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Contact Interactions and Gamma Convergence

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We introduce contact interactions defined by boundary conditions at the *contact manifold* $\Gamma \equiv \cup_{i,j}\{x_i = x_j\}$. There are two types of contact interactions, weak and strong. Both provide self-adjoint extensions of $\hat{H}_0$ the free hamiltonian restricted away from Γ. We analyze both of them by "lifting" the system to a space of more singular functions: the map is fractioning and mixing. In the new space we use tools of Functional Analysis. After returning to physical space we use Gamma convergence, a well-known variational tool. We prove that contact interactions are strong resolvent limits of potentials with finite range. Weak contact of one boson with two other bosons leads to the low-density Bose-Einstrin condensate. Simultaneous weak contact of three bosons produces the high-density condensate which has an Efimov sequence of bound states. In Low Energy Physics strong contact of one particle with another two produces an Efimov sequence of bound states (we will comment briefly on the relation with the effect with the same name in Quantum Mechanics). For N bosons strong contact gives a lower bound $-CN$ for the energy. A system of fermions in strong contact (unitary gas) has a positive hamiltonian. We give several examples in dimension 3,2,1. In the Appendix we describe the ground state of the Polaron.

Keywords: contact, gamma convergence, interaction, Efimov, Bose Einstein

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1. INTRODUCTION

In Classical Mechanics constraints we describe forces restricting the motion of two systems when they are in contact.

In Quantum Mechanics it is convenient to use the Heisenberg representation and describe the system by means of self-adjoint operators on some function space. Each self-adjoint operator has a domain of definition.

We consider first in some detail the dynamics in R^3 and later consider the case of dimension two and dimension one.

Contact (zero range) interactions in R^3 are defined by imposing that the wave function in the domain of the hamiltonian satisfies at the *coincidence manifold* Γ

$$\Gamma \equiv \cup_{i,j}\Gamma_{i,j} \qquad \Gamma_{i,j} \equiv \{x_i - x_j\} = 0, \quad i \neq j \, x_i \in R^3. \tag{1}$$

the boundary conditions

$$\phi(X) = \frac{C_{i,j}}{|x_i - x_j|} + D_{i,j} \quad i \neq j \tag{2}$$

These conditions were used already in 1935 by Bethe and Peirels [1] in the description of the interaction between a proton and neutron.

They were later used by Ter-Martirosian and Skorniakov [2] in their analysis of the three-body scattering within the Faddeev formalism. We shall call them Ter-Martrosian [2] boundary conditions. In the weak contact case *for each contact there is a zero-energy resonance* i.e., a function that *at infinity* is proportional to $\frac{1}{|x_i - x_j|}$.

For contact interactions it is easy to determine the spectra; the interest in the subject was renewed in Theoretical Physics by recent advances in the theoretical formulation of low energy physics and also by the flourishing of research on ultra-cold atoms interacting through potentials of very short range.

The T-M boundary conditions can be described by potentials $V_{i,j}(|x_i - x_j|)$ and $U_{i,j}(|x_i - x_j|)$ hat are *distributions supported by the boundary*

$$V_{i,j} = -C_{i,j}\delta(x_i - x_j) \quad \frac{d}{d\rho_{i,j}} U_{i,j}(\rho_{i,j}) = -D_{i,j}\delta(x_i - x_j)$$
$$C_{i,j} > 0 \;\; D_{i,j} > 0 \qquad \rho_{i,j} = |x_i - x_j| \qquad (3)$$

This can be verified by taking the scalar product with a function in the domain of $\hat{H}_0$ (the free hamiltonian restricted to functions that vanish in a neighborhood of Γ) and integrating by parts.

This condition implies a *very singular modification of the free dynamics* at coincidence hyper-planes.

At the boundary, the solutions are not in the domain of the free hamiltonian; *solution of the Schrödinger equation* is only meant in a weak sense, after averaging with a smooth function and integrating by parts.

The equation holds *in the sense of quadratic forms*. Quadratic form techniques play an important role.

2. STRONG AND WEAK CONTACT

We call *strong contacts* the self-adjoint extension characterized by $D_{i,j} = 0$ and *weak contacts* the one characterized by $C_{i,j} = 0$.

From a mathematical point of view, the resulting operators are *self-adjoint extensions* of the symmetric operator $\hat{H}_0$, the free hamiltonian for three particles of mass m defined on functions that have support away form Γ.

Notice in the case of strong contact both the free hamiltonian and the potential define quadratic forms (of opposite sign) on absolutely continuous functions.

The (negative) potential is defined on the larger class of continuous functions and there takes a finite value.

On continuous functions that are not absolutely continuous the quadratic form of the free hamiltonian "is infinite."

Therefore, in a two-particle system strong contact cannot be defined.

Weak contact can be defined but its domain contains a zero-energy resonance.

We shall prove that in three particle system *separate* strong contact of one particle with two other particles can be defined.

If the potential is sufficiently strong the system has an Efimov spectrum.

We consider mainly the case in which all particles have the same mass. In the case of strong contact of one of the particles with the other two the spectrum of Efimov type.

A small difference in the masses does not change the structure of the spectrum.

The (energy) scale is given by the mass and by the ratio between the strong and weak contact coefficients, if they are both present.

In the case of weak contact, the distributional potential at the boundary has the same scaling property under dilation as the kinetic energy. Therefore, the hamiltonian of weak contact is scale covariant under the dilation group.

This by itself is an indication of the presence of a resonance. Point interactions [3] can be seen as a weak contact interaction between two particles one of which is infinitely massive (with wave function concentrated in a point).

We emphasize that both strong and weak contact hamiltonian are needed to classify completely the zero range interactions.

We will prove that they produce *complementary and independent effects*.

Both effects are independent from those due to the possible presence of regular potentials (that may cause further resonances).

For the proof of independence, we shall use a general form for the resolvent of the interacting system, due to Krein (we use the Konno and Kuroda [4] improved version).

It should be noted that there is another "natural" extension of $\hat{H}_0$ which is obtained by imposing Dirichlet boundary conditions on some or all contact, manifold.

Imposing Dirichlet boundary conditions is an alternative procedure and *does not correspond* to the limit of attractive potentials V^ϵ.

Remark: To make a connection with the interaction trough two-body potentials, we will prove that the three-body strong contact hamiltonian is limited, in a strong resolvent sense, to Hamiltonians with two body potentials that scale as $V^\epsilon(|x|) = \frac{1}{\epsilon^3} V(\frac{|x|}{\epsilon})$.

Two body weak contact is the limit, in strong resolvent sense, of hamiltonian with potentials that scale as $U^\epsilon(|x|) = \frac{1}{\epsilon^2} U(\frac{|x|}{\epsilon})$ and have a zero energy resonance.

We will show that *in general* (and not only for point interactions [3]) weak contact requires the approximating hamiltonians to have a zero-energy resonance (infinite scattering length).

Since in the case of contact interactions the spectra and spectral properties can be given explicitly, contact interactions are a valuable tool for very short-range potentials.

We shall analyze in detail the case of the separate strong contact of a particle with other two (all particles have the same mass) and the case of the (weak) contact between two pairs of particles which are themselves in strong contact.

With the same formalism we analyze the case of three particles in which every particle has a weak contact interaction with the other two.

Notice that *simultaneous strong contact* of three particles leads to divergences [5].

3. MATHEMATICAL FORMULATION

From a mathematical point of view the problem of zero range interaction between three particles was first analyzed by Pavlov [6] who investigated the self-adjoint extensions defined by the condition that the wave function takes a finite value at the boundary $\Gamma_{i,j}$ (weak contact).

The problem was later studied by Minlos [7, 8] concentrating on (the physically relevant) case of two identical particles one interacting through "zero range potentials" with a particle of the same mass.

The same analysis applies to a system of three identical particles in which each has a strong contact with the other two. For the sake of simplicity all particles have mass one.

To analyze the system, we introduced a compact invertible map (the Krein map $\mathcal{K}$) to a space *of more singular functions*. The map depends on a positive parameter λ; this parameter will play a role in the semiclassical limit.

We call the map "Krein map" $\mathcal{K}$ because our steps are in the path of the theory of self-adjoint extensions of positive operators by Birman [9] and Krein [10], *but on the side of quadratic forms* as suggested in Klaus and Simon [11] (the advantage of this formalism is also remarked in Cassano and Pizzichillo [12]).

The idea of using this map came from reading [7, 8] and therefore we will call Minlos space $\mathcal{M}$ the image space.

The Krein operator is $(H_0+\lambda)^{-\frac{1}{2}}$.

The Krein map *acts differently* on the kinetic term and on the potential tem.

It acts on the kinetic energy (seen as operator) as follows $(H_0+\lambda) \to (H_0+\lambda)^{\frac{1}{2}}$.

It acts on the delta potential (seen as quadratic form) as follows $\delta \to (H_0+\lambda)^{-\frac{1}{2}}\delta(H_0+\lambda)^{-\frac{1}{2}}$.

Since for strong and weak contact the distributional potential "commutes" with the free hamiltonian (as seen by taking the Fourier transfor) in $\mathcal{M}$ the quadratic form is also the quadratic form of $\delta(H_0+\lambda^{-1}$ and this shows a formal relation with Birman [9] and Krein [10].

For very short distances the Krein map enhances the potential term with respect to the kinetic energy.

As a consequence, the quadratic form kinetic energy + potential may become unbounded below (the potential is attractive).

