frontiers in Physics

ORIGINAL RESEARCH
published: 17 January 2020
doi: 10.3389/fphy.2019.00214

Time-Dependent MHD Flow of Non-Newtonian Generalized Burgers' Fluid (GBF) Over a Suddenly Moved Plate With Generalized Darcy's Law

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Time-dependent magnetohydrodynamic (MHD) motion of a generalized Burgers' fluid (GBF) is investigated in this article. GBF is a highly complicated non-Newtonian fluid and is of highest degree in the class of rate type fluids. GBF is taken electrically conducting by using the restriction of small magnetic Reynolds number. Darcy's law has been used here in its generalized form using the GBF constitutive relation; hence, the medium is made porous. The impulsive motion in the fluid is induced due to sudden jerk of the plate. Exact expressions for velocity as well as for shear stress fields are obtained using the Laplace transform method. The solutions for hydrodynamic fluid (absence of MHD) in a non-porous medium as well as those for a Newtonian fluid (NF) executing a similar motion are also recovered. Results are sketched in terms of several plots and discussed for embedded parameters. It is found that the Hartmann number and porosity of the medium have strong influence on the velocity and shear stress fields.

Keywords: time-dependent flow, MHD, generalized Burgers' fluid, generalized Darcy's law, Laplace transform

OPEN ACCESS

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Specialty section:
This article was submitted to Mathematical Physics, a section of the journal Frontiers in Physics

Received: *28 September 2019*
Accepted: *21 November 2019*
Published: *17 January 2020*

Citation:
Alqahtani AM and Khan I (2020) Time-Dependent MHD Flow of Non-Newtonian Generalized Burgers' Fluid (GBF) Over a Suddenly Moved Plate With Generalized Darcy's Law. Front. Phys. 7:214. doi: 10.3389/fphy.2019.00214

INTRODUCTION

Most of the fluid problems (published literature), or fluid problems with heat transfer or heat and mass transfer together, are computed numerically due to the difficult nature of these problems. Indeed, the exact solutions for these problems are either not possible or quite difficult to obtain. These difficulties further increase if one is interested to solve such a problem using the integral transform techniques such as Laplace transform, Fourier transform, etc. In the Laplace transform, particularly the most difficult job is how to take the inversion. Therefore, some of the researchers are then using numerical inversion to somehow solve the inversion problem. However, such solutions are then not the so-called pure exact solutions. Among the interesting studies on exact solutions and, of course, the pioneering work includes the work of Rajagopal [1], where he studied non-Newtonian second-grade fluid for different flow motions and obtained exact solutions for each flow case. The flow was unsteady unidirectional and one-dimensional. Eight different flow cases were discussed. This work was then extended in 2007 by Hayat et al. [2] for the case of MHD flow and porous medium. More exactly, the fluid was taken electrically conducted and passing through a porous medium. They discussed seven different flow situations and obtained exact solution either by perturbation method or Fourier transform method. Other interesting studies on exact solutions include the work of Erdogan [3], Erdogan and Imrak [4], and Tan and Masuoka [5, 6]. Hayat et al. [7, 8] established for rotating flows exact analytic solutions for two different types of non-Newtonian fluids, namely, the second-grade fluid and the Maxwell fluid. They considered transient problems in both cases with combined effects of MHD and porosity.

The obtained exact solutions were discussed for various embedded parameters and concluded. Fetecau et al. [9] in a short note investigated analytically the Stokes' second problem (SSP) for Newtonian fluids (NF) flow. Fetecau and Fetecau [10] considered an unsteady problem of a Maxwell fluid (MF, non-Newtonian) over a rigid plate moved due to a sudden jerk. In another paper, Fetecau and Fetecau [11] extended the idea of MF to an Oldroyd-B fluid (OBF, non-Newtonian) and examined exact solutions for the first problem of Stokes'. Vieru et al. [12] also determined exact solution for the flow situation of an OBF over an infinite rigid plate.

In the group of viscoelastic fluids, Burgers' fluids and the corresponding generalized Burgers' fluids (GBFs) are less studied in the literature compared to other fluids in that group. Indeed, the resulting equations based on their complicated constitutive relations are not easy to handle. The exact solutions for these fluids problems are not possible unless we impose several assumptions. Even then, the exact solutions for these fluid problems are limited to certain well-known problems. Some famous fluid problems for Burgers or GBFs have been studied in Ravindran et al. [13], Hayat et al. [14], Khan et al. [15], Tong and Shan [16], Xue and Nie [17], Hayat et al. [18], Vieru et al. [19], Khan et al. [20–22], Fetecau et al. [23] and related references therein. However, for several other problems, such solutions are either too much complicated or even not possible. Such a complication even increases if the problem under consideration is composed of fractional differential equations, such as the problem considered in these articles on different aspects of sciences and engineering [24–36]. Some other related studies regarding fluid dynamics problems can be seen in Waqas et al. [37], Marin et al. [38], Jamil [39], and Jamil et al. [40, 41]. Roberts and Kaufman [42] is used for some of the Laplace inversion formulas needed for this work.

The main purpose of the present article is to study the time-dependent flow of GBF (incompressible) over an infinite (in horizontal-direction) rigid plate given sudden jerk. Simultaneous effects of MHD and porosity are also taken into consideration.

Exact analytic solutions are obtained for the dimensionless fluid velocity and non-trivial shear stress exerted by the fluid on the plate. Laplace transform is indeed a suitable method to solve this problem. Clearly, these solutions satisfy the given imposed conditions [initial and boundary conditions (IBCs)] and can produce other exact analytic solutions for other non-Newtonian fluids problems such as Burgers' fluids, OBFs, and Maxwell fluids performing a similar type of motion. Exact solutions for Newtonian fluids performing the same motion can also be obtained as a special case by vanishing all other non-Newtonian parameters. Graphical results are plotted and discussed for embedded parameters. Solutions for other fluids (generalized Burger fluids without MHD and porosity effects, Newtonian fluids) in limiting sense are also recovered.

