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New Investigation on the Generalized $\mathcal{K}$-Fractional Integral Operators

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The main objective of this paper is to develop a novel framework to study a new fractional operator depending on a parameter $\mathcal{K} > 0$, known as the generalized $\mathcal{K}$-fractional integral operator. To ensure appropriate selection and with the discussion of special cases, it is shown that the generalized $\mathcal{K}$-fractional integral operator generates other operators. Meanwhile, we derived notable generalizations of the reverse Minkowski inequality and some associated variants by utilizing generalized $\mathcal{K}$-fractional integrals. Moreover, two novel results correlate with this inequality, and other variants associated with generalized $\mathcal{K}$-fractional integrals are established. Additionally, this newly defined integral operator has the ability to be utilized for the evaluation of many numerical problems.

Keywords: Minkowski inequality, fractional integral inequality, generalized $\mathcal{K}$-fractional integrals, holder inequalitiy, generalized Riemann-Liouville fractional integral

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1. INTRODUCTION

Fractional calculus is truly considered to be a real-world framework, for example, a correspondence framework that comprises extravagant interfacing, has reliant parts that are utilized to achieve a bound-together objective of transmitting and getting signals, and can be portrayed by utilizing complex system models (see [1–8]). This framework is considered to be a mind-boggling system, and the units that create the whole framework are viewed as the hubs of the intricate system. An attractive characteristic of this field is that there are numerous fractional operators, and this permits researchers to choose the most appropriate operator for the sake of modeling the problem under investigation (see [9–13]). Besides, because of its simplicity in application, researchers have been paying greater interest to recently introduced fractional operators without singular kernels [2, 14, 15], after which many articles considering these kinds of fractional operators have been presented. These techniques had been developed by numerous mathematicians with a barely specific formulation, for instance, the Riemann-Liouville (RL), the Weyl, Erdelyi-Kober, Hadamard integrals, and the Liouville and Katugampola fractional operators (see [16–18]). On the other hand, there are numerous approaches to acquiring a generalization of classical fractional integrals. Many authors have introduced new fractional operators generated from general classical local derivatives (see [9, 19, 20]) and the references therein. Other authors have introduced a parameter and enunciated a generalization for fractional integrals on a selected space. These are called generalized $\mathcal{K}$-fractional integrals. For such operators, we refer to Mubeen and Habibullah [21] and Singh et al. [22] and the works cited in them. Inspired by these developments, future research can bring revolutionary thinking to provide novelties and produce variants concerning such fractional operators. Fractional integral inequalities are an appropriate device for enhancing the qualitative and quantitative properties of differential equations. There has been a continuous growth of interest in several areas of science: mathematics, physics, engineering, amongst others, and particularly, initial value problems, linear transformation stability, integral-differential equations, and impulse equations [23–30].

The well-known integral inequality, as perceived in Dahmani [31], is referred to as the reverse Minkowski inequality. In Nisar et al. [32, 33], the authors investigated numerous variants of extended gamma and confluent hypergeometric $\mathcal{K}$-functions and also established Gronwall inequalities involving the generalized Riemann-Liouville and Hadamard $\mathcal{K}$-fractional derivatives with applications. In Dahmani [25], Dahmani explored variants on intervals that are known as generalized $(\mathcal{K}, s)$-fractional integral operators for positive continuously decreasing functions for a certain family of $n(n \in \mathbb{N})$. In Chinchane and Pachpatte [34], the authors obtained Minkowski variants and other associated inequalities by employing Katugampola fractional integral operators. Recently, some generalizations of the reverse Minkowski and associated inequalities have been established via generalized $\mathcal{K}$ fractional conformable integrals by Mubeen et al. in [35]. Additionally, Hardy-type and reverse Minkowski inequalities are supplied by Bougoffa [36]. Aldhaifallah et al. [37], explored several variants by employing the $(\mathcal{K}, s)$-fractional integral operator.

In the present paper, the authors introduce a parameter and enunciate a generalization for fractional integrals on a selected space, which we name generalized $\mathcal{K}$-fractional integrals. Taking into account the novel ideas, we provide a new version for reverse Minkowski inequality in the frame of the generalized $\mathcal{K}$-fractional integral operators and also provide some of its consequences that are advantageous to current research. New outcomes are introduced, and new theorems relating to generalized $\mathcal{K}$-fractional integrals are derived that correlate with the earlier results.

The article is composed as follows. In the second section, we demonstrate the notations and primary definitions of our newly described generalized $\mathcal{K}$-fractional integrals. Also, we present the results concerning reverse Minkowski inequality. In the third section, we advocate essential consequences such as the reverse Minkowski inequality via the generalized $\mathcal{K}$-fractional integral. In the fourth section, we show the associated variants using this fractional integral.

2. PRELUDE

In this section, we demonstrate some important concepts from fractional calculus that play a major role in proving the results of the present paper. The essential points of interest are exhibited in the monograph by Kilbas et al. [20].

Definition 2.1. ([9, 20]) A function $\mathcal{Q}_1(\tau)$ is said to be in $L_{p,u}[0,\infty]$ space if

$$L_{p,u}[0,\infty) = \Big\{\mathcal{Q}_1 : \|\mathcal{Q}_1\|_{L_{p,u}[0,\infty)} = \left(\int_{\upsilon_1}^{\upsilon_2} |\mathcal{Q}_1(\eta)|^p \xi^u d\eta\right)^{\frac{1}{p}} < \infty,\ 1 \le p < \infty,\ u \ge 0\Big\}.$$

For $r = 0$,

$$L_p[0,\infty) = \Big\{\mathcal{Q}_1 : \|\mathcal{Q}_1\|_{L_p[0,\infty)} = \left(\int_{\upsilon_1}^{\upsilon_2} |\mathcal{Q}_1(\eta)|^p d\eta\right)^{\frac{1}{p}} < \infty,\ 1 \le p < \infty\Big\}.$$

Definition 2.2. ([38]) "Let $\mathcal{Q}_1 \in L_1[0,\infty)$ and Ψ be an increasing and positive monotone function on $[0,\infty)$ and also derivative Ψ' be continuous on $[0,\infty)$ and $\Psi(0) = 0$. The space $\chi^p_\Psi(0,\infty)$ $(1 \le p < \infty)$ of those real-valued Lebesgue measureable functions $\mathcal{Q}_1$ on $[0,\infty)$ for which

$$\|\mathcal{Q}_1\|_{\chi^p_\Psi} = \Big(\int_0^\infty |\mathcal{Q}_1(\eta)|^p \Psi'(\eta) d\eta\Big)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty$$

and for the case $p = \infty$

$$\|\mathcal{Q}_1\|_{\chi^\infty_\Psi} = ess \sup_{0\le\eta<\infty} \big[\Psi'(\eta)\mathcal{Q}_1(\eta)\big]".$$

In particular, when $\Psi(\lambda) = \lambda$ $(1 \le p < \infty)$, the space $\chi^p_\Psi(0,\infty)$ matches with the $L_p[0,\infty)$-space and, furthermore, if we take $\Psi(\lambda) = \ln\lambda$ $(1 \le p < \infty)$, the space $\chi^p_\Psi(0,\infty)$ concurs with $L_{p,u}[1,\infty)$-space.