Notice that the Krein is invertible; after inversion, the system was not changed.

The Krein map is only a tool to extract information, a magnifying glass

We distinguish between two cases.

a) Weak contact

   If in $\mathcal{M}$ the result is a unique strongly closed quadratic form, by rotation invariance it can be decomposed into *strictly convex* quadratic form.

   Their image under inversion of the Krein map are weakly closed strictly convex quadratic forms.

   Since the forms are bounded below (the inversion changes the topology of the space) they can be closed strongly [13] and define self-adjoint operators in "physical" space.

   Depending on the strength of the potential there may be a finite number of bound states.

b) Strong contact

   In position space in for $l=0$ by construction the potential term is the sum of a first order pole (with negative coefficient $-C_0$) and a smooth positive quadratic form Ξ_0 which is zero on the diagonal.

   The form Ξ_0 corresponds to a bounded positive operator; its contribution can be analyzed using perturbation theory.

   Since the Krein map is only an instrument to evidence *general features* of the interaction term *at small distance*, such bounded operators play no role in the following. We shall therefore neglect it.

If C_0 is sufficiently large one has a Weyl limit circle degeneracy [14] and a one-parameter family of self-adjoint extensions all unbounded below and each with an infinite negative point spectrum that diverges linearly. Inversion of the Krein map gives a family of quadratic forms *that are only weakly closed* and bounded below.

If there is one which admits a strong closure, this form defines a self-adjoint operator in physical space with an Efimov spectrum.

This form is obtained using Gamma convergence, a procedure often used in the analysis of finely fragmented materials (we will give later the definitions).

It is a minimization procedure for families of strictly convex forms (not necessary quadratic).

Gamma-convergence selects from a sequence of *strictly convex* forms, a unique one that has a strong closure. This selected form is called Gamma limit. The name *Gamma limit* is used because Gamma convergence is *a minimization process*.

Recall that the Gamma limit of a sequence of *strictly convex* weakly closed forms F_n in a topological space Y is the *unique* weakly closed quadratic form F such that for any subsequence the following holds

$$\forall y \in Y, \quad y_n \to y, \quad F(y) \leq liminf F(y_n) \quad limsup F(y_n) \geq F(y) \tag{4}$$

The limited form is strictly convex and therefore strongly closable.

The condition for the existence of the Gamma-limit is that the sequence be contained in a compact set of Y. In the present case Y has the Frechet topology given by Sobolev semi-norms.

Compactness of bounded sets is assured *by the absence of zero energy resonances*,

Therefore, there is a minimizing (Palais-Smale) sequence.

From the point of view of Functional Analysis, it is important that *Gamma convergence implies resolvent convergence* [15].

Notice that *this is precisely what is done in the study of composite materials: one first acquires information on the "small scale structure" and then draws conclusions at a macroscopic scale*.

We have noticed that the Krein map is *fractioning and mixing*.

This explains why inverting the Krein map requires tools from the theory of homogenization (Gamma convergence

[15]) a procedure often used in the analysis of finely fragmented materials.

If the interaction is strong enough there is an Efimov sequence of bound states.

It is easy to verify that these states (called "trimers" in the Physics literature) have increasingly larger essential support.

Therefore, only the first few members of the series can be detected experimentally.

For an outlook on experimental and theoretical results on the three and four body problem one can consult [16, 17].

4. STRONG AND WEAK CONTACT INTERACTIONS AS LIMITS

a) Strong contact

We prove that strong-contact hamiltonians are limited *in a strong resolvent sense* of finite range hamiltonians.

This makes contact hamiltonians a valuable tool in the study of interactions of a very small range.

We require that the potential $V(|x|)$ be of class C^1. It defines therefore a quadratic form in $\mathcal{H}^1$.

By duality, it is a bounded map from $\mathcal{H}^2$ to C^1 (this explains why we find hamiltonians that are bounded below).

We consider separately the restriction to irreducible representation of the rotation group (the approximating potentials are invariant under rotation).

The quadratic form associated to the potentials V^ϵ is a decreasing function of ϵ (the potential is negative).

Since there is no zero-energy resonance the sequence of the approximating hamiltonians belongs to a compact subset for topology given by the Sobolev semi-norms.

The potential V is negative therefore for any choice of $V \in C^1 \cap L^1$ the ϵ-dependent quadratic forms are *stricly* decreasing as function of ϵ.

A lower bound is the quadratic form of the contact interaction.

A decreasing sequence in a compact set with a lower bound always admits a converging subsequence.

If the sequence is strictly decreasing the limit point is unique.

If the potential is of class C^1 the limit of this converging minimizing subsequence belongs to the limit set of the contact interactions.

Since this set contains only one element for any choice of the L^1 norm of the approximating potentials, the limit is unique and coincides with the contact interaction with the same strength.

Gamma convergence implies strong resolvent convergence [15]. Therefore the sequence of self-adjoint operators with potentials $H_0 + V^\epsilon, \quad V^\epsilon \in C^1$ have in strong resolvent sense a limit which is the resolvent of the strong contact hamiltonian (which depends on the L^1 norm of the approximation potentials but *not on the shape*).

In turn strong resolvent convergence implies convergence of spectra and of the Wave Operator in Scattering Theory.

We have proved

Theorem 1: The hamiltonian of a system describing the strong contact interactions of a particle with two identical bosons is limit, in the strong resolvent sense, of hamiltonians with two body *negative* potentials of class C^1 that have support that shrinks to a point with law $V^\epsilon(|x|) = \frac{1}{\epsilon^3} V(\frac{|x|}{\epsilon})$. The limit hamiltonian is bounded below.

There are constant $C_1, \ C_2$ such that if $|V|_1 < C_1$ the negative spectrum is empty, if $C_1 \leq |V|_1 < C_2$ the strong contact hamiltonian has a finite negative spectrum while if $|V_1| \geq, C_2$ the negative spectrum is of Efimov type (the sequence of eigenvalues converges geometrically to zero).

In this latter case the eigenfunctions are centered on the barycenter of the system and have increasing support. $\diamond$

Remark: The same is true with the same proof in a three-particle system in which each pair has a separate strong contact interaction.

b) Weak contact

In the case of weak contact, the proof does not apply because the domain of the limit operators contains a zero-energy resonance and compactness in the topology of the Sobolev semi-norms fails.

This is the reason why the approximating potentials V must have a zero-energy resonance.

Since the potentials V^ϵ are obtained by scaling the resonance is independent from ϵ and can be chosen to be the same as the resonance of the weak contact hamiltonian.

Therefore, the domain of difference between the approximating potentials and the weak contact is inL^2 and one has compactness in the Sobolev semi-norms.

5. THE BIRMAN-KREIN-SCHWINGER FORMULA

The role on Gamma convergence in Quantum Mechanics is clearly seen considering the Birman-Krein -Schwinger formula for the difference of the resolvents of two self-adjoint operators H and H_0 both bounded below.

$$(H+\lambda)^{-1} - (H_0+\lambda)^{-1}) = (H_0+\lambda)^{-1} W_\lambda (H_0+\lambda)^{-1} \tag{5}$$

where λ is greater than the lower bound of the spectra.

W_λ is the *Krein kernel*, the quadratic form of a symmetric operator.

Usually H_0 is the free hamiltonian, but one can make other choices (for example H_0 may be the magnetic free hamiltonian, a positive operator).

The B-K-S formula, which is the basis for a perturbative analysis, clearly shows the role of Gamma convergence for strong contact in a non-perturbative setting.

This formula can be written as

$$(H+\lambda)^{-1} - (H_0+\lambda)^{-1}) = \{K_\lambda\}\{K_\lambda\}W_\lambda) \tag{6}$$

where $\{K_\lambda\}$ is the Krein map.

This formula is ill-defined as it stands because the image of the Krein map is a *weakly closed form* and the Krein map is defined only for strongly closed forms (self-adjoint operators).

If the image is closable in the strong topology one can take its closure before the second Krein map.

If not, one must select one of the weakly closed forms that has a strong closure.

Gamma convergence is the instrument to make this selection.

In the more general setting the formula should be replaced by

$$(H + \lambda)^{-1} - (H_0 + \lambda)^{-1}) = \{K_\lambda\}\Gamma(\{K_\lambda\}W_\lambda) \tag{7}$$

where the symbol Γ indicates that it is necessary to use Gamma convergence.

6. BOUNDARY CHARGES

An important aspect of contact interactions is that they are extension of H_0 that are *entirely due to "charges at the boundary."*

In the present case the boundary is *internal* i.e., they are the *contact manifolds*.

Compare with electrostatics: in that case the boundary has co-dimension one and the Krein map can be identified with the Weyl map from potentials to charges.

It is therefore natural to refer to Minlos space as the space of charges [18].

The distribution of "charges at the boundary" determine uniquely the self-adjoint extension; each function in the domain can be written as the sum of a part determined by the charges and a regular part in the domain of H_0. We sketch here the proof.

Let H be the self-adjoint extension that represent the contact interaction.

Choose λ in such a way that $H + \lambda I$ is invertible.

By construction, the quadratic form of $H + \lambda$ is the sum of the quadratic form of $H_0 + \lambda$ and a quadratic form in Krein space.

The elements in the form domain of the contact hamiltonian H are of the form $\psi = \phi + \zeta$ where $\phi \in D(H_0)$ and ζ is in Krein space [11].