PROBLEM FORMULATION AND INTEGRAL TRANSFORM SOLUTION

The problem formulation states that an incompressible flow strongly depends on time (unsteady flow) of a highly non-Newtonian fluid known as GBF lies in a semi-infinite porous space $y > 0$; i.e., the fluid is over a rigid plate kept at $y = 0$. The axes ($x-$axis and $y-$axis) are taken perpendicular to each other; i.e., the $x-$axis is taken in the flow direction while the $y-$axis is chosen normal to the direction of the flow. MHD effect is considered under which the fluid behaves like an electrically conducting liquid under the influence of an applied magnetic field such that the induced magnetic field is $v(0, t) = V, \quad v(y, t) \to 0 \text{ as } y \to \infty; \quad t > 0$, neglected assuming that magnetic Reynolds number is too small. GBF is initially taken at rest (for time $t = 0$); however, for time $t > 0$, the plate is give a sudden jerk (impulsive motion of the plate) and the fluid starts with the same impulsive motion. The scenario stated above is formulated in the form of partial differential equation with physical boundary and initial conditions as given below (for detailed analysis of the governing equation, one may refer to Xue and Nie [17] and Hayat et al. [18]):

$$\rho\left(1+\lambda\frac{\partial}{\partial t}+\gamma\frac{\partial^2}{\partial t^2}\right)\frac{\partial v}{\partial t} = \mu\left(1+\lambda_r\frac{\partial}{\partial t}+\gamma_1\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 v}{\partial y^2} - \delta B_0^2\left(1+\lambda\frac{\partial}{\partial t}+\gamma\frac{\partial^2}{\partial t^2}\right)v - \frac{\mu\varphi}{k}\left(1+\lambda_r\frac{\partial}{\partial t}+\gamma_1\frac{\partial^2}{\partial t^2}\right)v, \quad (1)$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\gamma\frac{\partial^2}{\partial t^2}\right)T(y,t) = \mu\left(1+\lambda_r\frac{\partial}{\partial t}+\gamma_1\frac{\partial^2}{\partial t^2}\right)\frac{\partial v(y,t)}{\partial y}, \quad (2)$$

$$v(0, t) = V, \quad v(y, t) \to 0 \text{ as } y \to \infty; \quad t > 0, \quad (3)$$

$$v(y, 0) = \frac{\partial v(y, 0)}{\partial t} = \frac{\partial^2 v(y, 0)}{\partial t^2} = 0; \quad y > 0. \quad (4)$$

in which v is the velocity component in $x-$ direction, ρ is the fluid density, μ is the dynamic viscosity, δ is the finite electrical conductivity of the fluid, $\varphi\,(0 < \varphi < 1)$ is the porosity, $k > 0$ is the permeability of the porous medium, λ and $\lambda_r\,(< \lambda)$ are respectively the relaxation and retardation times, γ and γ_1 are the material constants having the dimensions as the square of time, and V denotes the reference velocity.

The problem described by Equations (1)–(3), after using non-dimensional quantities, takes the following form:

$$\left(1+\frac{\partial}{\partial\tau}+\beta\frac{\partial^2}{\partial\tau^2}\right)\frac{\partial u(\xi,\tau)}{\partial\tau} = \begin{bmatrix}\left(1+\alpha\frac{\partial}{\partial\tau}+\beta_1\frac{\partial^2}{\partial\tau^2}\right)\frac{\partial^2 u(\xi,\tau)}{\partial\xi^2} - M^2\left(1+\frac{\partial}{\partial\tau}+\beta\frac{\partial^2}{\partial\tau^2}\right)u(\xi,\tau)\\ -\frac{1}{K}\left(1+\alpha\frac{\partial}{\partial\tau}+\beta_1\frac{\partial^2}{\partial\tau^2}\right)u(\xi,\tau), \quad \xi,\tau>0,\end{bmatrix} \quad (5)$$

$$\left(1+\frac{\partial}{\partial\tau}+\beta\frac{\partial^2}{\partial\tau^2}\right)s = \left(1+\alpha\frac{\partial}{\partial\tau}+\beta_1\frac{\partial^2}{\partial\tau^2}\right)\frac{\partial u}{\partial\tau}, \quad \xi,\tau>0, \quad (6)$$

$$u(0,\tau) = 1, \quad u(\xi,\tau) \to 0 \text{ as } \xi \to \infty \quad \tau > 0, \quad (7)$$

$$u(\xi, 0) = \frac{\partial u(\xi, 0)}{\partial \tau} = \frac{\partial^2 u(\xi, 0)}{\partial \tau^2} = 0, \quad \xi > 0, \tag{8}$$

where

$$\tau = \frac{t}{\lambda}, \ \xi = \frac{y}{c\lambda}, \ \ u = \frac{v}{V}, \ s = \frac{T}{\rho c V}, \ c = \sqrt{\frac{\mu}{\rho\lambda}}, \tag{9}$$

$$\alpha = \frac{\lambda_r}{\lambda}, \ \beta = \frac{\gamma}{\lambda^2}, \ \beta_1 = \frac{\gamma_1}{\lambda^2}, \ M^2 = \frac{\delta B_0^2 \lambda}{\rho}, \ \frac{1}{K} = \frac{\mu\varphi\lambda}{\rho k}. \tag{10}$$

In the transformed $q-$plane, Equations (5)–(8) give

$$\frac{d^2\overline{u}(\xi, q)}{d\xi^2} - \frac{\beta q^3 + a_0 q^2 + b_0 q + c_0}{\beta_1 q^2 + \alpha q + 1}\overline{u}(\xi, q) = 0, \tag{11}$$

$$\overline{u}(0, q) = \frac{1}{q}, \quad \overline{u}(\xi, q) \to 0 \text{ as } \xi \to \infty, \tag{12}$$

in which q is a Laplace transform parameter and

$$a_0 = M^2\beta + \frac{\beta_1}{K} + 1, \quad b_0 = 1 + M^2 + \frac{\alpha}{K}, \quad c_0 = M^2 + \frac{1}{K}, \tag{13}$$

$$\overline{u}(\xi, q) = L^{-1}\{u(\xi, \tau)\} = \int_0^\infty e^{-q\tau} u(\xi, \tau)\, d\tau.$$

