(D) - \sinh(D)\right)\frac{1}{2}A_1 Sech(D)^2\left(\cosh(D) - \sinh(D)\right)^2 - \frac{A_1(-4+\alpha)}{3\alpha}, \tag{43}$$

where $D = x + \frac{A_1 t(-4+\alpha)(8+\alpha)}{24\alpha\epsilon}$.

Case 2.

$$A_0 = -\frac{(h-2k)\left(-4+\sqrt{(-4+\alpha)^2}+\alpha\right)}{(-4+\alpha)\alpha},\ B_2 = 0,\ \beta = -2,$$
$$\epsilon = -\frac{(h-2k)\left(4\left(-4+\sqrt{(-4+\alpha)^2}\right)+\alpha^2\right)}{4h\sqrt{(-4+\alpha)^2}\alpha},\ B_1 = 0,$$
$$A_1 = \frac{6(h-2k)}{\sqrt{(-4+\alpha)^2}},\ A_2 = -\frac{3(h-2k)}{\sqrt{(-4+\alpha)^2}}, \tag{44}$$

we get

$$u_{12}(x,t) = \frac{6(h-2k)\,sech(ht-x)\left(\cosh(ht-x)+\sinh(ht-x)\right)}{\sqrt{(-4+\alpha)^2}} - \frac{3(h-2k)\,sech(ht-x)^2\left(\cosh(ht-x)+\sinh(ht-x)\right)^2}{\sqrt{(-4+\alpha)^2}} - \frac{(h-2k)\left(-4+\sqrt{(-4+\alpha)^2}+\alpha\right)}{(-4+\alpha)\alpha}. \tag{45}$$

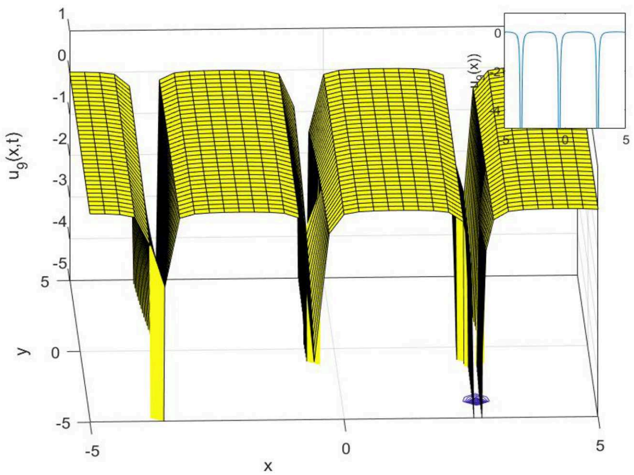

FIGURE 9 | The 3D, 2D, and contour surfaces of Equation (38) using $B_1 = 0.5$, $\alpha = 4$, and $\epsilon = 2$.

8. RESULT AND DISCUSSION

The powerful methods, namely the $(1/G')$ expansion method and the generalized exponential rational function method, are used to construct various analytical solutions for the Gilson-Pickering equation. Some results of the Gilson-Pickering equation have already been reported in the literature. Fan et al.

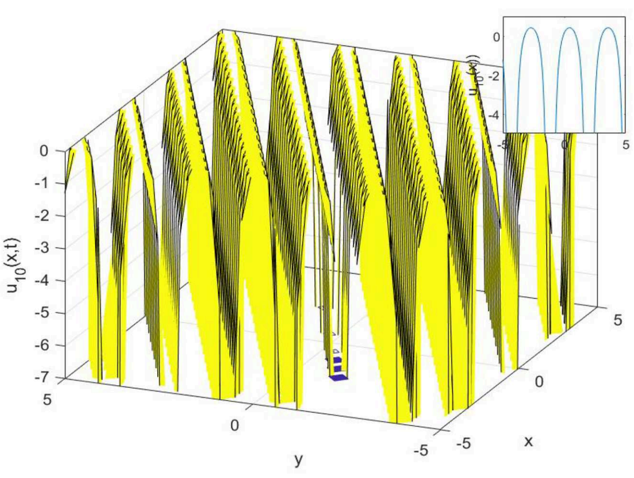

FIGURE 10 | The 3D, 2D, and contour surfaces of Equation (40) using $A_1 = 5$, $\alpha = 4$, and $\epsilon = 2$.