The action of H on elements in its domain is

$$(H + \lambda)\psi = (H_0 + \lambda)\phi \qquad \psi = \phi + \{K_\lambda\}\psi \tag{8}$$

The formal proof (modulo control of the domains) is as follows

$$\begin{aligned}&((H + \lambda)\psi, \frac{1}{H_0 + \lambda}(H + \lambda)\psi)\\ &\quad = ((H_0 + \lambda)\phi, \frac{1}{H + \lambda}(H_0 + \lambda)\phi)\\ &\quad = (\phi, (H_0 + \lambda)\phi) + (\{K_\lambda\}\phi, W_\lambda\{K_\lambda\}\phi)\end{aligned} \tag{9}$$

This is same procedure as for finite range potentials; Gamma convergence substantiates this formal argument.

Therefore, only "the space of charges" enters in the description of the domain.

Notice the analogy with electrostatics; the singular part is determined by the charges. The Weyl map takes the role of the Krein map. ♡

7. SOME REMARKS

I

It is worth stressing the connection with the *theory of boundary triples* [19].

This a generalization of the Weyl map in electrostatics from potential in a bounded set Ω in R^3 with regular boundaries to charges at the boundary $\partial\Omega$.

In this context the Krein map may be regarded as a Weyl map between "potentials" and "charges" (the charges belong to a space of more singular functions).

But in the present setting the "boundary charges" are placed on a boundary of co-dimension three (the contact manifold) and not on an external boundary of co-dimension one as in electrostatics (and in most of the papers on boundary triples).

For contact interactions the boundary *is internal* [20].

II

As is often the case for variational arguments for quadratic forms, the eigenfunctions of the minima *are not in the domain of the free hamiltonian*.

The minimum is obtained "by compensation" of the divergences of the kinetic and the potential contributions.

Since the eigenstates *are not in the domain of H_0 perturbation theory does not apply*.

The solutions of the Schrödinger equation with contact interaction belong to the space Ξ of functions that are at any time twice differentiable (more precisely in $\mathcal{H}^2$) away from the contact manifold but have a $\frac{1}{|x_i - x_k|}$ singularity at the contact manifold.

Since the Schrödinger flow is dispersive the entire set of solutions has this property.

Bound states are critical points of the energy functional (as in the classical case). Scattering and Wave operators are defined in the space Ξ.

III

It worth recalling that Gamma convergence gives simple results but since it is not a perturbative method it is difficult to evaluate the error made in using contact interactions. The error is of second order in ϵ (it is a minimization process) but the coefficient is not determined.

In the application to nuclear physics a comparison with experimental data [16, 17, 21] gives a reasonable agreement.

IV

One can take the limit in which the two masses of the particles that are not in contact is taken to be infinite; the resulting system is a particle in strong contact with two fixed points (a two-point interaction).

Also, this system has no zero-energy resonance.

On the contrary one cannot assign infinite mass to the particles which interact separately with the two other particles (two identical particles in strong contact with a fixed point); the procedure we follow gives in this limit a divergent result.

8. ON THE ROLE OF GAMMA CONVERGENCE

It is surprising that a formalism invented by de Giorgi more than 50 years ago to handle singular variational problems, and of common use in applied mathematics (and in industry) to treat finely fractured materials, plays an important role in such diverse fields as in the construction of self-adjoint extensions of positive symmetric operators, in the determination of the Efimov structure of three and four bodies systems in high energy physics, in explaining the role of zero energy (Fesbach) resonances (and the secret of the missing $\frac{1}{N}$) in the dilute case of Bose Einstein condensates and in finding the structure of the ground state in at high density.

9. SEMICLASSICAL LIMIT

The energy form for a three-body problem in which all particles have equal mass and each particle has a strong contact interaction with the other two can be evaluated on coherent states centered on the points $x_i, \quad i = 1, 2, 3$ and with "classical momentum" p_1, p_2, p_3.

The result is the "classical" energy functional of the newtonian three body problem.

The hamiltonian of the classical three-body problem admits infinitely many periodic solutions which correspond to critical points of the classical energy functional. Their energy decreases geometrically.

The three-body problem and its periodic solutions are well studied in Classical Mechanics.

In quantum mechanics for identical bosons in strong pairwise contact interactions there is an Efimov sequence of bound states i.e., infinite sequence of bound states with negative eigenvalues that decrease geometrically to zero (as in the classical case).

These are critical points of the quantum energy functional.

This suggests that the classical newtonian three body problem be the semiclassical limit of the quantum mechanical problem of strong contact interaction of one particle with the other two in a system of three identical particles.

Notice that variational problems are studied *using quadratic forms* (and not operators).

In the next section we shall show that the Krein map is related to the semiclassical limit.

10. RELATION OF THE KREIN MAP WITH SEMICLASSICAL LIMIT

We have indicated that in Milnos space the kinetic part of the hamiltonian is represented by $\sqrt{H_0 + \lambda}$ for some (arbitrary) positive λ.

In the three-body problem with strong contact in $\mathcal{M}$ the contact potential is represented by the quadratic form of an attractive Coulomb potential.

For the Krein map we can choose $\lambda > 0$ with the only requirement that the operator $H + \lambda$ is positive.

For λ large on can develop on a dense domain the square root as

$$\sqrt{H_0 + \lambda} = \sqrt{\lambda} + \frac{1}{2}\frac{H_0}{\sqrt{\lambda}} + O(\lambda^{-\frac{3}{2}}) \qquad (10)$$

Setting $\frac{1}{\sqrt{\lambda}} = \hbar$, apart from an irrelevant constant, this is to first order in $\hbar$ the free hamiltonian of the quantum system.

When evaluated on coherent states this is the classical free hamiltionian.

When λ is large, in $\mathcal{M}$ strong contact potentials are represented by the Coulomb potential $-\frac{C}{|x_i - x_i|} \quad C > 0$.

Therefore, for λ very large the map $\mathcal{K}_\lambda$ can be related to a semi-classical limit and in $\mathcal{M}$ the free hamiltonian tends (a part for a large constant) to the Classical hamiltonian.

In the semiclassical limit the free hamiltonian is scaled by a factor to $\hbar^{-2}$ and the Coulomb potential is scaled by a factor $\hbar^{-1}$.

If we identify the radius of the potential (the parameter ϵ) with $\hbar$ (both have the dimension of a length) the limit $\hbar = \epsilon \to 0$ gives contact interaction at a quantum scale, Coulomb interaction at a semiclassical scale.

Therefore, the Newtonian three body problem can be considered semiclassical limit of the quantum three body problem with pairwise strong contact interaction.

Addition of a magnetic potential is represented as usual with the substitution $i\nabla \to i\nabla + A$.

11. WEAK CONTACT

Now we consider the case of weak contact.

Since the weak contact potential has the same scaling properties under dilations as the kinetic energy in order to have an hamiltonian which is bounded below there can be at most as many weak contacts as particles.

In R^3 with Riemann stricture weak contact of two particles as self-adjoint extension can occur *only if there is a zero-energy resonance* (infinite scattering length for the approximation potential).

If one of the particles has infinite mass (so that it may be considered as a fixed point) weak contact is called *point interaction*.

In Adami et al. [22], it is proved that this extension cannot exist in a three-dimensional manifold with a sub-Riemmannian manifold so that the operator defined on $R^3 - \{0\}$ is essentially self-adjoint.

In spite of the richness of the mathematics it has produced weak contact of two particles has severe limitations in the physical applications.

This is not the case for weak contact of two particles in a three-particle system (the difference is that the position of the barycenter of the two-particle subsystem cannot be fixed).

In this case there is a self-adjoint extension which has a bound state.

One can also consider *simultaneous* weak contact of three particles. In this case the extension has an Efimov spectrum.

We will discuss this case when we will analyze the Bose gas in the high-density regime.

We shall consider first the case of two particles in weak contact.

Now the [2] boundary conditions require that functions in the domain take a finite value at the boundary.

In the study of weak contact, we can proceed as in the case of strong contact and introduce the Minlos space.

The Krein map is induced again by the operator $(H_0 + \lambda)^{-\frac{1}{2}}$.

This corresponds to fractioning *but there is no mixing*.

In $\mathcal{M}$ the kinetic energy is represented by the operator $(H_0 + \lambda)^{\frac{1}{2}}$ and the potential has a $log(|x_i - x_j|)$ singularity at the coincidence manifold.

The hamiltonian is covariant under dilation

Due to scale covariance this is the behavior of the wave function also at large distances; this is the origin of the zero-energy resonance.

Inverting the Krein map one has in physical space a weakly closed quadratic form bounded below with a $\frac{1}{|x_i - x_j|}$ behavior *at large distances* and therefore a zero-energy resonance (the Krein map does not alter the long-distance behavior).

It is strongly closed [13] and corresponds to a self-adjoint operator bounded below and with a zero-energy resonance.

The hamiltonian and the Krein map are invariant under rotations. Therefore, one can study separately the irreducible components in the angular momentum sectors.

In each of them the weak contact hamiltonian is the limit in strong resolvent sense of the hamiltonian with the approximation potentials V^ϵ *but in the $L = 0$ sector a zero energy resonance must be subtracted away* before on can use compactness to prove the existence of the limit.

This explains why in the case of weak contact of two particles *the approximating potentials must have a zero-energy resonance*.