The transformed solution of Equation (11) under the boundary conditions (12) gives

$$\overline{u}(\xi, q) = \frac{1}{q}\exp\left[-\xi\sqrt{\frac{\beta q^3 + a_0 q^2 + b_0 q + c_0}{\beta_1 q^2 + \alpha q + 1}}\right]. \tag{14}$$

In obtaining $u(\xi, \tau) = L^{-1}\{\overline{u}(\xi, q)\}$, we write Equation (14) as

$$\overline{u}(\xi, q) = \overline{u}_1(q)\ \overline{u}_2(\xi, q), \tag{15}$$

with

$$\overline{u}_1(q) = \frac{1}{q}, \tag{16}$$

$$\overline{u}_2(\xi, q) = \exp\left(-\xi\sqrt{w(q)}\right);$$
$$w(q) = \frac{\beta q^3 + a_0 q^2 + b_0 q + c_0}{\beta_1 q^2 + \alpha q + 1}. \tag{17}$$

Expressing $u_1(\tau) = L^{-1}\{\overline{u}_1(q)\}$, $u_2(\xi, \tau) = L^{-1}\{\overline{u}_2(\xi, q)\}$, Equation (16) after Laplace inversion gives

$$u_1(\tau) = 1. \tag{18}$$

To find $u_2(\xi, \tau) = L^{-1}\{\overline{u}_2(\xi, q)\}$, using the inversion formula for compound functions

$$L^{-1}\{F[w(q)]\} = \int_0^\infty f(u)\ g(u, \tau)\ du, \tag{19}$$

where $f(\tau) = L^{-1}\{F(q)\}$ and $g(u, \tau) = L^{-1}\{e^{-uw(q)}\}$. Choosing $f(\xi, q) = e^{-\xi\sqrt{q}}$, then

$$f(\xi, \tau) = L^{-1}\{e^{-\xi\sqrt{q}}\} = \frac{\xi}{2\tau\sqrt{\pi\tau}}\exp\left(\frac{-\xi^2}{4\tau}\right); \quad \xi > 0 \tag{20}$$

and

$$u_2(\xi, \tau) = L^{-1}\{\overline{u}_2(\xi, q)\} = \int_0^\infty f(\xi, u)\ g(u, \tau)\ du$$
$$= \frac{\xi}{2\sqrt{\pi}}\int_0^\infty \frac{1}{u\sqrt{u}}\exp\left(\frac{-\xi^2}{4u}\right) g(u, \tau)\ du. \tag{21}$$

In order to find $g(u, \tau) = L^{-1}\{e^{-uw(q)}\}$, we express $w(q)$ as follows

$$w(q) = b_1 + a_1 q + \frac{\eta_1}{q - q_1} + \frac{\eta_2}{q - q_2}, \tag{22}$$

$$a_1 = \frac{\beta}{\beta_1}, \ b_1 = \left(a_0 - \frac{\alpha\beta}{\beta_1}\right)\frac{1}{\beta_1}, \ c_1 = b_0 - \frac{\beta}{\beta_1} - \frac{\alpha}{\beta_1}\left(a_0 - \frac{\alpha\beta}{\beta_1}\right),$$
$$d_1 = c_0 + \left(a_0 - \frac{\alpha\beta}{\beta_1}\right)\frac{1}{\beta_1}, \ \eta_1 = \frac{c_1 q_1 + d_1}{q_1 - q_2}, \ \eta_2 = -\frac{c_1 q_2 + d_1}{q_1 - q_2}, \tag{23}$$

where q_1 and q_2 are the roots of the equation $\beta_1 q^2 + \alpha q + 1 = 0$. Thus,

$$g(u, \tau) = e^{-u\eta_0} L^{-1}\left\{\exp\left(-\frac{ua}{d}q\right)\ [1 - H_1(q) - H_2(q) + H_1(q)\ H_2(q)]\right\},$$

with

$$H_1(q) = 1 - \exp\left(-\frac{u\eta_1}{q - q_1}\right) \quad \text{and}$$
$$H_2(q) = 1 - \exp\left(-\frac{u\eta_2}{q - q_2}\right).$$

Let us denote

$$h_1(\tau) = L^{-1}\{H_1(q)\} = \sqrt{\frac{\eta_1 u}{\tau}}e^{q_1\tau} J_1\left(2\sqrt{\eta_1 u\tau}\right), \tag{24}$$

and

$$h_2(\tau) = L^{-1}\{H_2(q)\} = \sqrt{\frac{\eta_2 u}{\tau}} e^{q_2\tau} J_1\left(2\sqrt{\eta_2 u\tau}\right), \tag{25}$$

where $J_1(\cdot)$ denotes the Bessel function of first kind of order one and then finally one has

$$\begin{aligned} g(u,\tau) = {} & \delta(\tau - ua_1)\, e^{-ub_1} \\ & -\sqrt{\eta_1 u}\int_0^\tau \frac{\delta(s-ua_1)}{\sqrt{\tau-s}} e^{q_1(\tau-s)} J_1\left(2\sqrt{\eta_1 u(\tau-s)}\right) \mathrm{d}s \\ & -\sqrt{\eta_2 u}\int_0^\tau \frac{\delta(s-ua_1)}{\sqrt{\tau-s}} e^{q_2(\tau-s)} J_1\left(2\sqrt{\eta_2 u(\tau-s)}\right) \mathrm{d}s \\ & +u\sqrt{\eta_1\eta_2}\int_0^\tau\int_0^s \\ & \times \frac{\delta(\tau-s-ua_1)}{\sqrt{\sigma(s-\sigma)}} e^{q_1\sigma+q_2(s-\sigma)} J_1\left(2\sqrt{\eta_1 u\sigma}\right) \\ & J_1\left(2\sqrt{\eta_2 u(s-\sigma)}\right) \mathrm{d}s\,\mathrm{d}\sigma \end{aligned} \tag{26}$$

and $L^{-1}\{e^{-\alpha q}\} = \delta(\tau-\alpha)$. Here $\delta(\cdot)$ indicates the Dirac delta function.