Now, we present a new fractional operator that is known as the generalized $\mathcal{K}$-fractional integral operator of a function in the sense of another function Ψ.

Definition 2.3. Let $\mathcal{Q}_1 \in \chi^q_\Psi(0,\infty)$, and let Ψ be an increasing positive monotone function defined on $[0,\infty)$, containing continuous derivative $\Psi'(\lambda)$ on $[0,\infty)$ with $\Psi(0) = 0$. Then, the left- and right-sided generalized $\mathcal{K}$-fractional integral operators of a function $\mathcal{Q}_1$ in the sense of another function Ψ of order $\eta > 0$ are stated as:

$$\big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{\upsilon_1^+,\tau}\mathcal{Q}_1\big)(\lambda) = \frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_{\upsilon_1}^{\lambda} \Psi'(\eta)(\Psi(\lambda) - \Psi(\eta))^{\frac{\rho}{\mathcal{K}}-1}\mathcal{Q}_1(\eta)d\eta, \quad \upsilon_1 < \lambda \tag{2.1}$$

and

$$\big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{\upsilon_2^-,\tau}\mathcal{Q}_1\big)(\lambda) = \frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_{\lambda}^{\upsilon_2} \Psi'(\eta)(\Psi(\eta) - \Psi(\lambda))^{\frac{\rho}{\mathcal{K}}-1}\mathcal{Q}_1(\eta)d\eta, \quad \lambda < \upsilon_2, \tag{2.2}$$

where $\rho \in \mathbb{C}, \Re(\rho) > 0$, and $\Gamma_{\mathcal{K}}(\lambda) = \int_0^\infty \eta^{\lambda-1}e^{-\frac{\eta^{\mathcal{K}}}{\mathcal{K}}}d\eta$, $\Re(\lambda) > 0$ is the $\mathcal{K}$-Gamma function introduced by Daiz and Pariguan [39].

Remark 2.1. Several existing fractional operators are just special cases of (2.1) and (2.2).
(1) Choosing $\mathcal{K}=1$, it turns into the both sided generalized RL-fractional integral operator [20].
(2) Choosing $\Psi(\lambda)=\lambda$, it turns into the both-sided $\mathcal{K}$-fractional integral operator [21].
(3) Choosing $\Psi(\lambda)=\lambda$ along with $\mathcal{K}=1$, it turns into the both-sided RL-fractional integral operators.
(4) Choosing $\Psi(\lambda)=\log\lambda$ along with $\mathcal{K}=1$, it turns into the both-sided Hadamard fractional integral operators [9, 20].
(5) Choosing $\Psi(\lambda)=\frac{\lambda^{\beta}}{\beta}, \beta>0$, along with $\mathcal{K}=1$, it turns into the both-sided Katugampola fractional integral operators [17].
(6) Choosing $\Psi(\lambda)=\frac{(\lambda-a)^{\beta}}{\beta}, \beta>0$ along with $\mathcal{K}=1$, it turns into the both-sided conformable fractional integral operators defined by Jarad et al. [2].
(7) Choosing $\Psi(\lambda)=\frac{\lambda^{u+v}}{u+v}$ along with $\mathcal{K}=1$, it turns into the both-sided generalized conformable fractional integrals defined by Khan et al. [40].

Definition 2.4. Let $\mathcal{Q}_1\in\chi^{q}_{\Psi}(0,\infty)$, and let Ψ be an increasing positive monotone function defined on $[0,\infty)$, containing continuous derivative $\Psi'(\lambda)$ in $[0,\infty)$ with $\Psi(0)=0$. Then, the one-sided generalized $\mathcal{K}$-fractional integral operator of a function $\mathcal{Q}_1$ in the sense of another function Ψ of order $\eta>0$ is stated as:

$$({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1)(\lambda)=\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_0^{\lambda}\Psi'(\eta)(\Psi(\lambda)-\Psi(\eta))^{\frac{\rho}{\mathcal{K}}-1}\mathcal{Q}_1(\eta)d\eta,\quad \eta>0, \tag{2.3}$$

where $\Gamma_{\mathcal{K}}$ is the $\mathcal{K}$-Gamma function.

In Set et al. [41] proved the Hermite-Hadamard and reverse Minkowski inequalities for an RL-fractional integral. The subsequent consequences concerning the reverse Minkowski inequalities are the motivation of work finished to date concerning the classical integrals.

Theorem 2.5. Set et al. [41] For $s\geq 1$, let $\mathcal{Q}_1,\mathcal{Q}_2$ be two positive functions on $[0,\infty)$. If $0<\varsigma\leq\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}\leq\Omega, \lambda\in[\upsilon_1,\upsilon_2]$, then

$$\Big(\int_{\upsilon_1}^{\upsilon_2}\mathcal{Q}_1^s(\lambda)d\lambda\Big)^{\frac{1}{s}}+\Big(\int_{\upsilon_1}^{\upsilon_2}\mathcal{Q}_2^s(\lambda)d\lambda\Big)^{\frac{1}{s}}\leq\frac{1+\Omega(\varsigma+2)}{(\varsigma+1)(\Omega+1)}\Big(\int_{\upsilon_1}^{\upsilon_2}\big(\mathcal{Q}_1+\mathcal{Q}_2\big)^s(\lambda)d\lambda\Big)^{\frac{1}{s}}.$$

Theorem 2.6. Set et al. [41] For $s\geq 1$, let $\mathcal{Q}_1,\mathcal{Q}_2$ be two positive functions on $[0,\infty)$. If $0<\varsigma\leq\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}\leq\Omega, \lambda\in[\upsilon_1,\upsilon_2]$, then

$$\Big(\int_{\upsilon_1}^{\upsilon_2}\mathcal{Q}_1^s(\lambda)d\lambda\Big)^{\frac{2}{s}}+\Big(\int_{\upsilon_1}^{\upsilon_2}\mathcal{Q}_2^s(\lambda)d\lambda\Big)^{\frac{2}{s}}\geq\Big(\frac{(1+\Omega)(\varsigma+1)}{\Omega}-2\Big)\Big(\int_{\upsilon_1}^{\upsilon_2}\mathcal{Q}_1^s(\lambda)d\lambda\Big)^{\frac{1}{s}}\Big(\int_{\upsilon_1}^{\upsilon_2}\mathcal{Q}_2^s(\lambda)d\lambda\Big)^{\frac{1}{s}}.$$

In Dahmani [31], introduced the subsequent reverse Minkowski inequalities involving the RLFI operators.