Remark: The case of a weak interaction *in a two-particle system* is discussed in Albeverio and Hoegh-Krohn [3] using methods of Functional analysis in the case when one of the particles has infinite mass. This particle may be considered as a fixed point (point interaction).

The presence of a zero-energy resonance implies a singularity of the resolvent at zero momentum and this requires an accurate and difficult estimate of the zero-energy limit in the B.K.S. formula for the difference of two resolvents [3].

In Albeverio and Hoegh-Krohn [3], this analysis is presented for the weak contact interaction of a particle with a fixed point (a particle of infinite mass) but the same analysis can be done for the case of weak contact interaction of two particles in the reference frame of the barycenter.

12. A PARTICLE IN WEAK CONTACT WITH A PAIR OF IDENTICAL PARTICLES

Consider now the case of a particle *in weak contact* with a pair of identical particles.

We use the same Krein map as the case of strong contact. Therefore, it corresponds to fragmenting the wave functions and in mixing the two channels.

Again, we restrict attention to product states.

The Krein map is mixing and fractioning.

In Minlos space the boundary potentials are represented by a function that has a logarithmic singularity at coincidence hyperplanes (the derivative in polar coordinates has a $\frac{1}{|x_i - x_j|}$ singularity).

The boundary potential and the kinetic energy transform covariantly under dilation.

Therefore, the boundary potential in Minlos space behaves *also at infinity* as $log(|x_i - x_j|)$.

The kinetic energy is still represented by $\sqrt{H_0 + \lambda}$.

Lifting to physical space one has a unique a three-body operator. In the B.K.S. formula for the difference between the resolvent of weak contact and the free resolvent, at the origin in momentum space one has the inverse of a two-by-two matrix with zero on the diagonal.

The matrix is therefore invertible and has a negative eigenvalue (one may say that the two zero energy resonances conspire to give a bound state).

Therefore, if the potential of the weak contact is strong enough *the system has a bound state and no zero energy resonances*.

The same occurs for the sequence of approximating potentials with a zero-energy resonance.

Since there is no zero energy resonance in the difference, the sequence in physical space of the difference the quadratic of the weak contact and that form associated to the potentials V^ϵ is compact in bounded sets in the Sobolev topology it converges to zero when $\epsilon \to 0$.

It has a (Palais.Smale) limit that represents

Proposition 1: A particle *in weak contact* with a pair of identical particles is represented by a self-adjoint operator with one bound state *and no zero energy resonances*.

It is the limit of the hamiltonians with potential V^ϵ that scale as $V^\epsilon(x) = \frac{1}{\epsilon^2} V(\frac{|x|}{\epsilon})$.

There may zero energy resonances due to additional regular potentials, but we shall prove that their contribution is independent of that of weak contact.

A direct study "in physical space" of the limit is not difficult. We sketch some details (based on an unpublished manuscript with A. Michelangeli).

From the analysis of B.K.S it follows that the resolvent $R(z) = \frac{1}{H+z}$ of H satisfies

$$R(z) - R_0(z) = [R_0(z)A_\epsilon^*](1 - Q_\epsilon z))^{-1}[A_\epsilon R_0(z)] \tag{11}$$

$$A_\epsilon = \sqrt{V_1^\epsilon + V_2^\epsilon} \quad Q_\epsilon(z) = A_\epsilon \frac{1}{R_0}(z) A_\epsilon^* \quad R_0(z) = H_0 - \epsilon z \tag{12}$$

If V_1 and V_2 are of class C^1, under the scaling $V \rightarrow V^\epsilon(x) = \epsilon^{-2}V(\frac{x}{\epsilon})$ one has

$$lim_{\epsilon\rightarrow 0}[\sqrt{V_1^\epsilon(y_1)+V_2^\epsilon(y_2)} - \sqrt{V_1^\epsilon(y_1)} - \sqrt{V_2^\epsilon(y_2)}] = 0 \quad (13)$$

and therefore the "overlap" vanishes when $\epsilon \rightarrow 0$ and one can substitute $A = \sqrt{V^\epsilon(y_1)} + \sqrt{V^\epsilon(y_2)}$.

Since A_ϵ is the sum of two terms, one has four summands.

To estimate the limit $\epsilon \rightarrow 0$ perform *in the integral over internal variables* a scaling $x \rightarrow \epsilon^{\frac{3}{2}}x$.

The two integrations implicit in the right-hand side of (12) provide a factor ϵ^3; the product of the two potential provide a factor ϵ^{-4}.

Therefore, to find the limit one can neglect all contributions that are of order >1 in ϵ.

To first order in ϵ there is only a separate contributions of the zero energy resonances in each channel.

This is an invertible two-by-two symmetric matrix with zero on the diagonal. It has therefore a negative eigenvalues.

Substitution in the B.K.S. formula for the difference of two resolvents this produces a bound state if the potential is strong enough and ϵ is sufficiently small.

The limit $\epsilon \rightarrow 0$ is the resolvent of the hamiltonian of the system made of a particle of mass 1 in weak contact interaction with two identical bosons of mass one.

Remark 1: The scaling $x \rightarrow \epsilon^{\frac{3}{2}}x$ that enters in the rescaling *of the integral over the internal variables* transforms weak contact interaction into strong contact.

We shall come back to this point when we will discuss the Bose-Einstein gas in the high-density case.

Remark 2: The result does not depend on the masses of the particles provided that they are not all zero.

If two of the particles have zero mass the bound state is the Polaron [23]; we will consider this case in detail in **Appendix 1**.

If two of the particles have infinite mass the system represents weak contact interaction with two fixed point (point interaction with two fixed points).

This system has zero energy resonances and therefore the Wave operator is a bounded map $L^p \rightarrow L^q$ for $1 < p \leq q < \infty$.

13. THREE PARTICLES IN PAIRWISE WEAK CONTACT: LOW DENSITY BOSE-EINSTEIN CONDENSATE

The Bose-Einstein condensate is a relatively dilute gas of identical bosons in weal contact.

The density is such that the probability to find a third particle nearby is negligible.

Still, due to the zero-energy resonance, (a long-range effect) the presence of a third particle is essential (the particle we consider has a weak contact with *two particles*).

We have seen in the preceding section there is a bound state.

We shall call Ω_w this bound state.

Weak contact is the limit of an attractive potentials of very short radius ϵ and a zero-energy resonance.

If ϵ is very small *and if the gas is diluted* one can choose $\epsilon^{-1} = N$ and regard the subsystems as composed of only three particles in weak contact. The bound state Ω_w is stable because the hamiltonian of the two-particle subsystem is positive (and have a zero-energy resonance).

A zero energy (Fesbach) resonance is required for the interaction of the two-body pairs. Once formed, the pairs are stable.

Since the gas is very diluted the probability that *all three particles are very close* is negligible (notice the interaction has range ϵ).

But the particles are identical and satisfy Bose-Einstein statistics.

Their state is therefore "entangled," and each pair has equal probability to be in weak contact.

Since the particles are identical, *it is as if* the system be composed of separate pairs of particles.

The ground state of the system of $2N$ particles is the tensor product of the vectors $\otimes\Omega_w^i$ for all different two-body pairs (properly symmetrized since the particles are identical bosons).

The error is of order $\frac{1}{N}$.

Since the two (identical) bosons in the pair are in (weak) contact and each of them satisfies the Schrödinger equation with as potential the density of the other, each of two bosons satisfies the Gross-Pitaewskii equation for a Bose-Einstein condensate with an *effective coupling potential* due to the presence of a zero-energy resonance [24].

Remark 1: In Benedikter et al. [25] to have the right scaling one adds an extra N factor (N is the number of particles).

This scaling is justified with the assumption that *each particle contributes for a fraction* $\frac{1}{N}$.

In our approach the correct scaling is a consequence of weak contact.

The probability of having a correlation with a third particle (and therefore with another pair) vanishes as ϵ and this is the basis for the proof of condensation in Khowles and Pickl [26]. Since weak contact is limit of a potential with very short range $\frac{1}{N}$ and the gas is very diluted the error term in neglecting the interaction with the other pairs is proportional to $\frac{1}{N}$ and one can use perturbation theory to describe the interaction between pairs.

To first order the ground state is a collection of non-interacting weakly bound pairs.

Choosing $\epsilon \equiv \frac{1}{N}$ permits a Fock space analysis.

We will not analyze further here this problem .

14. BOSE-EINSTEIN CONDENSATE, THE HIGH-DENSITY CASE, THE NEW GROUND STATE

Consider now the high-density case.

The particles are now *simultaneously* in weak contact.

The interaction s represented, before taking the limit $\epsilon \to 0$ by the hamiltonian

$$H_{int} = H_0 + \sum_{i \neq j \neq k} \frac{1}{\epsilon^2} V(\frac{|x_i - x_j|}{\epsilon}) \tag{14}$$

In Krein space we can use perturbation theory.

In the perturbation formula for the resolvent the terms that depend only on two of the potentials give the same result of the weak contact interaction of one particle with a pair.

We are interested in the contribution of terms *that depend on all three potentials*.

In this contribution we can "artificially" take away two ϵ from the denominator of one of the potential and "give" an ϵ^{-1} factor to each of the other two (this artifice does not alter the result).

The remaining potential now plays no role.

Redistributing the ϵ is an artifice but it leads to the conclusion that at high density *the ground state of system* is better described considering a system of three particles one of which is in *strong* contact with the other two.

The strong contact interaction takes place *separately* with the two particles; since the particles are identical one has a gas two particles strongly bound.