Insertion of Equation (26) into Equation (21) leads to the following result:

$$\begin{aligned} u_2(\xi,\tau) = {} & \frac{\xi}{2\sqrt{\pi}}\int_0^\infty \frac{\delta(\tau-ua_1)}{u\sqrt{u}} \exp\left(\frac{-\xi^2}{4u} - b_1 u\right) \mathrm{d}u \\ & -\frac{\sqrt{\eta_1}\xi}{2\sqrt{\pi}}\int_0^\tau\int_0^\infty \frac{\delta(s-ua_1)}{u\sqrt{\tau-s}} \\ & \times \exp\left(\frac{-\xi^2}{4u} + q_1(\tau-s) - b_1 u\right) \\ & J_1\left(2\sqrt{\eta_1 u(\tau-s)}\right) \mathrm{d}u\,\mathrm{d}s \\ & -\frac{\sqrt{\eta_2}\xi}{2\sqrt{\pi}}\int_0^\tau\int_0^\infty \frac{\delta(s-ua_1)}{u\sqrt{(\tau-s)}} \\ & \exp\left(\frac{-\xi^2}{4u} + q_2(\tau-s) - b_1 u\right) \\ & J_1\left(2\sqrt{\eta_2 u(\tau-s)}\right) \mathrm{d}u\,\mathrm{d}s \\ & +\frac{\sqrt{\eta_1\eta_2}\xi}{2\sqrt{\pi}}\int_0^\tau\int_0^s\int_0^\infty \frac{\delta(\tau-s-ua_1)}{\sqrt{u\sigma(s-\sigma)}} \\ & \exp\left(\frac{-\xi^2}{4u} + q_1\sigma + q_2(s-\sigma) - b_1 u\right) \\ & \times J_1\left(2\sqrt{\eta_1 u\sigma}\right) J_1\left(2\sqrt{\eta_2 u(s-\sigma)}\right) \mathrm{d}u\,\mathrm{d}s\,\mathrm{d}\sigma. \end{aligned} \tag{27}$$

Taking into consideration Equations (27) and (18), one obtains

$$\begin{aligned} u(\xi,\tau) = {} & \frac{\xi}{2\sqrt{\pi}}\int_0^\tau\int_0^\infty \frac{\delta(s-ua_1)}{u\sqrt{u}} \exp\left(\frac{-\xi^2}{4u} - b_1 u\right) \mathrm{d}u\,\mathrm{d}s \\ & -\frac{\sqrt{\eta_1}\xi}{2\sqrt{\pi}}\int_0^\tau\int_0^s\int_0^\infty \frac{\delta(\sigma-ua_1)}{u\sqrt{s-\sigma}} \\ & \exp\left(\frac{-\xi^2}{4u} + q_1(s-\sigma) - b_1 u\right) \\ & J_1\left(2\sqrt{\eta_1 u(s-\sigma)}\right) \mathrm{d}u\,\mathrm{d}s\,\mathrm{d}\sigma \\ & -\frac{\sqrt{\eta_2}\xi}{2\sqrt{\pi}}\int_0^\tau\int_0^s\int_0^\infty \frac{\delta(\sigma-ua_1)}{u\sqrt{s-\sigma}} J_1\left(2\sqrt{\eta_2 u(s-\sigma)}\right) \\ & \exp\left(\frac{-\xi^2}{4u} + q_2(s-\sigma) - b_1 u\right) \mathrm{d}u\,\mathrm{d}s\,\mathrm{d}\sigma \\ & +\frac{\sqrt{\eta_1\eta_2}\xi}{2\sqrt{\pi}}\int_0^\tau\int_0^s\int_0^\sigma\int_0^\infty \frac{\delta(s-\sigma-ua_1)}{\sqrt{u\eta(\sigma-\eta)}} \\ & J_1\left(2\sqrt{\eta_1 u\eta}\right) J_1\left(2\sqrt{\eta_2 u(\sigma-\eta)}\right) \\ & \times \exp\left(\frac{-\xi^2}{4u} + q_1\eta + q_2(\sigma-\eta) - b_1 u\right) \mathrm{d}u\,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}\eta. \end{aligned} \tag{28}$$

Setting $u = dv/a$ into Equation (28) and using the following property:

$$\int_a^b f(x)\,\delta(x-x_0)\,dx = \begin{cases} f(x_0) & \text{for } x \in [a,b), \\ 0 & \text{for } x \notin [a,b) \end{cases} \tag{29}$$

we arrive at the following result:

$$\begin{aligned} u(\xi,\tau) = {} & \frac{\xi\sqrt{a_1}}{2\sqrt{\pi}}\int_0^\tau \frac{1}{s\sqrt{s}} \exp\left(\frac{-a_1\xi^2}{4s} - \frac{b_1 s}{a_1}\right) \mathrm{d}s \\ & -\frac{\sqrt{\eta_1}\xi}{2\sqrt{\pi}}\int_0^\tau\int_0^s \frac{1}{\sigma\sqrt{s-\sigma}} \\ & \exp\left(\frac{-a_1\xi^2}{4\sigma} + q_1(s-\sigma) - \frac{b_1\sigma}{a_1}\right) \\ & \times J_1\left(2\sqrt{\frac{\eta_1}{a_1}\sigma(s-\sigma)}\right) \mathrm{d}s\,\mathrm{d}\sigma \\ & -\frac{\sqrt{\eta_2}\xi}{2\sqrt{\pi}}\int_0^\tau\int_0^s \frac{\exp\left(\frac{-a_1\xi^2}{4\sigma} + q_2(s-\sigma) - \frac{b_1\sigma}{a_1}\right)}{\sigma\sqrt{s-\sigma}} \\ & \times J_1\left(2\sqrt{\frac{\eta_2}{a_1}\sigma(s-\sigma)}\right) \mathrm{d}s\,\mathrm{d}\sigma \\ & +\frac{\sqrt{\eta_1\eta_2}\xi}{2\sqrt{a_1}\pi}\int_0^\tau\int_0^s\int_0^\sigma \frac{J_1\left(2\sqrt{\frac{\eta_1}{a_1}\eta(s-\sigma)}\right)}{\sqrt{\eta(s-\sigma)(\sigma-\eta)}} \\ & \times \exp\left(\frac{-a_1\xi^2}{4(s-\sigma)} + q_1\eta + q_2(\sigma-\eta) - \frac{b_1}{a_1}(s-\sigma)\right) \\ & J_1\left(2\sqrt{\frac{\eta_2}{a_1}(s-\sigma)(\sigma-\eta)}\right) \mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}\eta. \end{aligned} \tag{30}$$