Theorem 2.7. Dahmani [31] For $\rho\in\mathbb{C},\ \Re(\rho)>0,\ s\geq 1$, and let $\mathcal{Q}_1,\mathcal{Q}_2$ be two positive functions on $[0,\infty)$ such that, for all $\lambda>0,\ \mathcal{T}^{\rho}_{\upsilon_1^+}\mathcal{Q}_1^s(\lambda)<\infty,\ \mathcal{T}^{\rho}_{\upsilon_1^+}\mathcal{Q}_2^s(\lambda)<\infty$. If $0<\varsigma\leq\frac{\mathcal{Q}_1(\lambda)}{\mathcal{Q}_2(\lambda)}\leq\Omega,\eta\in[\upsilon_1,\lambda]$, then

$$\Big(\mathcal{T}^{\rho}_{\upsilon_1^+}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}}+\Big(\mathcal{T}^{\rho}_{\upsilon_1^+}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{1}{s}}\leq\frac{1+\Omega(\varsigma+2)}{(\varsigma+1)(\Omega+1)}\Big(\mathcal{T}^{\rho}_{\upsilon_1^+}\big(\mathcal{Q}_1+\mathcal{Q}_2\big)^s(\lambda)\Big)^{\frac{1}{s}}.$$

Theorem 2.8. Dahmani [31] For $\rho\in\mathbb{C},\ \Re(\rho)>0,\ s\geq 1$, and let $\mathcal{Q}_1,\mathcal{Q}_2$ be two positive functions on $[0,\infty)$ such that, for all $\lambda>0,\ \mathcal{T}^{\rho}_{\upsilon_1^+}\mathcal{Q}_1^s(\lambda)<\infty,\ \mathcal{T}^{\rho}_{\upsilon_1^+}\mathcal{Q}_2^s(\lambda)<\infty$. If $0<\varsigma\leq\frac{\mathcal{Q}_1(\lambda)}{\mathcal{Q}_2(\lambda)}\leq\Omega,\eta\in[\upsilon_1,\lambda]$, then

$$\Big(\mathcal{T}^{\rho}_{\upsilon_1^+}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{2}{s}}+\Big(\mathcal{T}^{\rho}_{\upsilon_1^+}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{2}{s}}\geq\Big(\frac{(1+\Omega)(\varsigma+2)}{\Omega}-2\Big)\Big(\mathcal{T}^{\rho}_{\upsilon_1^+}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}}\Big(\mathcal{T}^{\rho}_{\upsilon_1^+}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{1}{s}}.$$

3. REVERSE MINKOWSKI INEQUALITY VIA GENERALIZED $\mathcal{K}$-FRACTIONAL INTEGRALS

Throughout the paper, it is supposed that all functions are integrable in the Riemann sense. Also, this segment incorporates the essential contribution for obtaining the proof of the reverse Minkowski inequality via the newly described generalized $\mathcal{K}$-fractional integrals defined in section (2.4).

Theorem 3.1. For $\mathcal{K}>0,\rho\in\mathbb{C},\Re(\rho)>0$ and $s\geq 1$, and let two positive functions $\mathcal{Q}_1,\mathcal{Q}_2$ be defined on $[0,\infty)$. Assume that Ψ is an increasing positive monotone function on $[0,\infty)$ having derivative Ψ' and is continuous on $[0,\infty)$ with $\Psi(0)=0$ such that, for all $\lambda>0$, ${}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1^s(\lambda)<\infty$ and ${}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_2^s(\lambda)<\infty$. If $0<\varsigma\leq\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}\leq\Omega$ for $\varsigma,\Omega\in\mathbb{R}^+$ and for all $\eta\in[0,\lambda]$, then

$$\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}}+\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{1}{s}}\leq\theta_1\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\big(\mathcal{Q}_1+\mathcal{Q}_2\big)^s(\lambda)\Big)^{\frac{1}{s}} \tag{3.1}$$

with $\theta_1=\frac{\Omega(\varsigma+1)+(\Omega+1)}{(\varsigma+1)(\Omega+1)}$.

Proof: Under the given conditions $\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}\leq\Omega,\ 0\leq\eta\leq\lambda$, it can written as

$$\mathcal{Q}_1(\eta)\leq\Omega\big(\mathcal{Q}_1(\eta)+\mathcal{Q}_2(\eta)\big)-\Omega\mathcal{Q}_1(\eta),$$

which implies that

$$(\Omega+1)^s\mathcal{Q}_1^s(\eta)\leq\Omega^s\big(\mathcal{Q}_1(\eta)+\mathcal{Q}_2(\eta)\big)^s. \tag{3.2}$$

If we multiply both sides of (3.2) by $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}$ and integrate w.r.t η over $[0,\lambda]$, one obtains

$$\frac{(\Omega+1)^s}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_0^\lambda \Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{Q}_1^s(\eta)d\eta$$
$$\leq \frac{\Omega^s}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_0^\lambda \Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}\big(\mathcal{Q}_1(\eta)+\mathcal{Q}_2(\eta)\big)^s d\eta. \tag{3.3}$$

Accordingly, it can be written as

$$\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}} \leq \frac{\Omega}{\Omega+1}\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\big(\mathcal{Q}_1+\mathcal{Q}_2\big)^s(\lambda)\Big)^{\frac{1}{s}}. \tag{3.4}$$

In contrast, as $\varsigma\mathcal{Q}_2(\lambda)\leq\mathcal{Q}_1(\lambda)$, it follows

$$\Big(1+\frac{1}{\varsigma}\Big)^s\mathcal{Q}_2^s(\eta)\leq\Big(\frac{1}{\varsigma}\Big)^s\big(\mathcal{Q}_1(\eta)+\mathcal{Q}_2(\eta)\big)^s. \tag{3.5}$$

Again, taking the product of both sides of (3.5) with $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}$ and integrating w.r.t η over $[0,\lambda]$, we obtain

$$\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{1}{s}} \leq \frac{1}{\varsigma+1}\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\big(\mathcal{Q}_1+\mathcal{Q}_2\big)^s(\lambda)\Big)^{\frac{1}{s}}. \tag{3.6}$$

The desired inequality (3.1) can be obtained from 3.4 and 3.6. □

Inequality (3.1) is referred to as the reverse Minkowski inequality related to the generalized $\mathcal{K}$-fractional integral.