Remark that the presence of a third particle is mandatory to define strong contact. The role of the third particle is to prevent free motion for the barycenter of the two particles in strong contact.

Call Ω_s the ground state. To first order the ground state of the high-density Bose-Einstein gas is $\otimes_i \Omega_s^i$.

It is not related to the ground state $\otimes_i \Omega_w^i$ of the diluted Bose-Einstein gas.

Since the two (identical) bosons in the pair are in strong contact, each of them satisfies the Schrödinger equation with as potential the density of the other i.e., the focusing cubic Schrödinger equation (and not the Gross-Pitaewskii equation which has *a different effective coupling constant* due to the presence of a zero energy resonance) [24].

15. EFIMOV EFFECT IN QUANTUM MECHANICS

The Efimov effect in Quantum Mechanics is the presence of an Efimov sequence of bound states for a particle that moves in a potential that is the sum of two potentials which taken separately have a zero-energy resonance.

In spite of the same name, the effects have *totally different origin*.

They lead to the same result because in the two cases there is the same balance of kinetic and potentials energies.

We assign +2 for each of the three particles particle (since they satisfy a second order differential equation) and –3 two strong contacts (the power of ϵ^{-1} in the scaling). If the difference is zero one has Efimov spectrum.

In the quantum mechanical case there is only one second order differential operator and there are two weak contacts with two resonances; for the counting of weights this is the same as a weak contact with the resulting bound state.

One assigns - 2 to weak contact with bound state. The net sum is zero as in the three-particle case with strong contact.

Therefore, one can expect to have the same effect (this counting is not a substitute for proofs but, in spite of its empirical flavor, provides very efficient indications).

A proof of the Efimov effect in Quantum Mechanics for contact interactions can be obtained using the Krein map.

The Krein map is well defined.

Setting equal to one the mass of the particle in Minlos space the kinetic term is $\sqrt{-\frac{1}{2}\Delta + \lambda}$.

The two zero energy resonances conspire to a give a bound state and this gives a negative quadratic form with a singularity of degree –1 at the origin in position space.

This leads to a Weyl limit circle singularity as in the three particles case.

Inverting the Krein map and making use of Gamma convergence one obtains the Efimov spectrum as in the three-particle case.

Remark: Notice that we have used only the existence of two zero energy resonances and the fact that the Schr odinger equation is of second order; therefore, this analysis through the Krein map applies as well to the case of smooth potentials, leading to an alternative proof usual Efimov effect in QM [27, 28].

In the same way one can analyze the case of the Pauli equation for non-relativistic spinors.

The Pauli equation is a first order differential equation (for spinors) with positive generator.

In one dimension one can use as contact potential the delta function; this gives a contact of weak type at the vertex for a system of three particles that move on a Y-shaped graph and interact at the vertex.

There is a bound state since there are three operators and only two weak contacts (in one dimension the delta function represents a weak contact interaction because it scales as the differential operator).

At the end we shall return briefly to this system.

16. STRONG AND WEAK CONTACT ARE INDEPENDENT

Theorem 2: In three dimensions for $N \geq 3$ contact interactions and weak-contact interactions contribute *separately* and *independently* to the spectral properties and to the boundary conditions at the contact manifold.

Contact interaction contribute to the Efimov part of the spectrum and to the T-M boundary condition $\frac{c_{i,j}}{|x_j - x_i|}$ at the boundary $\Gamma \equiv \cup_{i,j} \Gamma_{i,j}$.

Weak-contact interactions contribute to the constant terms at the boundary and may contribute to the (finite) negative part spectrum. ◇

Remark: This theorem states that all results of the weak-contact case (in particular for point interactions) remain valid when strong contact interactions are added.

Proof. For an unified presentation (which includes also the proof that the addition of a regular potential does not change the picture) it is convenient to use a symmetric presentation due to Konno and Kuroda [4] (who generalize previous work by Krein and Birman) for hamiltonians that can be written in the form

$$H = H_0 + H_{int} \qquad H_{int} = B^* A \tag{15}$$

where $B, \ A$ are densely defined closed operators with $D(A) \cap D(B) \subset D(H_0)$ and such that, for every z in the resolvent set of H_0, the operator $A \frac{1}{H_0+z} B^*$ has a bounded extension, denoted by $Q(z)$. We give details in the case $N = 3$.

Since we consider the case of attractive forces, and therefore negative potentials it is convenient to denote by $-V_k(|y|)$ the two body potentials.

The particle's coordinates are $x_k \in R^3, k = 1, 2, 3$.

We take the interaction potential to be of class C^1 and set

$$V^\epsilon(X) = \sum_{i \neq j} [V_1^\epsilon(|x_i - x_j|) + V_2^\epsilon(|x_i - x_j|) \tag{16}$$

where V_1 and V_2 are negative and $V_3(|y)$ is a regular potential.

For each pair of indices i, j we define $V_1^\epsilon(|y|)) = \frac{1}{\epsilon^3} V_1(\frac{|y|}{\epsilon})$ and $V_2^\epsilon(|y|) = \frac{1}{\epsilon^2} V_2(\frac{|y|}{\epsilon})$.

The limit corresponds, respectively to contact and weak-contact.

We define $B^\epsilon = A^\epsilon = \sqrt{-V^\epsilon}$.

For $\epsilon > 0$ using Krein resolvent formula one can give explicitly the operator B^ϵ as *convergent power series* of products of the free resolvent $R_0(z), Rez > 0$ and the square roots of the sum of potentials $V_k^\epsilon \ \ k = 1, 2, 3$. One has then for the resolvent $R(z) \equiv \frac{1}{H+z}$ the following form [4]

$$R(z) - R_0(z) = [R_0(z) B^\epsilon][1 - Q^\epsilon(z)]^{-1}[B^\epsilon R_0(z)] \qquad z > 0 \tag{17}$$

with

$$R_0(z) = \frac{1}{H_0 + z} \qquad Q^\epsilon(z) = B^\epsilon \frac{1}{H_0 + z} B^\epsilon \tag{18}$$

If $\epsilon > 0$ the Born series converges and the resolvent can be cast in the Konno-Kuroda form [4], where the operator B is given as (convergent) power series of convolutions of the potential U^ϵ and V_1^ϵ with the resolvent of H_0.

In general

$$\sqrt{V_1^\epsilon(|y|) + U^\epsilon(|y|)} \neq \sqrt{V_1^\epsilon(|y|)} + \sqrt{U^\epsilon(|y|)} \tag{19}$$

and in the Konno-Kuroda formula for the resolvent of the operator H_ϵ one loses separation between the two potentials V_1^ϵ and U^ϵ.

Notice however that, if V_1^ϵ and U^ϵ are of class C^1 , the L^1 norm of U^ϵ vanishes as $\epsilon \to 0$ uniformly on the support of V_1^ϵ.

By the Cauchy inequality one has

$$lim_{\epsilon \to 0} \| \sqrt{V_1^\epsilon(y)} . \sqrt{U^\epsilon(y)} \|^1 = 0 \tag{20}$$

Therefore, if the limit exists the strong and weak contact interactions act independently.

In the same way one proves the independence of the strong and weak contact interactions from the regular interaction.

The weak-contact part has a limit in strong resolvent sense.

The limit is unconditional, i.e., it does not depend on the particular denumerable subsequence $\epsilon_n \to 0$ used.

The strong contact part has a limit along minimizing sequences by Theorem 1.

Therefore, the joint limit exits along these minimizing sequences. ♡

17. WEAK-CONTACT CASE: SEPARATION OF THE REGULAR PART

Consider now *separate* weak-contact interaction of a particle with a pair of identical particles.

We allow for the presence of a "regular part" represented by a smooth two body L^1 potential of finite range and call *singular part* the quasi contact interaction and the resonance.

Theorem 3: For a weak-contact interactions the singular term (pure weak-contact) and the regular term in the two-body part of the interaction *contribute separately* to the spectral structure of the hamiltonian. ◇

Proof. For the proof we use again the Konno-Kuroda resolvent formula but now for a system with potentials $V_2^\epsilon + V_3$.

Recall that set $V_2^\epsilon(|x|) = \frac{1}{\epsilon^2} V_2(\frac{|x|}{\epsilon})$.

The Konno-Kuroda formula is now for $Re(z) > 0$ and $R_0^\epsilon(z) = H_0 + \epsilon z$

$$\frac{1}{H_\epsilon + z} - \frac{1}{H_0 + z} = -\frac{1}{H_0 + z} Q^\epsilon B^\epsilon Q^\epsilon \frac{1}{H_0 + z} \tag{21}$$

$$B^\epsilon = \sqrt{V_2^\epsilon + V_3} \quad Q^\epsilon(z) = B^\epsilon \frac{1}{R_0}(z) B^\epsilon \quad R_0(z) = (H_0 + \epsilon z)^{-1} \tag{22}$$

One can now repeat the procedure in Theorem 3.

By assumption V_2 and V_3 are of class C^1 and as $\epsilon \to 0$ on the support of V_2^ϵ the L^2 norm of V^3 is of order ϵ.

Therefore

$$\| (\sqrt{V_3^\epsilon} + \sqrt{V_2^\epsilon})^2 - V_3^\epsilon - V_2^\epsilon \| = 0(\epsilon) \tag{23}$$

We conclude that in limit the potentials V_2 and V_3 contribute *additively* to spectral properties.