Now, the expression for the shear stress can be easily found from Equation (6) and hence finally we get.

$$\begin{aligned} s(\xi,\tau) = {} & \sqrt{\frac{a_1}{\pi\tau}} \exp\left(\frac{-a_1\xi^2}{4\tau} - \frac{b_1\tau}{a_1}\right) \\ & -\frac{\sqrt{\eta_1}}{a_1\sqrt{\pi}}\int_0^\tau \frac{1}{\sqrt{\tau-s}} \end{aligned}$$

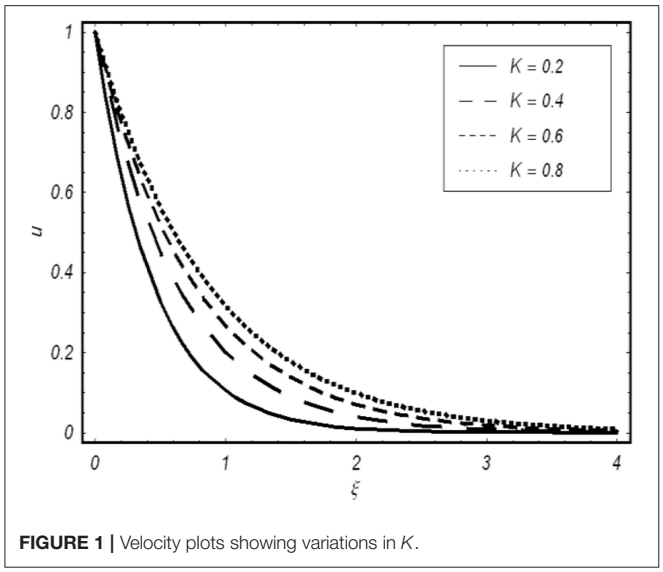

FIGURE 1 | Velocity plots showing variations in K.

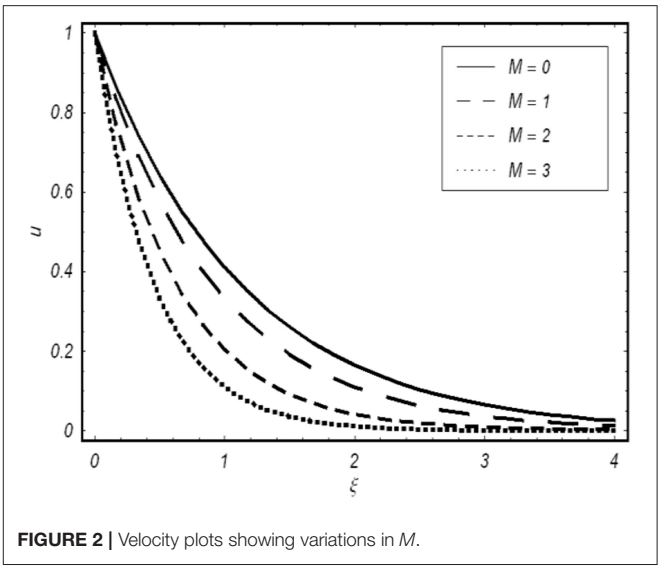

FIGURE 2 | Velocity plots showing variations in M.

$$\exp\left(\frac{-a_1\xi^2}{4s}+q_1(\tau-s)-\frac{b_1 s}{a_1}\right)J_1\left(2\sqrt{\frac{\eta_1}{a_1}s(\tau-s)}\right)\mathrm{d}s$$
$$-\frac{\sqrt{\eta_2}}{a_1\sqrt{\pi}}\int_0^\tau\frac{1}{\sqrt{\tau-s}}\exp\left(\frac{-a_1\xi^2}{4s}+q_2(\tau-s)-\frac{b_1 s}{a_1}\right)$$
$$J_1\left(2\sqrt{\frac{\eta_2}{a_1}s(\tau-s)}\right)\mathrm{d}s$$
$$+\frac{\sqrt{\eta_1\eta_2}}{a_1\sqrt{a_1\pi}}\int_0^\tau\int_0^s\frac{\sqrt{\tau-s}}{\sqrt{\sigma(s-\sigma)}}\qquad(31)$$
$$\exp\left(\frac{-a_1\xi^2}{4(\tau-s)}+q_1\sigma+q_2(s-\sigma)-\frac{b_1}{a_1}(\tau-s)\right)$$
$$\times J_1\left(2\sqrt{\frac{\eta_1}{a_1}\sigma(\tau-s)}\right)J_1\left(2\sqrt{\frac{\eta_2}{a_1}(\tau-s)(s-\sigma)}\right)\mathrm{d}s\,\mathrm{d}\sigma.$$