Theorem 3.2. For $\mathcal{K}>0,\rho\in\mathbb{C},\Re(\rho)>0$ and $s\geq1$, let two positive functions $\mathcal{Q}_1,\mathcal{Q}_2$ be defined on $[0,\infty)$. Assume that Ψ is an increasing positive monotone function on $[0,\infty)$ having derivative Ψ' and is continuous on $[0,\infty)$ with $\Psi(0)=0$ such that, for all $\lambda>0$, ${}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1^s(\lambda)<\infty$ and ${}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_2^s(\lambda)<\infty$. If $0<\varsigma\leq\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}\leq\Omega$ for $\varsigma,\Omega\in\mathbb{R}^+$ and for all $\eta\in[0,\lambda]$, then

$$\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{2}{s}}+\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{2}{s}}$$
$$\geq\theta_2\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}}\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{1}{s}} \tag{3.7}$$

with $\theta_2=\frac{(\varsigma+1)(\Omega+1)}{\Omega}-2$.

Proof: Multiplying 3.4 and 3.6 results in

$$\frac{(\varsigma+1)(\Omega+1)}{\Omega}\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}}\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{1}{s}}$$
$$\leq\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\big(\mathcal{Q}_1+\mathcal{Q}_2\big)^s(\lambda)\Big)^{\frac{2}{s}}. \tag{3.8}$$

Involving the Minkowski inequality, on the right side of (3.8), we get

$$\frac{(\varsigma+1)(\Omega+1)}{\Omega}\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}}\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{1}{s}}$$
$$\leq\Big(\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}}+\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{1}{s}}\Big)^2. \tag{3.9}$$

From 3.9, we conclude that

$$\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{2}{s}}+\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{2}{s}}$$
$$\geq\Big(\frac{(\varsigma+1)(\Omega+1)}{\Omega}-2\Big)\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}}\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{1}{s}}.$$

□

4. CERTAIN ASSOCIATED INEQUALITIES VIA THE GENERALIZED $\mathcal{K}$-FRACTIONAL INTEGRAL OPERATOR

Theorem 4.1. For $\mathcal{K}>0,\rho\in\mathbb{C},\Re(\rho)>0,\ s,r\geq1,\frac{1}{s}+\frac{1}{r}=1$ and let two positive functions $\mathcal{Q}_1,\mathcal{Q}_2$ be defined on $[0,\infty)$. Assume that Ψ is an increasing, positive monotone function on $[0,\infty)$ having derivative Ψ' and is continuous on $[0,\infty)$ with $\Psi(0)=0$ such that, for all $\lambda>0$, ${}^{\Psi}\mathcal{T}^{\eta,\mathcal{K}}_{0^+,\tau}\mathcal{Q}_1^s(\lambda)<\infty$ and ${}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_2^s(\lambda)<\infty$. If $0<\varsigma\leq\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}\leq\Omega$ for $\varsigma,\Omega\in\mathbb{R}^+$ and for all $\eta\in[0,\lambda]$, then

$$\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1(\lambda)\Big)^{\frac{1}{s}}\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_2(\lambda)\Big)^{\frac{1}{r}}$$
$$\leq\Big(\frac{\Omega}{\varsigma}\Big)^{\frac{1}{sr}}\Big(\Big({}^{\Psi}\mathcal{T}^{\rho,\mathcal{K}}_{0^+,\lambda}\mathcal{Q}_1^{\frac{1}{s}}(\lambda)\mathcal{Q}_2^{\frac{1}{r}}(\lambda)\Big). \tag{4.1}$$

Proof: Under the given condition $\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}\leq\Omega,\ 0\leq\eta\leq\lambda$, it can be expressed as

$$\mathcal{Q}_1(\eta)\leq\Omega\mathcal{Q}_2(\eta),$$

which implies that

$$\mathcal{Q}_2^{\frac{1}{r}}(\eta)\geq\Omega^{-\frac{1}{r}}\mathcal{Q}_1^{\frac{1}{r}}(\eta). \tag{4.2}$$

Taking the product of both sides of (4.2) by $\mathcal{Q}_1^{\frac{1}{s}}(\eta)$, we are able to rewrite it as follows:

$$\mathcal{Q}_1^{\frac{1}{s}}(\eta)\mathcal{Q}_2^{\frac{1}{r}}(\eta)\geq\Omega^{-\frac{1}{r}}\mathcal{Q}_1(\eta). \tag{4.3}$$

Multiplying both sides of (4.3) with $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}$ and integrating w.r.t η over $[0,\lambda]$, one obtains

$$\frac{\Omega^{-\frac{1}{r}}}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_0^\lambda\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{Q}_1(\eta)d\eta$$

$$\geq \frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)} \int_0^{\lambda} \Psi'(\eta)\big(\Psi(\lambda) - \Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1} \mathcal{Q}_1^{\frac{1}{s}}(\eta)\mathcal{Q}_2^{\frac{1}{r}}(\eta)d\eta. \quad (4.4)$$

As a consequence, we can rewrite as follows

$$\Omega^{\frac{-1}{sr}}\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1(\lambda)\Big)^{\frac{1}{s}} \leq \Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1^{\frac{1}{s}}(\lambda)\mathcal{Q}_1^{\frac{1}{r}}(\lambda)\Big)^{\frac{1}{s}}. \quad (4.5)$$

Similarly, as $\varsigma\mathcal{Q}_2(\eta) \leq \mathcal{Q}_1(\eta)$, it follows that

$$\varsigma^{\frac{1}{s}}\mathcal{Q}_2^{\frac{1}{s}}(\eta) \leq \mathcal{Q}_1^{\frac{1}{s}}(\eta). \quad (4.6)$$

Again, taking the product of both sides of (4.6) by $\mathcal{Q}_2^{\frac{1}{s}}(\eta)$ and using the relation $\frac{1}{s}+\frac{1}{r}=1$ gives

$$\varsigma^{\frac{1}{s}}\mathcal{Q}_2(\eta) \leq \mathcal{Q}_1^{\frac{1}{s}}(\eta)\mathcal{Q}_2^{\frac{1}{s}}(\eta). \quad (4.7)$$