The potential V_2 (weak-contact) may contribute for a finite or infinite number of elements of the spectrum (depending on the masses and the coupling constants), the potential V_3 gives a contribution to the spectral measure.

In both case there are no singularities at the bottom of the (absolutely) continuous spectrum.

This proves Theorem 3. ♡

Remark: The zero-energy resonance we have found *is due to a very sort range potential* (in the limit, zero range) whereas the possible resonances of the regular potentials are due *to their very long range*.

The presence of one does not interfere with the presence of the other.

18. CONNECTION WITH OTHER APPROACHES

Heat kernel renormalization

We comment briefly on the relation with the "heat kernel" renormalization introduced in Erman and Turgut [29].

Start with the identity

$$\int_0^{\tau} e^{-H_0+\lambda} dt = \tau + \frac{1}{H_0+\lambda} + O(\frac{1}{\tau}) \tag{24}$$

where H_0 is the free hamiltonian.

The heat kernel renormalization consists by definition in taking the limit $\tau \to \infty$ and neglecting the divergent constant.

Since $H_0 + \lambda$ commutes (formally) with a delta distribution (a constant in Fourier space) the heat kernel renormalization of the potential V^ϵ for the three-body system

$$V^\epsilon \to \frac{1}{\sqrt{H_0-\lambda}} V^\epsilon \frac{1}{\sqrt{H_0+\lambda}} \tag{25}$$

may be defined as a regularization map for $\epsilon > 0$ and in the limit $\epsilon \to 0$ is the interaction potential in $\mathcal{M}$.

Recall that the Krein map is a "fractioning" of the "wave function" (the wave function becomes more singular) *while switching the channels* which results in mixing.

In this case this "renormalization" consists in using Gamma convergence after the inversion of the Krein map (i.e., in physical space).

This is clearly a non-perturbative scheme and does not require "removing infinities."

We recall that Gamma-convergence is equivalent to resolvent convergence i.e., roughly speaking, convergence under the assumption that one considers sequences of states on which the hamiltonian stays bounded (this is the role of renormalization).

Interior boundary conditions

This approach has been proposed recently, mostly in view of a second-quantization scheme [20, 30, 31].

With different wording and different analytical techniques this approach has some similarities with the one which is developed here.

After all, the boundary conditions at the contact manifold are "interior boundary conditions."

In an **Appendix** we develop a second quantization scheme (similar to that in Lampart [30]) adapted to the self-adjoint extensions discussed here.

It is a "baby second quantization scheme," adapted to the three-body contact interaction for one massive and two massless particles, in which only the zero mass particles undergo second quantization.

Notice that a quantum mechanical three body problem arises naturally if creation and annihilation operators are "partially dequantized" by choosing for *two of the zero mass particles* the ground state of a system in which the zero mass particles are in strong contact interaction with the massive one.

The ground state of the system is then obtained choosing for the remaining particles the vacuum state of a suitable representation of the c.c.r.

The resulting ground state is a model for the *polaron* [32], the ground state of the Nelson model [23]. We discuss this model in the **Appendix**. We can also in the same way find the excited states below the continuum by choosing different bound states of the three-body problem and the vacuum of another suitable representation.

Changing the bound state changes also the representation.

Notice that this procedure limits the role of strong contact interaction in quantization problems to *linear couplings* of a particle and a quantized field.

19. DETAILS FOR SOME SIGNIFICANT CASES IN THREE DIMENSIONS

We study in the following systems of non-relativistic bosons and fermions that satisfy the Schrödinger equation.

Later we will consider on a lattice electrons which are fermions and satisfy the Pauli equation.

Since the Pauli equation is a first order differential equation, weak contact plays the same role as strong contact in the bosonic case.

For boson we shall discuss in what follow, both for strong and weak contact , some relevant case; they are sufficient to draw conclusions on a system of an arbitrary number of particles.

In particular we shall consider in three dimensions

I) A particle of mass m in strong or weak contact interaction with two identical bosons of unit mass.
II) A particle of mass 1 in strong contact with two fermions of the same mass.
III) Two pairs of identical bosons in strong contact whose barycenters are in weak contact
IV) The same problem for fermions,
V) N pairs of boson or fermions in strong contact.

In the case of strong contact for fermions we prove that the hamiltonian *is positive for any value of N*. This system is called *Unitary gas*.

In the case of bosons, the (negative) lower bound of the spectrum is linear in N.

I

Consider first the case of a particle of mass m in strong contact interaction with two identical bosons of unit mass.

Setting again for simplicity $\lambda = 0$ the quadratic form in $\mathcal{M}$ is the sum of two terms

$$Q = Q_1 + Q_2 \tag{26}$$

where

$$Q_1(\phi) = \frac{m}{m+1}(\phi, \sqrt{H_0}\phi) \tag{27}$$

while the kernel of Q_2 is

$$Q_2(p,q) = -\frac{\frac{2}{1+m}(p.q)}{(p^2+q^2)^- \frac{1}{(1+m)}(p.q)} \tag{28}$$

Again, this kernel quadratic reaches its minimum value at $q = p$.

In Fourier transform one has

$$B(m)\sqrt{-\Delta} - D(m)\frac{1}{|x|} + \Xi''(m) \tag{29}$$

where $B(m), D(m)$ are suitable positive functions of the parameter m and Ξ'' is a positive form with a smooth kernel that vanishes on the diagonal.

We consider only the case $\Xi = 0$. The contribution of Ξ is small and does not alter the conclusions.

Following Derezinky and Richard [14] proves that for each eigenvalue l of the angular momentum there are threshold N_l^*, N_l^{**} such that for $m > N_l^{**}$ the spectrum is absolutely continuous and positive.

For $N_l^* < m \leq N_l^{**}$ there is a continuous family of self-adjoint extensions, each with a negative eigenvalue, and for $0 < m \leq N_l^*$ the negative spectrum is pure point and accumulates geometrically to $-\infty$ (a Weyl limit circle effect).

One can verify that for equal masses $N_0^{**} > 1$ while $N_l^* < 1$ for all $l \geq 1$.

Therefore, in the equal mass case there is a family of extensions; for each of them there is a family of bound states with energies that diverge linearly to $-\infty$.

Inverting the Krein map by Gamma convergence one has

Proposition 2: The hamiltonian of a pair identical bosons in strong contact interaction with a third particle of the same mass has an Efimov sequence of bound states if the interaction is strong enough. The support of the wave functions is larger for decreasing energy; the wave functions belong to the form domain. ◇

II

We consider next in R^3 the case of a particle of mass m in strong contact with a pair of fermions with the same mass.

The analysis proceeds as in the strong contact case but since the contact is weak the integral in the integrand in Q_2 must be anti-symmetrized.

Now one has

$$(\phi, Q_1\phi) = \frac{m}{m+1}(\phi, \sqrt{H_0 + \lambda}\phi) \tag{30}$$

while the kernel of Q_2 is

$$Q_2(p,q) = \frac{2}{1+m}\frac{1}{|p-q|^2}(p^2+q^2)^2 - \frac{4}{(1+m)^2(p.q)^2} \tag{31}$$

These are the quadratic forms in $\mathcal{M}$ that correspond, respectively the kinetic energy and to the distributional potential.

For the study of spectral properties, it is convenient to notice that the kernel of Q_2 is positive (it has a positive maximum at $p = q$).

Since the Krein map is positivity preserving in physical space the operator is positive.

Proposition 3: The hamiltonian of a pair of fermions of mass m which are in strong contact with a third particle of the same mass has a positive spectrum. ◇

20. III, STRONG AND WEAK CONTACT IN A FOUR BOSONS SYSTEM

We have analyzed the case of three particles.

Consider now a four bosons system. We assume that there is a strong contact of any particle with two other particles ad in addition there is a weak contact between the barycenters of any two pairs.

Notice that the total degree of the kinetic terms is eight (two for each particle) and the total degree of the interaction term is also eight (three for each strong contact and two for the weak contact).

Therefore, we expect to have an Efimov sequence of four-bound states (quadrimers).

The analysis is simple in momentum space.

The explicit expressions of these forms in momentum space were known to R.Minlos (private communication).

We choose as coordinates the difference of the coordinates of the particles in strong contact and the difference of the coordinates of the barycenters.

The strong interactions within a triple gives a contribution that we have already analyzed.

The only difference is the presence of the weak interaction between the barycenters. In Minlos space the kinetic energy is represented by $\sqrt{H_0 + \lambda}$.

The interaction is the sum of three terms C_i, $i = 1, 2, 3$.

C_1 and C_2 are the images in $\mathcal{M}$ of the convolution of the four-particle Green function with the strong interaction potentials.

$$\begin{aligned}(\phi, C_1\phi) &= (\phi, C_2\phi)\\ &= \int dkdsdw\bar{\phi}(k,w)\frac{\phi(s,w)+\phi(k,s)}{k^2+s^2+w^2+(k,s)+(k,w)+(s,w)}\end{aligned} \tag{32}$$

Contributions C_1, C_2 refer to the three-particle case, i.e., a particle in strong contact with two other particles.

It is different from the case of three particles we have considered before because of the presence of a fourth particle.

The presence of a fourth particle is irrelevant because it only enters the definition of the Krein map, which is inverted at the end.

C_3 is a genuine four particle term which is not present in the three-particle sector. It describes the (weak) interaction of the two barycenters.