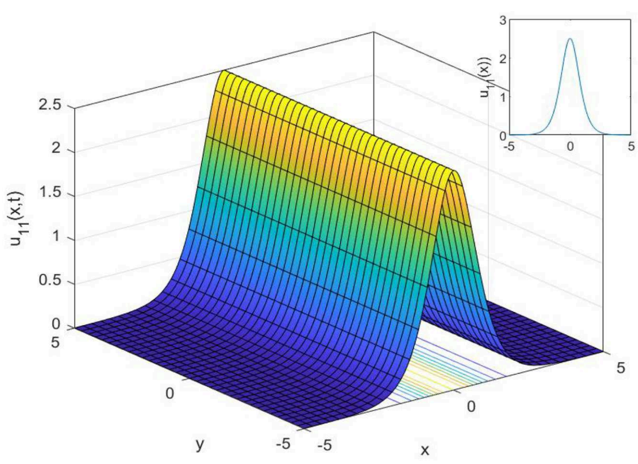

FIGURE 11 | The 3D, 2D, and contour surfaces of Equation (43) using $A_1 = 5$, $\alpha = 4$, and $\epsilon = 2$.

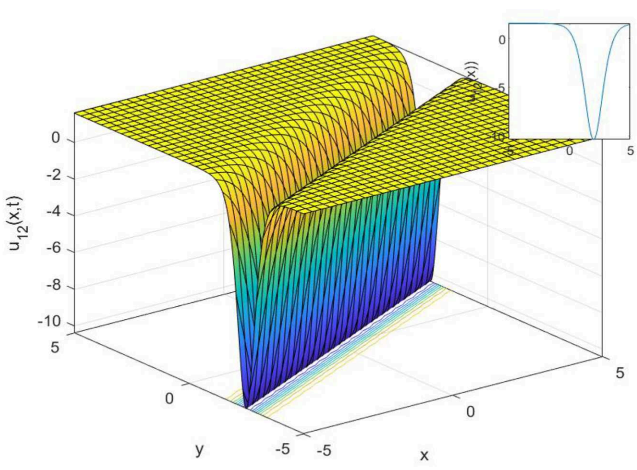

FIGURE 12 | The 3D, 2D, and contour surfaces of Equation (45) using $k = 3$, $\alpha = 5$, and $h = 2$.

[28] used (G'/G) and the ansatz method and found the solitary wave solutions to Equation (1). Baskonus [29] investigated the Gilson-Pickering equation by using the first integral method. Zabihi and Saffarian [30] implemented the simplified (G'/G) expansion method to reveal the hyperbolic, trigonometric function, and rational function solutions. Singla and Gupta [31] reported some new complex soliton solutions to Equation (1) with the aid of the Bernoulli sub-equation function method. Camsssa et al. [32] used a not a knot meshless method to obtain numerical solutions to Equation (1). Fuchssteiner and Fokas [33] performed Lie symmetry analysis and found conservation laws for the space-time fractional Gilson-Pickering equation. In this article, we obtained the singular, compound singular, complex, topological, and non-topological wave solutions to the studied equation. It is known that non-topological solutions detect waves with an intensity lower than the background, topological solutions with such a maximum intensity higher than the background, and singular solutions that are waves with discontinuous derivatives.

9. CONCLUSION

In this study, we have successfully applied the $(1/G')$ expansion method and the generalized exponential rational function method to find new exact solutions for the Gilson-Pickering equation. In order to convert the governing equation into a NODE, a traveling wave transformation has been implemented. Various analytical solutions of the proposed model have been constructed such as singular solutions, as shown in **Figures 1**, **2**, **3**, compound singular solution, as seen in **Figure 4**, complex solution, as seen in **Figure 5**, as well as a singular solution, can be shown in **Figure 6**. The non-topological solution, as shown in **Figure 7**, topological solutions, as shown in **Figure 8**, and compound singular solutions, as seen in **Figures 9**, **10**. Also, topological solution and non-topological solution as seen in **Figures 11**, **12**, respectively. Compared with the results reported in Fan et al. [28], Baskonus [29], Zabihi and Saffarian [30], Singla and Gupta [31], Camsssa et al. [32], and Fuchssteiner and Fokas [33], the solutions obtained are novel. Both methods are efficient for solving complex nonlinear partial differential equations, but, by using the generalized exponential rational function method, we can get more solutions than with the $(1/G')$ expansion method. Furthermore, the 2D, 3D, and contour surfaces are plotted for all obtained solutions by selecting suitable values for the parameters.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

RY and SN suggested the problem first. KA drafted the first version of the problem statement with the help of HD. All authors made several suggestions to make improvements in the problem statement and contributed to the development of solution in their best possible ways.

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