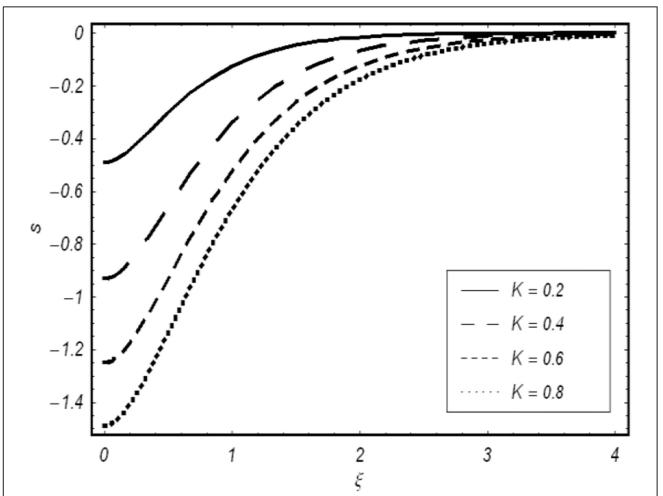

FIGURE 3 | Shear stress plots showing variations in K.

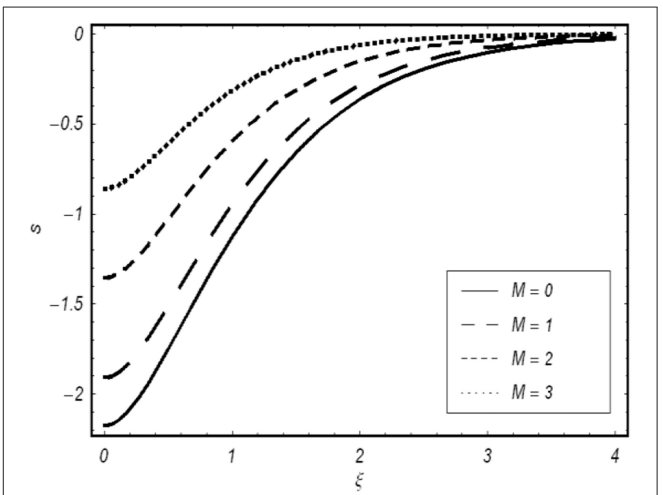

FIGURE 4 | Shear stress plots showing variations in M.

LIMITING CASES

Absence of MHD and Porosity

In limiting sense, when the magnetic effect is absent ($M = 0$) and the medium is non-porous, then the above solutions take the following forms:

$$u(\xi,\tau)=\frac{\xi\sqrt{\beta}}{2\sqrt{\beta_1\pi}}\int_0^\tau\frac{1}{s\sqrt{s}}\exp\left(\frac{-\beta\xi^2}{4\beta_1 s}-\left(\frac{1}{\beta}-\frac{\alpha}{\beta_1}\right)s\right)\mathrm{d}s$$
$$-\frac{\sqrt{\eta_3}\xi}{2\sqrt{\pi}}\int_0^\tau\int_0^s\frac{1}{\sigma\sqrt{s-\sigma}}$$
$$\exp\left(\frac{-\beta\xi^2}{4\beta_1\sigma}+q_1(s-\sigma)-\left(\frac{1}{\beta}-\frac{\alpha}{\beta_1}\right)\sigma\right)$$
$$\times J_1\left(2\sqrt{\frac{\beta_1\eta_3}{\beta}\sigma(s-\sigma)}\right)\mathrm{d}s\,\mathrm{d}\sigma$$
$$-\frac{\sqrt{\eta_4}\xi}{2\sqrt{\pi}}\int_0^\tau\int_0^s\frac{1}{\sigma\sqrt{s-\sigma}}J_1\left(2\sqrt{\frac{\beta_1\eta_4}{\beta}\sigma(s-\sigma)}\right)$$
$$\times\exp\left(\frac{-\beta\xi^2}{4\beta_1\sigma}+q_2(s-\sigma)-\left(\frac{1}{\beta}-\frac{\alpha}{\beta_1}\right)\sigma\right)\mathrm{d}s\mathrm{d}\sigma$$

$$+\frac{\sqrt{\beta_1\eta_3\eta_4}\xi}{2\sqrt{\beta\pi}}\int_0^\tau\int_0^s\int_0^\sigma \exp\left(\frac{-\beta\xi^2}{4\beta_1(s-\sigma)}+q_1\eta+q_2(\sigma-\eta)-\left(\frac{1}{\beta}-\frac{\alpha}{\beta_1}\right)(s-\sigma)\right)\Big/\sqrt{\eta(s-\sigma)(\sigma-\eta)}\times J_1\left(2\sqrt{\frac{\beta_1\eta_3}{\beta}\eta(s-\sigma)}\right)J_1\left(2\sqrt{\frac{\beta_1\eta_4}{\beta}(s-\sigma)(\sigma-\eta)}\right)\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}\eta. \quad (32)$$

$$\begin{aligned}s(\xi,\tau)=&\sqrt{\frac{\beta}{\beta_1\pi\tau}}\exp\left(\frac{-\beta\xi^2}{4\beta_1\tau}-\left(\frac{1}{\beta}-\frac{\alpha}{\beta_1}\right)\tau\right)\\&-\frac{\beta_1\sqrt{\eta_3}}{\beta\sqrt{\pi}}\int_0^\tau\frac{1}{\sqrt{\tau-s}}\exp\left(\frac{-\beta\xi^2}{4\beta_1 s}+q_1(\tau-s)-\left(\frac{1}{\beta}-\frac{\alpha}{\beta_1}\right)s\right)\\&\times J_1\left(2\sqrt{\frac{\beta_1\eta_3}{\beta}s(\tau-s)}\right)\mathrm{d}s\\&-\frac{\beta_1\sqrt{\eta_4}}{\beta\sqrt{\pi}}\int_0^\tau\frac{1}{\sqrt{\tau-s}}J_1\left(2\sqrt{\frac{\beta_1\eta_4}{\beta}s(\tau-s)}\right)\mathrm{d}s\\&\times\exp\left(\frac{-\beta\xi^2}{4\beta_1 s}+q_2(\tau-s)-\left(\frac{1}{\beta}-\frac{\alpha}{\beta_1}\right)s\right)\\&+\frac{\beta_1\sqrt{\beta_1\eta_3\eta_4}}{\beta\sqrt{\beta\pi}}\int_0^\tau\int_0^s\frac{\sqrt{\tau-s}}{\sqrt{\sigma(s-\sigma)}}\\&\times\exp\left(\frac{-\beta\xi^2}{4\beta_1(\tau-s)}+q_1\sigma+q_2(s-\sigma)-\left(\frac{1}{\beta}-\frac{\alpha}{\beta_1}\right)(\tau-s)\right)\\&J_1\left(2\sqrt{\frac{\beta_1\eta_3}{\beta}\sigma(\tau-s)}\right)\\&J_1\left(2\sqrt{\frac{\beta_1\eta_4}{\beta}(\tau-s)(s-\sigma)}\right)\mathrm{d}s\,\mathrm{d}\sigma,\end{aligned} \quad (33)$$