If we multiply both sides of (4.7) by $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\Psi'(\eta)\big(\Psi(\lambda) - \Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}$ and integrate w.r.t η over $[0,\lambda]$, we obtain

$$\varsigma^{\frac{1}{sr}}\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_2(\lambda)\Big)^{\frac{1}{r}} \leq \Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1^{\frac{1}{s}}(\lambda)\mathcal{Q}_1^{\frac{1}{r}}(\lambda)\Big)^{\frac{1}{r}}. \quad (4.8)$$

Finding the product between (4.5) and (4.8) and using the relation $\frac{1}{s}+\frac{1}{r}=1$, we get the desired inequality (4.1). □

Theorem 4.2. For $\mathcal{K}>0, \rho\in\mathbb{C}, \Re(\rho)>0,\ s,r\geq 1, \frac{1}{s}+\frac{1}{r}=1$, and let two positive functions $\mathcal{Q}_1,\mathcal{Q}_2$ be defined on $[0,\infty)$. Assume that Ψ is an increasing, positive monotone function on $[0,\infty)$ having derivative Ψ' and is continuous on $[0,\infty)$ with $\Psi(0)=0$ such that, for all $\lambda>0$, ${}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\mathcal{K}}\mathcal{Q}_1^s(\lambda)<\infty$ and ${}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_2^s(\lambda)<\infty$. If $0<\varsigma\leq\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}\leq\Omega$ for $\varsigma,\Omega\in\mathbb{R}^+$ and for all $\eta\in[0,\lambda]$, then

$$\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1(\lambda)\mathcal{Q}_1(\lambda)\Big) \leq \theta_3\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1^s+\mathcal{Q}_2^s\big)(\lambda)\Big) + \theta_4\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1^r+\mathcal{Q}_2^r\big)(\lambda)\Big) \quad (4.9)$$

with $\theta_3=\frac{2^{s-1}\Omega^s}{s(\Omega+1)^s}$ and $\theta_4=\frac{2^{r-1}}{r(\varsigma+1)^r}$.

Proof: Under the assumptions, we have the subsequent identity:

$$(\Omega+1)^s\mathcal{Q}_1^s(\eta) \leq \Omega^s(\mathcal{Q}_1+\mathcal{Q}_2)^s(\eta). \quad (4.10)$$

Multiplying both sides of (4.10) by $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\Psi'(\eta)\big(\Psi(\lambda) - \Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}$ and integrating w.r.t η over $[0,\lambda]$, one obtains

$$\frac{(\Omega+1)^s}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)} \int_0^{\lambda} \Psi'(\eta)\big(\Psi(\lambda) - \Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{Q}_1^s(\eta)d\eta$$

$$\leq \frac{\Omega^s}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)} \int_0^{\lambda} \Psi'(\eta)\big(\Psi(\lambda) - \Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}(\mathcal{Q}_1+\mathcal{Q}_2)^s(\eta)d\eta. \quad (4.11)$$

Accordingly, it can be written as

$${}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1^s(\lambda) \leq \frac{\Omega^s}{(\Omega+1)^s}\ {}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1+\mathcal{Q}_2\big)^s(\lambda). \quad (4.12)$$

In contrast, as $0<\varsigma\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)},\ 0<\eta<\lambda$, it follows

$$(\varsigma+1)^r\mathcal{Q}_2^r(\eta) \leq (\mathcal{Q}_1+\mathcal{Q}_2)^r(\eta). \quad (4.13)$$

Again, taking the product of both sides of (4.13) with $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\Psi'(\eta)\big(\Psi(\lambda) - \Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}$ and integrating w.r.t η over $[0,\lambda]$, one obtains

$${}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_2^r(\lambda) \leq \frac{1}{(\varsigma+1)^r}\ {}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1+\mathcal{Q}_2\big)^r(\lambda). \quad (4.14)$$

Considering Young's inequality,

$$\mathcal{Q}_1(\eta)\mathcal{Q}_2(\eta) \leq \frac{\mathcal{Q}_1^s(\eta)}{s}+\frac{\mathcal{Q}_2^r(\eta)}{r}. \quad (4.15)$$

If we multiply both sides of (4.15) with $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\Psi'(\eta)\big(\Psi(\lambda) - \Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}$ and integrate w.r.t η over $[0,\lambda]$, we obtain

$${}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1\mathcal{Q}_2\big)(\lambda) \leq \frac{{}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1^s(\lambda)}{s}+\frac{{}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_2^r(\lambda)}{r}. \quad (4.16)$$

Invoking (4.12) and (4.14) into (4.16), we obtain

$$\begin{aligned}
&{}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1\mathcal{Q}_2\big)(\lambda)\\
&\leq \frac{{}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1^s(\lambda)}{s}+\frac{{}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_2^r(\lambda)}{r}\\
&\leq \frac{\Omega^s}{(\Omega+1)^s}\ {}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1+\mathcal{Q}_2\big)^s(\lambda)\\
&+\frac{1}{(\varsigma+1)^r}\ {}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1+\mathcal{Q}_2\big)^r(\lambda).
\end{aligned} \quad (4.17)$$

Using the inequality $(\mu+\nu)^z\leq 2^{z-1}(\mu^z+\nu^z),\ z>1,\ \mu,\nu>0$, one obtains

$${}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1+\mathcal{Q}_2\big)^s(\lambda) \leq 2^{s-1}\ {}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1^s+\mathcal{Q}_2^s\big)(\lambda) \quad (4.18)$$

and

$${}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1+\mathcal{Q}_2\big)^r(\lambda) \leq 2^{r-1}\ {}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1^r+\mathcal{Q}_2^r\big)(\lambda). \quad (4.19)$$

The desired (4.9) can be established from (4.17), (4.18) and (4.19) jointly.