The corresponding quadratic form in M is

$$(\phi, C_3\phi) = -\int dwdsdk \frac{\bar{\phi}(k,s)\phi(w - \frac{k+s}{2}, -w - \frac{k+s}{2})}{w^2 + \frac{3}{4}(k^2+s^2) + \frac{1}{2}(k,s)} \quad (33)$$

(this quadratic form was known to R. Minlos, private communication).

The form has a simpler structure when written as a function of the difference of coordinates of the barycenters of the two pairs. In these coordinates it is the image *in the four-particle sector* of an "effective" weak contact interaction between the barycenters of the two pairs. Weak contact between the two barycenters gives a system with at most a finite number of bound states.

Therefore, in Krein space the system is described by a two-parameters family of operators which have an infinite number of bound states with eigenvalues that diverge linearly to ∞ and a self-adjoint operator wit at most a finite number of bound states.

Inverting the Krein map one obtains a two parameter families of weakly closed forms.

By construction, the form is invariant under rotation but also under permutation of the particles.

We decompose again in irreducible representation of the rotation group and quotient it by the permutation group.

Each component is now strictly convex, and we can use Gamma convergence to extract a convergent subsequence.

This give a unique weakly closed quadratic form bounded below that can be closed strongly and provides a self-adjoint operator with an Efimov spectrum.

Since it is a four-body system it describes an Efimov sequence of *quaternions*.

Therefore

Proposition 4: *If the interaction is strong enough a system of two pairs of bosons in strong contact and with a weak contact between the barycenter has an Efimov sequence of quaternions.* ◇

Four-body Efimov states have been reported experimentally [16, 21].

21. IV. THE CASE OF FERMIONS

Consider the system of two pairs of identical spin $\frac{1}{2}$ fermions of mass one which satisfy the Schrödinger equation and in contact interaction. Spin $\frac{1}{2}$ is required because antisymmetry of the wave functions of parallel spins is zero at contact. The generalization to N identical spin $\frac{1}{2}$ fermions will describe the unitary gas.

In $\mathcal{M}$ the quadratic form of the system is the sum a term C_0 which represents the kinetic part of form minus three forms C_1, C_2 C_3.

The explicit expressions of these forms in momentum space were known to R.Minlos (private communication).

C_1 and C_2 are the images in $\mathcal{M}$ of the convolution of the four-particle Green function with two delta singularities of the potential between two fermions with opposite spin.

$$\begin{aligned}(\phi, C_1\phi) &= (\phi, C_2\phi) \\ &= \int dkdsdw\bar{\phi}(k,w) \frac{\phi(s,w) + \phi(k,s)}{k^2 + s^2 + w^2 + (k,s) + (k,w) + (s,w)}\end{aligned} \quad (34)$$

As in the tree particles case, when written in space coordinates they have a Coulomb type singularity in different variables related to the possible triples. But now due to antisymmetry the coefficient of the Coulomb term is positive.

C_3 is a genuine four particle term which is not present in the three-particle sector.

It represents an effective interaction between the barycenters of the two pairs.

Notice that a pair of fermions with opposite spin has the symmetry a boson.

The corresponding quadratic form in M is

$$(\phi, C_3\phi) = -\int dwdsdk \frac{\bar{\phi}(k,s)\phi(w - \frac{k+s}{2}, -w - \frac{k+s}{2})}{w^2 + \frac{3}{4}(k^2+s^2) + \frac{1}{2}(k,s)} \quad (35)$$

It has a simpler expression when written as a function of the difference of coordinates of the barycenters of the two pairs. In these coordinates it is the image *in the four-particle sector of* $\mathcal{M}$ of an "effective" contact interaction between the barycenters of two pairs with opposite spin. Notice that only *pairs of particles* enter this term.

The form can be decomposed into a symmetric and antisymmetric part under interchange *of the two pairs*.

Only the kinetic energy contributes to the antisymmetric part; this part is positive.

Also, the symmetric term of is positive.

Therefore, the quadratic form is positive.

Since the Krein map is positivity preserving the same is true in physical space and the system is described by a positive hamiltonian (with a zero-energy bound state).

Proposition 5: *The operator associated to a system two pairs of identical fermions in strong contact and such that the barycenters are in weak contact is a positive self-adjoint operator in* $L^2(R^{12})$. *Its hamiltonian is the limit, in the strong resolvent sense, of a sequence of approximating hamiltonians with potentials of decreasing support.* ◇

Remark: In Michelamgeli and Pfeiffer [33], positivity of the spectrum was conjectured with the aid of a computer.

V: the case of N particles. We have considered so far the cases $N = 3$ and $N = 4$.

Consider now the case of N identical bosons.

For N particles the negative part of the spectrum is *entirely due either to a strong contact of tree bodies or to the four-body contact described in Proposition 4*.

Therefore

Proposition 6: The energy of a gas of N bosons in strong contact interaction is bounded below by $-CN$ which the positive constant C depends on the strength of the interactions.

The system described by any number of identical fermions in weak or strong interaction is described by a positive hamiltonian. This system is often call often called *Unitary gas*. ◇

22. TWO DIMENSIONS; SIMULTANEOUS PAIRWISE WEAK CONTACT

In the two-dimensional case strong contact interaction is a distributional potential $\delta(|x_i - x_j|)$ at the coincidence manifold.

It is the limit of the interaction through two-body potentials of that scale as $V^\epsilon(|y|) = \frac{1}{\epsilon^2} V(\frac{|y|}{\epsilon})$.

The Krein map is the same as in three-dimensional case.

Again, in Minlos space the free hamiltonian H_0 is represented by $(H_0 + \lambda)^{\frac{1}{2}}$ and the potentials differ from $-\frac{C}{|x_i x_j|}$ for a smooth positive quadratic form.

One has therefore

Propostion 7: *In two dimensions strong simultaneous pairwise weak contact interaction of three bosons is represented by the potential* $-C\sum \delta(|x_i - x_k)$, $C > 0$.

It is well defined in physical space through Gamma convergence. It is the limit for $\epsilon \to 0$ *of potentials that scale as* $V^\epsilon(|x|) = \frac{1}{\epsilon^2} V(\frac{|x|}{\epsilon})$

The system has an Efimov sequence of bound states. ◇

Since there are no zero energy resonances the mapping properties of the Wave operator in physical space are $L^p \to L^q$ for $1 < p \leq q < \infty$.

This result has been obtained also for regular potentials in Erdogan et al. [34]).

23. TWO DIMENSIONS, SEPARATE STRONG CONTACT

Consider now a system of three identical bosons in which each has a strong contact with the other two.

We describe in detail the hamiltonian of the resulting system.

To study the structure of the operator we study its quadratic form and assume a before that the particles are *identical bosons*. The wave function in the frame of reference of the barycenter is best written as a function of one radial coordinate r and two Euler coordinates on S^3.

We define $r^2 = (|x_1 - x_3|)^2 + (|x_2 - x_3|)^2 \quad x_k \in R^2, \quad r \in R^+$.

In the Theoretical Physics literature this coordinates are called "homogeneous."

The quadratic forms we consider have the same structure as in the case of three dimensions but in two dimension the singularities are different.

Again, we use a Krein map with the compact operator $\sqrt{H_0 + \lambda}^{-\frac{1}{2}}$ (for each particle there are two contacts).

For simplicity we take $\lambda = 0$ in the following formulae.

If we denote by $x_k \in R^2 \quad k = 1, 2, 3$ the coordinates of the three points with $x_1 + x_2 + x_0 = 0$ one has in $\mathcal{M}$ for the quadratic forms

$$Q(\phi) = Q_0(\phi) + Q_1(\phi) \tag{36}$$

where

$$Q_0 = (\phi, \sqrt{H_0 + \lambda})\phi) \tag{37}$$

In the center of mass, using Fourier coordinates conjugated with $x_1 - x_3$ and $x_2 - x_3$, the *kernel* of Q_1 is

$$\frac{1}{(q_1^2 + q_2^2 + (q_1, q_2) + \lambda)(q_1 + q_2)^2 + \lambda} \qquad q_i \in R^2 \tag{38}$$

Setting

$$(x_1 - x_3)^2 + (x_2 - x_3)^2 = r^2 \quad r \in R^+ \tag{39}$$

the kernel Q_1 can be written in spatial homogeneous coordinates as integral over S^3 of a kernel $-C\frac{1}{r} + W(x_1, x_2, \lambda)$ where $C > 0$ and W is a smooth kernel which vanished in the diagonal.

On now proceeds as in the three-body case in R^3 with weak contact interactions.

Proposition 8: In two dimensions the pairwise strong contact of three identical bosons is represented by a hamiltonian which, if the interaction is strong enough, has a bound state.

Since there are no zero energy resonances the Wave Operator for the system is a bounded map from L^p *to* L^p *for all* $1 < p < \infty$. ◇

24. ONE DIMENSION. LATTICE STRUCTURE. THE FERMI SEA

The purpose of the following section is manly to have a rough picture of the Fermi sea.

We follow the usual description according to which nuclear forces the *conduction electrons* to move on a graph-like stricture with Y-shaped vertices.

The nucleus can be considered fixed at the center of the cell have a weak contact with the two inner electrons (weak contact at a larger scale is Coulomb interaction).

The system has therefore a bound state of energy $-K$ and the two internal electrons have a wave function (essentially) localized at the center of the cell.

Conduction electrons move on the graph and satisfy the Pauli equation (a first order differential equation for a two-component spinor).