with the following expressions for η_3 and η_4:

$$\eta_3=\frac{\left(1-\frac{\beta}{\beta_1}-\frac{\alpha}{\beta_1}+\frac{\alpha^2\beta}{\beta_1^2}\right)q_1+\frac{\alpha\beta}{\beta_1^2}-\frac{1}{\beta_1}}{q_1-q_2},$$
$$\eta_4=-\frac{\left(1-\frac{\beta}{\beta_1}-\frac{\alpha}{\beta_1}+\frac{\alpha^2\beta}{\beta_1^2}\right)q_2+\frac{\alpha\beta}{\beta_1^2}-\frac{1}{\beta_1}}{q_1-q_2}. \quad (34)$$

It is important to note that if we put $M=\frac{1}{K}=0$ into the governing Equation (6) and solve along with Equation (7) with the prescribed boundary and initial conditions, we get the same expressions for velocity and shear stress as given above.

Newtonian Fluid

For Newtonian fluid, we make λ, λ_r, γ, and γ_1 equal to zero or equivalently $\lambda=\lambda_r=\gamma=\gamma_1$, then the solutions (30) and (31) reduce to

$$u(\xi,\tau)=\frac{\xi}{2\sqrt{\pi}}\int_0^\tau\frac{1}{s\sqrt{s}}\exp\left(\frac{-\xi^2}{4s}-\left(M^2+\frac{1}{K}\right)s\right)\mathrm{d}s, \quad (35)$$

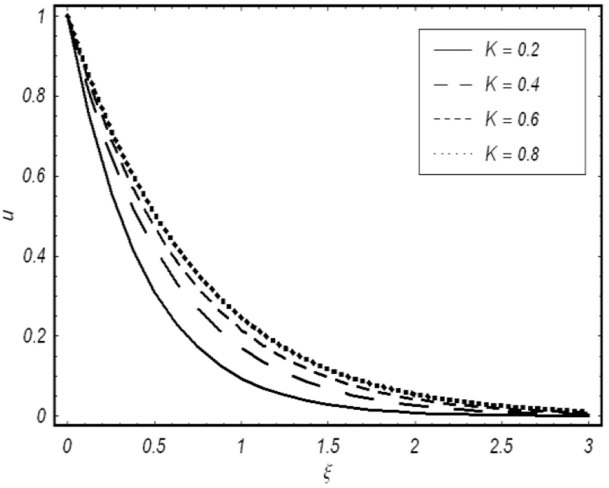

FIGURE 5 | Velocity plots showing variations in K (Newtonian fluid).

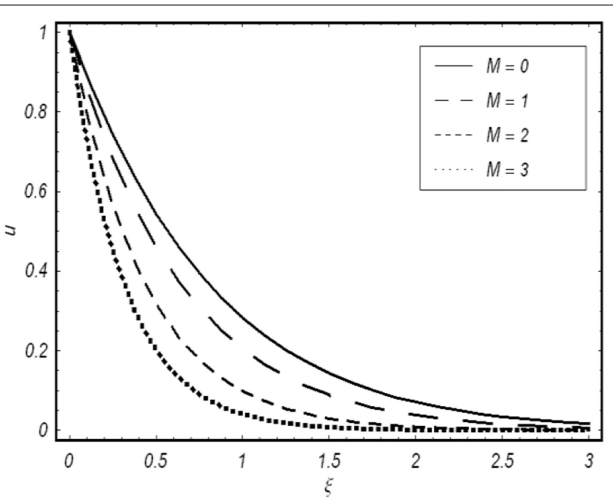

FIGURE 6 | Velocity plots showing variations in M (Newtonian fluid).

$$s(\xi,\tau)=\frac{1}{\sqrt{\pi\tau}}\exp\left(\frac{-\xi^2}{4\tau}-\left(M^2+\frac{1}{K}\right)\tau\right). \quad (36)$$

Now, taking $\lambda=\lambda_r=\gamma=\gamma_1$ in the governing Equation (6) and solving the resulting equations with the given boundary and initial conditions, we get

$$\overline{u}(\xi,q)=\frac{1}{q}\exp\left(-\xi\sqrt{q}\right)=\overline{u_1}(q)\,\overline{u_2}(\xi,q) \quad (37)$$

where

$$\overline{u_1}(q)=\frac{1}{q}\quad\text{and}\quad\overline{u_2}(\xi,q)=\exp\left(-\xi\sqrt{q+M^2+\frac{1}{K}}\right). \quad (38)$$

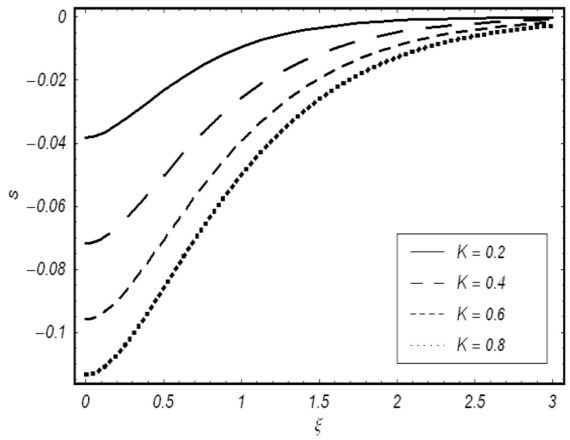

FIGURE 7 | Shear stress plots showing variations in K (Newtonian fluid).