Theorem 4.3. For $\mathcal{K}>0, \rho\in\mathbb{C}, \Re(\rho)>0,\ s,r\geq 1, \frac{1}{s}+\frac{1}{r}=1$ and let two positive functions $\mathcal{Q}_1,\mathcal{Q}_2$ be defined on $[0,\infty)$. Assume that Ψ is an increasing positive monotone function on $[0,\infty)$ having derivative Ψ' and is continuous on $[0,\infty)$ with $\Psi(0)=0$ such that, for all $\lambda>0$, ${}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\mathcal{K}}\mathcal{Q}_1^s(\lambda)<\infty$ and

${}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_2^s(\lambda) < \infty$. If $0 < \zeta < \varsigma \leq \frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)} \leq \Omega$ for $\varsigma, \Omega \in \mathbb{R}^+$ and for all $\eta \in [0,\lambda]$, then

$$\frac{\Omega+1}{\Omega-\zeta}\Big(\mathfrak{J}_{\Psi}^{\lambda}\big(\mathcal{Q}_1(\lambda)-\mathcal{Q}_2(\lambda)\big)\Big) \leq \Big(\mathfrak{J}_{\Psi}^{\lambda}\mathcal{Q}_1(\lambda)\Big)^{\frac{1}{s}} + \Big(\mathfrak{J}_{\Psi}^{\lambda}\mathcal{Q}_2(\lambda)\Big)^{\frac{1}{s}} \leq \frac{\varsigma+1}{\varsigma-\zeta}\Big(\mathfrak{J}_{\Psi}^{\lambda}\big(\mathcal{Q}_1(\lambda)-\mathcal{Q}_2(\lambda)\big)\Big)^{\frac{1}{s}}. \tag{4.20}$$

Proof: Using the hypothesis $0 < \zeta < \varsigma \leq \Omega$, we get

$$\varsigma\zeta \leq \Omega\zeta \Rightarrow \varsigma\zeta+\varsigma \leq \varsigma\zeta+\Omega \leq \Omega\zeta+\Omega \Rightarrow (\Omega+1)(\varsigma-\zeta) \leq (\varsigma+1)(\Omega-\zeta).$$

It can be concluded that

$$\frac{\Omega+1}{\Omega-\zeta} \leq \frac{\varsigma+1}{\varsigma-\zeta}.$$

Further, we have that

$$\varsigma-\zeta \leq \frac{\mathcal{Q}_1(\eta)-\zeta\mathcal{Q}_2(\eta)}{\mathcal{Q}_2(\eta)} \leq \Omega-\zeta$$

implies that

$$\frac{\big(\mathcal{Q}_1(\eta)-\zeta\mathcal{Q}_2(\eta)\big)^s}{(\Omega-\zeta)^s} \leq \mathcal{Q}_2^s(\eta) \leq \frac{\big(\mathcal{Q}_1(\eta)-\zeta\mathcal{Q}_2(\eta)\big)^s}{(\varsigma-\zeta)^s}. \tag{4.21}$$

Again, we have that

$$\frac{1}{\Omega} \leq \frac{\mathcal{Q}_2(\eta)}{\mathcal{Q}_1(\eta)} \leq \frac{1}{\varsigma} \Rightarrow \frac{\varsigma-\zeta}{\zeta\varsigma} \leq \frac{\mathcal{Q}_1(\eta)-\zeta\mathcal{Q}_2(\eta)}{\zeta\mathcal{Q}_1(\eta)} \leq \frac{\Omega-\zeta}{\zeta\Omega}$$

implies that

$$\Big(\frac{\Omega}{\Omega-\zeta}\Big)^s\big(\mathcal{Q}_1(\eta)-\zeta\mathcal{Q}_2(\eta)\big)^s \leq \mathcal{Q}_1^s(\eta) \leq \Big(\frac{\varsigma}{\varsigma-\zeta}\Big)^s\big(\mathcal{Q}_1(\eta)-\zeta\mathcal{Q}_2(\eta)\big)^s. \tag{4.22}$$

If we multiply both sides of (4.21) with $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}$ and integrate w.r.t η over $[0,\lambda]$, we obtain

$$\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)(\Omega-\zeta)^s}\int_0^{\lambda}\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}\big(\mathcal{Q}_1(\eta)-\zeta\mathcal{Q}_2(\eta)\big)^s d\eta \leq \frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_0^{\lambda}\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{Q}_2^s(\eta)d\eta \leq \frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)(\varsigma-\zeta)^s}\int_0^{\lambda}\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}\big(\mathcal{Q}_1(\eta)-\zeta\mathcal{Q}_2(\eta)\big)^s d\eta.$$

Accordingly, it can be written as

$$\frac{1}{\Omega-\zeta}\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1(\lambda)-\zeta\mathcal{Q}_2(\lambda)\big)^s\Big)^{\frac{1}{s}} \leq \Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}} \leq \frac{1}{\varsigma-\zeta}\Big(\mathfrak{J}_{\Psi}^{\lambda}\big(\mathcal{Q}_1(\lambda)-\zeta\mathcal{Q}_2(\lambda)\big)^s\Big)^{\frac{1}{s}}. \tag{4.23}$$

In a similar way with (4.22), one obtains

$$\frac{\Omega}{\Omega-\zeta}\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1(\lambda)-\zeta\mathcal{Q}_2(\lambda)\big)^s\Big)^{\frac{1}{s}} \leq \Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}} \leq \frac{\varsigma}{\varsigma-\zeta}\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1(\lambda)-\zeta\mathcal{Q}_2(\lambda)\big)^s\Big)^{\frac{1}{s}}. \tag{4.24}$$

The desired inequality (4.20) can be established by adding (4.23) and (4.24). □

Theorem 4.4. For $\mathcal{K} > 0, \rho \in \mathbb{C}, \Re(\rho) > 0,\ s, r \geq 1, \frac{1}{s}+\frac{1}{r} = 1$ and let two positive functions $\mathcal{Q}_1, \mathcal{Q}_2$ be defined on $[0,\infty)$. Assume that Ψ is an increasing positive monotone function on $[0,\infty)$ having derivative Ψ' and is continuous on $[0,\infty)$ with $\Psi(0) = 0$ such that, for all $\lambda > 0$, ${}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\mathcal{K}}\mathcal{Q}_1^s(\lambda) < \infty$ and ${}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_2^s(\lambda) < \infty$. If $0 \leq d \leq \mathcal{Q}_1(\eta) \leq \mathcal{D}$ and $0 \leq f \leq \mathcal{Q}_2(\eta) \leq \mathcal{F}$ for $\varsigma, \Omega \in \mathbb{R}^+$ and for all $\eta \in [0,\lambda]$, then

$$\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}} + \Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{1}{s}} \leq \theta_5\Big(\mathfrak{J}_{\Psi}^{\lambda}(\mathcal{Q}_1+\mathcal{Q}_2)^s(\lambda)\Big)^{\frac{1}{s}} \tag{4.25}$$

with $\theta_5 = \frac{\mathcal{D}(d+\mathcal{F})+\mathcal{F}(\mathcal{D}+f)}{(\mathcal{D}+f)(d+\mathcal{F})}$.