The generator is the (positive) Pauli operator

$$P \equiv i\sigma.\nabla + mI, \quad m > 0 \tag{40}$$

(σ_i are the Pauli matrices and I is the unit matrix).

Notice that the structure of the graph is entirely due to the position of the nuclei. The vertices are Y-shaped.

The interaction of the conduction electrons on the lattice takes place at the vertices.

The lattice stricture forces the conduction electrons to change direction at the vertices; before and after the vertex the conduction electrons *are closer to the nucleus*.

This can be represented by a (negative) potential. The kinetic energy is not changed; in this sense the interaction is attractive.

We describe it by a weak contact at the vertex and therefore there is a zero-energy resonance.

Since there are two electrons moving on the lattice this gives a bound state.

Since the momentum is discontinuous at the vertex the interaction depends on both position a momentum.

We represent this by allowing the (negative) energy of this bound state to be in an interval $[-c, 0)$.

In an extended crystal by the Pauli exclusion principle all these states are occupied: *this is the Fermi sea*.

The electrons "near the surface" have negligible energy. The wave functions are essentially flat, and they have "a Dirac spectrum."

In presence of an electric field along the edge, since the electrons are charged particles, a flow of current is generated. Spins at the two ends of an edge form a magnetic dipole; in presence of a magnetic field the orientation of the dipole is changed.

At the semiclassical scale one can introduce smooth magnetic fields (in the previous scale they correspond to discontinuous potentials); at this scale the motion of electrons *on the surface of the Fermi sea* is seen as classical motion of point particles which satisfy the laws of *classical* electrodynamics [35]. The formalism we have described leaves room also to the "magnetic" Pauli operator.

Of course, at the semiclassical level in presence of electromagnetic fields the Fermi surface can have a complicated structure and the description of dynamics of a point on the Fermi surface may require a refined analysis [35].

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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APPENDIX 1

A. THE POLARON

We treat the Polaron problem ([N][G][FS]) in the context of second quantizaton.

Second quantization can be thought as Weyl quantization for a system with an infinite number of particles.

Lebesgue measure is substituted by a measure on function space (Gauss measure in the Bose case).

Functions on phase space can be represented by the coefficients of their Fourier transforms.

Very roughly speaking for bosons in second quantization a wave function f is substituted with of a scalar field $\Psi(f) = a(f) + a^*(\bar{f})$ where $a(f)$ (resp. $a^*(\bar{f})$) destroys (resp. creates) a particle with wave f. Both terms are linear in f.

The field satisfies the (non-relativistic) commutation relations $[\Psi(\bar{f}), \Psi(g)] = (f, g)$.

In the Fock representation one postulates the existence of a vector Ω (the "vacuum") such that $a(f)\Omega = 0 \ \forall f$ in the Hilbert space.

Fock space is the space generated by repeated action of the $a^*(f)$ on Ω (this justifies the name "creation operators").

We shall use the formalism of second quantization and denote by $a((k)$ (resp. $a^*(k)$) the annichilation (resp. creation) of a zero mass particle "of momentum k" (we omit the more precise definition).

In the following we consider the contact interaction of the particle of mass m with any two non relativistic zero mass particles in the second quantization formulation for the field.

This system is called *polaronic* and the ground state is the *polaron* [N].

We take the interaction to be weak contact of the massive particle with any two of the zero mass particles.

We approximate the interaction by using the two-body potential $V^\epsilon = \frac{1}{\epsilon^2} V(\frac{|x_i - x|}{\epsilon})$ where $V \in C^1$.

$$H^\epsilon = H_0 + \int V^\epsilon(x - y_1)\Psi(y_1)dy_1 + \int V^\epsilon(x - y_2)\Psi(y_2)dy_2$$
$$H_0 = -\frac{1}{2m}\Delta_x + \int \omega(p)a^*(p)a(p)dp \tag{A1}$$

where $\omega(p) = |p|^2$ and the $a(k)$ satisfy the c.c.r.

The limit $\epsilon \to 0$ is the contact interaction of the particle with the field.

We have proved that this system has a bound state Ψ. We denote by $\hat{H}$ the hamiltonian.

We use the formalism of second quantization *paying attention* to the fact that for zero mass particles there infinitely *inequivalent* representations of the c.c.r.

A vector of finite energy in the Hilbert space may contain an infinity of zero mass particles with smaller and smaller momentum (this is known as *infrared problem*).

We denote by H the limit hamiltonian. It describes the contact interaction of the massive particle with the cloud of zero mass particles.

To find the structure of the ground state (the Polaron) we will "partially dequantize" the field by choosing properly the state *of two of the zero mass particles* (and therefore the representation of the c.c.r. since the zero mass particles are identical).

We have previously remarked that the weak contact interaction of a particle of mass m with two particles of zero mass leads to a bound state.

Let $\Phi(x)$ be the wave function.

To find the ground state of the combined system we fiber the second quantization space of the zero mass particles *choosing as parameter the position of the particle of mass m*.

We choose the representation by defining annichilation operators

$$A_x(y) = a(y) - \Phi(x) \tag{A2}$$

For each value of x the (distribution valued) operators $A_x(y)$ satisfies the same c.c.r as the operators $a(y)$ but the two representations are *inequivalent*.

Different values of the position of the particle of mass m correspond to a different "infrared behavior" of the mass zero field.

If one writes the Hamiltonian as a function of the field $A_x(y)$ one obtains

$$H = \hat{H} + \int \omega(p)A_x^*(p)A_x(p)dp$$

Remark: In the Theoretical Physics literature this operation goes under the name of "completing the square" and the particle of positive mass is now "dressed" with the a particles

In order to minimize the energy one must choose the Fock representation for A_x for every x. .

The minimum of energy is obtained on the vaccum.

It is convenient therefore to write the relation between A_x and a in the following way

$$a(y) = A_x(y) + \Phi(x) \tag{A3}$$

The self-adjoint operator $\hat{H}$ has a ground state Φ.

There is no coupling.

Therefore, the ground state Ψ of the entire system (i.e., the *polaron*) is at each point x the product $\Phi \times \Omega_x$, where Ω_x is the vacuum in the A_x representation, properly symmetrized. .

By construction the A_x representation is *inequivalent* to the a representation.

The ground state of the system (the polaron) is a "cloud" of *infinitely many* mass zero identical particle with distribution that depends on the wave function $\Omega(x)$.

The cloud *depends on the coordinate of the heavy particle*. [N] [F,S], [L,S], [S].

APPENDIX 2

B. A FIELD THEORY APPROACH

In the following we make some (tentative) comments on a *Field Theory* approach.

In case of the Polaron we have chosen a non relativistic quantization for the Bose field since the particle is non relativistic.

The theory is hybrid since Fock space is used for the mass zero field but not for the particle.

On can place contact interaction in a fully relativistic setting in the context of Relativistic Quantum Field Theory.

Notice that "being in contact at a given time" is a relativistic invariant statement.

Define the Krein map as in the non relativistic case but now with the free relativistic hamiltonian H_{rel} of Relativistic Field Theory (a positive operator *in Fock space*).

The Krein map acts differently on the kinetic part and on the interaction term, *and also this is a relativistic invariant statement*.

Gamma convergence is a minimization procedure, and therefore a relativistic invariant.

The Fock representation is a Gaussian measure space in the case of bosons, Berezin-Segal measure space in the case of fermions.

We consider here only the case of bosons.

In relativistic Fock (r-Fock space) space the free hamiltonian for bosons is $H_0 = \sum_n (-\Delta_n + m_n)^{\frac{1}{2}}$.

In order to have "strong contact" with interaction hamiltonian density formally given by "$\phi^3(x)$" one must use a space in which the hamiltonian is a second order differential operator and therefore one must use a Fock space based on $\mathcal{H}^{-\frac{1}{2}}$.

We call this space *nr- Fock space* (non-relativistic Fock space) As measure spaces, the two Fock spaces *are not equivalent*.

In nr-Fock space the free hamiltonian is $\hat{H}_0 = \sum_n (-\Delta_n + m_n)^{\frac{1}{2}}$.

As usual we use the Krein as a way to explore the system.

In the non relativistic we introduce (weak) contact by a $::\phi^3(x)::$ interaction density as in Quantum Mechanics.

With the symbol $::.::$ we denote the normal ordered defined by the condition [L,S,T,T] that the last term (from the right) in the product is an annichilation operator and the first one is a creation operator (so that the choice is only in the middle term).

It a relativistically invariant definition. It has the consequence that the vacuum is invariant.

By construction the interaction hamiltonian has matrix elements only between states that contain at least one particle each.

Let $\psi_1(x)$ and $\psi_2(x)$ the wave functions of these two particles. Without loss of generality we choose $\psi_1 = \psi_2 \equiv \psi(x)$.

One is therefore back to contact of a particle with two identical particles.

We have discussed this case at length. Depending on the strength of the contact the system has a bound state or an Efimov sequence of bound states. .

We must now pass to the r-Fock space.

The topology of r-Foch space as measure space is weaker than that of nr-Fock space (the topology in r-Foch space is given by the relativistic hamiltonian, a first order differential operator, while the topology in a nr-Fock space is given by the non relativistic hamiltonian, a second order differential operator).

But also the covariance is different and the two effect concel.

Therefore, in r-Fock space there are bound states, states of fixed energy in any reference frame.

They correspond to particles.