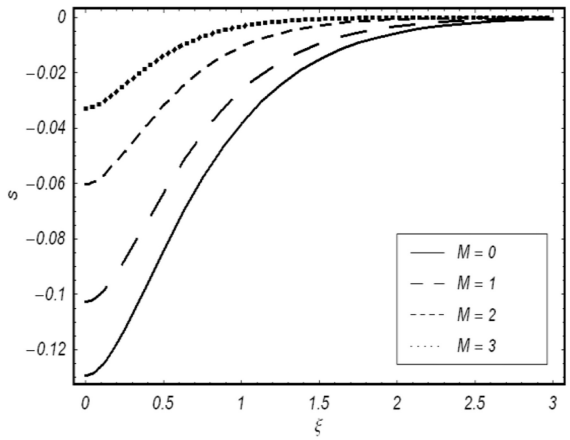

FIGURE 8 | Shear stress plots showing variations in M (Newtonian fluid).

Taking the Laplace inverse of Equation (38), we get

$$u_1(\tau) = 1, \quad u_2(\xi, \tau) = \frac{\xi}{2\tau\sqrt{\pi\tau}} \exp\left(\frac{-\xi^2}{4\tau} - \left(M^2 + \frac{1}{K}\right)\right). \quad (39)$$

The convolution product of $u_1(\tau) = 1$ and $u_2(\xi, \tau)$ gives

$$u(\xi, \tau) = \frac{\xi}{2\sqrt{\pi}} \int_0^\tau \frac{1}{s\sqrt{s}} \exp\left(\frac{-\xi^2}{4\tau} - \left(M^2 + \frac{1}{K}\right) s\right) \mathrm{d}s. \quad (40)$$

The corresponding shear stress can be easily found by using Equation (6); i.e.,

$$\bar{s}(\xi, q) = -\frac{\exp\left(-\xi\sqrt{w(q)}\right)}{\sqrt{w(q)}}; \quad w(q) = q + M^2 + \frac{1}{K}. \quad (41)$$

Using a similar method as in the case of velocity, the final expression for the shear stress is given as follows:

$$s(\xi, \tau) = \frac{1}{\sqrt{\pi\tau}} \exp\left(\frac{-\xi^2}{4\tau} - \left(M^2 + \frac{1}{K}\right)\tau\right). \quad (42)$$

Here, we noted that in both cases, i.e., from the final solutions given by Equations (30) and (31) and from the governing Equations (5) and (6), we obtained the same exact results for velocity and shear stress given by Equations (35), (36), (40), and (42), respectively. Indeed, this provides a useful check of correctness.

NUMERICAL RESULTS AND DISCUSSION

Figure 1 is plotted for $K = 0.2, 0.4, 0.6, 0.8$ when $M = 0.2$, $\alpha = 0.9$, $\beta_1 = 0.5$, $\beta = 0.8$ and $\tau = 0.5$, whereas **Figures 2**, **4** are sketched for $M = 0, 1, 2, 3$ when $K = 2$, $\alpha = 0.9$, $\beta_1 = 0.5$, $\beta = 0.8$, and $\tau = 0.5$. **Figures 1–4** have been displayed to see the influence of Hartmann number M and porosity parameters K on the fluid velocity and the corresponding shear stress of a GBF. To check the effects of M and K on the fluid velocity and related shear stress for a Newtonian fluid, **Figures 5–8** are sketched. **Figures 5**, **7** are plotted for different values of K when $M = 0.2$ and $\tau = 0.5$, whereas **Figures 6**, **8** are prepared for various values of M when $K = 2$ and $\tau = 0.5$. Note that **Figures 1–8** provide a comparison of velocity field and the related shear stress for the case of GBF with that of a Newtonian fluid. **Figure 1** shows the influence of K on the Burgers' fluid velocity; it can be noticed that velocity increases with the increasing values of K, due to the decrease in opposing forces. In **Figure 2**, the impact of M is shown on fluid velocity; from this figure, it is noticed that velocity is a decreasing function of M. This is because the greater values of M enhance the Lorentz forces, which are the opposing forces. The same behavior is noticed in **Figures 5**, **6** for Newtonian fluid. **Figure 3** is plotted in order to show the effect of K on shear stress; the shear stress decreases with the increasing values of K. The behavior of shear stress is noticed for different values of M in **Figure 4**. It is observed that the shear stress increases with the increasing values of M. **Figures 7**, **8** also show the same behavior of shear stress for Newtonian fluid.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

AA formulated the problem. IK solved the problem and discussed results.

FUNDING

This research was funded by Deanship of Scientific Research at Princess Nourah bint Abdulrahman University (Grant No. 39/S/269: ٣٩/ص/٢٦٩).

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

NOMENCLATURE

v (m/s)	Velocity component in the $x-$ direction
λ (s)	Relaxation time
ρ (kg/m^3)	Fluid density
λ_r ($< \lambda$) (s)	Retardation time
μ (kg/m s)	Dynamic viscosity
γ (s^2) and γ_1 (s^2)	Material constants having the dimensions as the square of time
δ (s^{3}A^2/kg m^3)	Finite electrical conductivity
V (m/s)	Reference velocity
φ ($0 < \varphi < 1$)	Porosity
B_0 (kg/s^{2}A)	Applied magnetic field
$k > 0$ (m^2)	The permeability of the porous medium
$v(y, t)$ (m/s)	Fluid velocity
$T(y, t)$ (kg/s^{2}m)	Shear tress
$u(\xi, \tau)$ (m/s) ands (ξ, τ) (Pa)	Dimensionless fluid velocity and shear stress