Proof: Under the assumptions, it pursues that

$$\frac{1}{\mathcal{F}} \leq \frac{1}{\mathcal{Q}_2(\lambda)} \leq \frac{1}{f}. \tag{4.26}$$

Taking the product between (4.26) and $0 \leq d \leq \mathcal{Q}_1(\eta) \leq D$, we have

$$\frac{d}{\mathcal{F}} \leq \frac{\mathcal{Q}_1(\lambda)}{\mathcal{Q}_2(\lambda)} \leq \frac{\mathcal{D}}{f}. \tag{4.27}$$

From (4.27), we get

$$\mathcal{Q}_2^s(\eta) \leq \Big(\frac{\mathcal{F}}{d+\mathcal{F}}\Big)^s\big(\mathcal{Q}_1(\eta)+\mathcal{Q}_2(\eta)\big)^s \tag{4.28}$$

and

$$\mathcal{Q}_1^s(\eta) \leq \Big(\frac{\mathcal{D}}{f+\mathcal{D}}\Big)^s\big(\mathcal{Q}_1(\eta)+\mathcal{Q}_2(\eta)\big)^s. \tag{4.29}$$

If we multiply both sides of (4.28) with $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}$ and integrate w.r.t η over $[0,\lambda]$, we obtain

$$\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_0^\lambda \Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{Q}_2^s(\eta)d\eta$$
$$\leq \frac{\mathcal{F}^s}{(d+\mathcal{F})^s\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_0^\lambda \Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}\big(\mathcal{Q}_1(\eta)+\mathcal{Q}_2(\eta)\big)^s d\eta.$$

Likewise, it can be composed as

$$\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{1}{s}} \leq \frac{\mathcal{F}}{d+\mathcal{F}}\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}(\mathcal{Q}_1+\mathcal{Q}_2)^s(\lambda)\Big)^{\frac{1}{s}}. \quad (4.30)$$

In the same way with (4.29), we have

$$\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}} \leq \frac{\mathcal{D}}{f+\mathcal{D}}\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}(\mathcal{Q}_1+\mathcal{Q}_2)^s(\lambda)\Big)^{\frac{1}{s}}. \quad (4.31)$$

The desired inequality (4.25) can be established by adding (4.30) and (4.31). □

Theorem 4.5. For $\mathcal{K}>0, \rho\in\mathbb{C}, \Re(\rho)>0$, $s\geq 1$, and let two positive functions $\mathcal{Q}_1,\mathcal{Q}_2$ be defined on $[0,\infty)$. Assume that Ψ is an increasing positive monotone function on $[0,\infty)$ having derivative Ψ' and is continuous on $[0,\infty)$ with $\Psi(0)=0$ such that, for all $\lambda>0$, ${}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\mathcal{K}}\mathcal{Q}_1^s(\lambda)<\infty$ and ${}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_2^s(\lambda)<\infty$. If $0<\theta<\varsigma\leq\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}\leq\Omega$ for $\varsigma,\Omega\in\mathbb{R}^+$ and for all $\eta\in[0,\lambda]$, then

$$\frac{1}{\Omega}\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1(\lambda)\mathcal{Q}_2(\lambda)\Big)$$
$$\leq \frac{1}{(\varsigma+1)(\Omega+1)}\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\big(\mathcal{Q}_1+\mathcal{Q}_2\big)^2(\lambda)\Big)$$
$$\leq \frac{1}{\varsigma}\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1(\lambda)\mathcal{Q}_2(\lambda)\Big). \quad (4.32)$$

Proof: Using $0<\varsigma\leq\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}\leq\Omega$, it follows that

$$(\varsigma+1)\mathcal{Q}_2(\eta)\leq\mathcal{Q}_1(\eta)+\mathcal{Q}_2(\eta)\leq\mathcal{Q}_2(\eta)(\Omega+1). \quad (4.33)$$

Also, it follows that $\frac{1}{\Omega}\leq\frac{\mathcal{Q}_2(\eta)}{\mathcal{Q}_1(\eta)}\leq\frac{1}{\varsigma}$, which yields

$$\mathcal{Q}_1(\eta)\Big(\frac{\Omega+1}{\Omega}\Big)\leq\mathcal{Q}_1(\eta)+\mathcal{Q}_2(\eta)\leq\mathcal{Q}_1(\eta)\Big(\frac{\varsigma+1}{\varsigma}\Big). \quad (4.34)$$

Finding the product between (4.33) and (4.34), we have

$$\frac{\mathcal{Q}_1(\eta)\mathcal{Q}_2(\eta)}{\Omega}\leq\frac{(\mathcal{Q}_1(\eta)+\mathcal{Q}_2(\eta))^2}{(\varsigma+1)(\Omega+1)}\leq\frac{\mathcal{Q}_1(\eta)\mathcal{Q}_2(\eta)}{\varsigma}. \quad (4.35)$$

If we multiply both sides of (4.28) with $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}$ and integrate w.r.t η over $[0,\lambda]$, we obtain

$$\frac{1}{\Omega\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_0^\lambda \Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{Q}_1(\eta)\mathcal{Q}_2(\eta)d\eta$$
$$\leq\theta_6\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_0^\lambda \Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}(\mathcal{Q}_1(\eta)+\mathcal{Q}_2(\eta))^2d\eta$$
$$\leq\frac{1}{\varsigma\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_0^\lambda \Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{Q}_1(\eta)\mathcal{Q}_2(\eta)d\eta$$

with $\theta_6=\frac{1}{(\varsigma+1)(\Omega+1)}$.
Likewise, the required outcome (4.32) can be finished up. □

Theorem 4.6. For $\mathcal{K}>0, \rho\in\mathbb{C}, \Re(\rho)>0$, $s\geq 1$, and let two positive functions $\mathcal{Q}_1,\mathcal{Q}_2$ be defined on $[0,\infty)$. Assume that Ψ is an increasing positive monotone function on $[0,\infty)$ having derivative Ψ' and is continuous on $[0,\infty)$ with $\Psi(0)=0$ such that, for all $\lambda>0$, ${}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\mathcal{K}}\mathcal{Q}_1^s(\lambda)<\infty$ and ${}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_2^s(\lambda)<\infty$. If $0<\theta<\varsigma\leq\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}\leq\Omega$ for $\varsigma,\Omega\in\mathbb{R}^+$ and for all $\eta\in[0,\lambda]$, then

$$\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}}+\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{1}{s}}$$
$$\leq 2\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{H}^s\big(\mathcal{Q}_1(\lambda),\mathcal{Q}_2(\lambda)\big)\Big)^{\frac{1}{s}}, \quad (4.36)$$

where $\mathcal{H}\big(\mathcal{Q}_1(\eta),\mathcal{Q}_2(\eta)\big)=\max\Big\{\Omega\big(\frac{\Omega}{\varsigma}+1\big)\mathcal{Q}_1(\lambda)-\Omega\mathcal{Q}_2(\lambda),\frac{(\varsigma+\Omega)\mathcal{Q}_2(\lambda)-\mathcal{Q}_1(\lambda)}{\varsigma}\Big\}$.

Proof: Under the given conditions $0<\varsigma\leq\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}\leq\Omega$, $0\leq\eta\leq\lambda$, can be written as

$$0<\varsigma\leq\Omega+\varsigma-\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}, \quad (4.37)$$

and

$$\Omega+\varsigma-\frac{\mathcal{Q}_1(\eta)}{\mathcal{Q}_2(\eta)}\leq\Omega. \quad (4.38)$$

From (4.35) and (4.38), we obtain

$$\mathcal{Q}_2(\eta)<\frac{(\Omega+\varsigma)\mathcal{Q}_2(\eta)-\mathcal{Q}_1(\eta)}{\varsigma}\leq\mathcal{H}\big(\mathcal{Q}_1(\eta),\mathcal{Q}_2(\eta)\big), \quad (4.39)$$

where $\mathcal{H}\big(\mathcal{Q}_1(\eta),\mathcal{Q}_2(\eta)\big)=\max\Big\{\Omega\big(\frac{\Omega}{\varsigma}+1\big)\mathcal{Q}_1(\lambda)-\Omega\mathcal{Q}_2(\lambda),\frac{(\varsigma+\Omega)\mathcal{Q}_2(\lambda)-\mathcal{Q}_1(\lambda)}{\varsigma}\Big\}$.
From hypothesis, it also follows that $0<\frac{1}{\Omega}\leq\frac{\mathcal{Q}_2(\eta)}{\mathcal{Q}_1(\eta)}\leq\frac{1}{\varsigma}$ implies that

$$\frac{1}{\Omega}\leq\frac{1}{\Omega}+\frac{1}{\varsigma}-\frac{\mathcal{Q}_2(\eta)}{\mathcal{Q}_1(\eta)} \quad (4.40)$$

and

$$\frac{1}{\Omega}+\frac{1}{\varsigma}-\frac{\mathcal{Q}_2(\eta)}{\mathcal{Q}_1(\eta)}\leq\frac{1}{\varsigma}. \tag{4.41}$$

From (4.40) and (4.41), we obtain

$$\frac{1}{\Omega}\leq\frac{\left(\frac{1}{\Omega}+\frac{1}{\varsigma}\right)\mathcal{Q}_1(\eta)-\mathcal{Q}_2(\eta)}{\mathcal{Q}_1(\eta)}\leq\frac{1}{\varsigma}, \tag{4.42}$$

which can be composed as

$$\begin{aligned}\mathcal{Q}_1(\eta)&\leq\Omega\Big(\frac{1}{\Omega}+\frac{1}{\varsigma}\Big)\mathcal{Q}_1(\eta)-\Omega\mathcal{Q}_2(\eta)\\&=\frac{\Omega(\Omega+\varsigma)\mathcal{Q}_1(\eta)-\Omega^2\varsigma\mathcal{Q}_2(\eta)}{\varsigma\Omega}\\&=\Big(\frac{\Omega}{\varsigma}+1\Big)\mathcal{Q}_1(\eta)-\Omega\mathcal{Q}_2(\eta)\\&\leq\Omega\Big[\Big(\frac{\Omega}{\varsigma}+1\Big)\mathcal{Q}_1(\eta)-\Omega\mathcal{Q}_2(\eta)\Big]\\&\leq\mathcal{H}\big(\mathcal{Q}_1(\eta),\mathcal{Q}_2(\eta)\big).\end{aligned} \tag{4.43}$$

We can compose from (4.40) and (4.43)

$$\mathcal{Q}_1^s(\eta)\leq\mathcal{H}^s\big(\mathcal{Q}_1(\eta),\mathcal{Q}_2(\eta)\big), \tag{4.44}$$

$$\mathcal{Q}_2^s(\eta)\leq\mathcal{H}^s\big(\mathcal{Q}_1(\eta),\mathcal{Q}_2(\eta)\big). \tag{4.45}$$

Multiplying both sides of (4.44) by $\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}$ and integrating w.r.t η over $[0,\lambda]$, one obtains

$$\begin{aligned}&\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_0^\lambda\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{Q}_1^s(\eta)d\eta\\&\leq\frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\rho)}\int_0^\lambda\Psi'(\eta)\big(\Psi(\lambda)-\Psi(\eta)\big)^{\frac{\rho}{\mathcal{K}}-1}\mathcal{H}^s\big(\mathcal{Q}_1(\eta),\mathcal{Q}_2(\eta)\big)d\eta.\end{aligned}$$

Likewise, it can be composed as

$$\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_1^s(\lambda)\Big)^{\frac{1}{s}}\leq\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{H}^s\big(\mathcal{Q}_1(\lambda),\mathcal{Q}_2(\lambda)\big)\Big)^{\frac{1}{s}}. \tag{4.46}$$

Repeating the same procedure as above, for (4.45), we have

$$\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{Q}_2^s(\lambda)\Big)^{\frac{1}{s}}\leq\Big({}^{\Psi}\mathcal{T}_{0^+,\lambda}^{\rho,\mathcal{K}}\mathcal{H}^s\big(\mathcal{Q}_1(\lambda),\mathcal{Q}_2(\lambda)\big)\Big)^{\frac{1}{s}}. \tag{4.47}$$

The desired inequality (4.36) is obtained from (4.46) and (4.47). □

5. CONCLUSION

This article succinctly expresses the newly defined fractional integral operator. We characterize the strategy of generalized $\mathcal{K}$-fractional integral operators for the generalization of reverse Minkowski inequalities. The outcomes presented in section 3 are the generalization of the existing work done by Dahmani [31] for the RL-fractional integral operator. Also, the consequences in section 3 under certain conditions are reduced to the special cases proved in Set al. [41]. The variants built in section 4 are the generalizations of the existing results derived in Sulaiman [42]. Additionally, our consequences will reduce to the classical results established by Sroysang [43]. Our consequences with this new integral operator have the capacities to be used for the assessment of numerous scientific issues as utilizations of the work, which incorporates existence and constancy for the fractional-order differential equations.

AUTHOR CONTRIBUTIONS

All authors contributed to each part of this work equally, read, and approved the final manuscript.

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