



New Reflections on Gravitational Duality

Hugo García-Compeán^{1*}, Octavio Obregón^{2*} and Cupatitzio Ramirez³

¹ Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, Mexico City, Mexico, ² División de Ciencias e Ingenierías, Departamento de Física, Universidad de Guanajuato, Guanajuato, Mexico, ³ Facultad de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, Puebla, Mexico

OPEN ACCESS

Edited by:

Ashkbiz Danehkar,
Harvard-Smithsonian Center for
Astrophysics, United States

Reviewed by:

Alexander Alexandrovich Reshetnyak,
Institute of Strength Physics and
Materials Science (ISPMS SB RAS),
Russia

Sergio Hörtnner,
Universidad Autónoma de Madrid,
Spain

Sayantani Choudhury,
Max-Planck-Institut für
Gravitationsphysik, Germany

*Correspondence:

Hugo García-Compeán
compean@fis.cinvestav.mx
Octavio Obregón
octavio@fisica.ugto.mx

Specialty section:

This article was submitted to
High-Energy and Astroparticle
Physics,
a section of the journal
Frontiers in Physics

Received: 15 June 2018

Accepted: 23 August 2018

Published: 19 September 2018

Citation:

García-Compeán H, Obregón O and
Ramirez C (2018) New Reflections on
Gravitational Duality.
Front. Phys. 6:102.
doi: 10.3389/fphy.2018.00102

In general terms duality consists of two descriptions of one physical system by using degrees of freedom of different nature. There are different kinds of dualities and they have been extremely useful to uncover the underlying strong coupling dynamics of gauge theories in various dimensions and those of the diverse string theories. Perhaps the oldest example exhibiting this property is Maxwell theory, which interchanges electric and magnetic fields. An extension of this duality involving the sources is also possible if the magnetic monopole is incorporated. At the present time a lot has been understood about duality in non-abelian gauge theories as in the case of $\mathcal{N} = 4$ supersymmetric gauge theories in four dimensions or in the Seiberg-Witten duality for $\mathcal{N} = 2$ theories. Moreover, a duality that relates a gravitational theory (or a string theory) and a conformal gauge theory, as in the case of gauge/gravity correspondence, have been also studied with considerable detail. The case of duality between two gravitational theories is the so called *gravitational duality*. At the present time, this duality has not been exhaustively studied, however some advances have been reported in the literature. In the present paper we give a general overview of this subject. In particular we will focus on non-abelian dualities, applied to various theories of gravity as developed by the authors, based in the Rocek-Verlinde duality procedure. Finally, as a new development in this direction, we study the gravitational duality in Hitchin's gravity in seven and six dimensions and their relation is also discussed.

Keywords: form gravity, duality-weak strong and gravitational duality, topological M-theory, topological gravity, field theory

1. INTRODUCTION

In general terms, *duality* consists of two descriptions of a physical system through different degrees of freedom. There are different kinds of dualities and they have been extremely useful to uncover the underlying dynamics of strong coupling gauge theories in various dimensions and those of the diverse string theories [1].

In some cases this relation involves the inversion of the coupling constant in such a way that non-perturbative phenomena of the original degrees of freedom can be mapped to a perturbative theory of the dual degrees of freedom. This duality is termed S-duality and it will be the subject of the present article in the context of gravitational theories. Here we will not intend to give an exhaustive, complete and detailed overview on this subject, which is out of our scope. Thus for a general review, see for instance [2, 3] and references therein.

The paradigmatic example satisfying the property of duality is the Maxwell theory with magnetic charges and magnetic currents. At the present time a lot has been understood about duality in

gauge theories. The most relevant examples are the $\mathcal{N} = 4$ supersymmetric gauge theories in four dimensions, where the Montonen-Olive duality [4] was proved [5], and the Seiberg-Witten duality for $\mathcal{N} = 2$ theories (see for instance, [6]). Further, a duality that relates a gravitational theory (or a string theory) and a conformal gauge theory, as in the case of gauge/gravity correspondence, has also been studied with quite detail [7]. However, much of the general work on duality has been done using the Feynman functional integral and it has not been considered as a well established mathematical result. This has motivated many mathematicians and mathematical physicists to work on rigorous proofs to support these results.

The case of duality between two gravitational theories is the so called *gravitational duality*. At the present time this duality has not been exhaustively understood and it is a conjectured symmetry existing in some gravity theories or in the gravitational sector of some higher-dimensional supergravity or superstring theories; some advances have been reported in the literature [2, 3]. In fact, the reach of this analysis is not comparable with those obtained in supersymmetric Yang-Mills theories or superstring theories. In these latter theories the power of strong/weak duality and T-duality in superstring theories has allowed to compute many non-trivial observables carrying much information on the system [8].

Very recently new advances in duality have been done in the context of condensed matter systems, see for instance [9]. There is a conjecture asserting that a fermionic system coupled to a Chern-Simons field is dual to a Chern-Simons gauged Wilson-Fisher bosonic theory. This conjecture has been proved for the case of negative mass deformation of the fermionic theory [9].

In Dijkgraaf et al. [10] the so called *topological M-theory* has been proposed. This is a gravity theory of three-forms on a seven-dimensional manifold of G_2 -holonomy, which has been regarded as a master system from which it is possible to obtain, by dimensional reduction, the different *form theories of gravity* in lower dimensions, six, four and three. Among the known theories of gravity are: In six dimensions, on a Calabi-Yau manifold, there are two theories of gravity describing the moduli of complex structures and the Kähler cone, i.e., the so called Kodaira-Spencer gravity [11], and the Kähler gravity [12] respectively. Further, in four dimensions there is Plebański gravity [13] and in three dimensions Chern-Simons gravity [14–16].

The action principle for topological M-theory is Hitchin's functional defined as the integration of the volume form in seven dimensions, constructed with the G_2 -holonomy invariant forms or calibrations [17]. Volume functionals in six dimensions also have been constructed by Hitchin [18]. These actions reflect the possibility of constructing volumes in terms of the Kähler structure or the complex structure of the underlying six-manifold. In Dijkgraaf et al. [10] it was showed that these six-dimensional Hitchin's functionals are related to the topological string theories of A and B types. Also topological M-theory has been proposed as a master system to derive all the form gravity theories in dimensions lower or equal to seven [10]. This theory has motivated recent work in exploring some new relations between Hitchin's functionals in seven and six-dimensions and form gravity theories in four and three dimensions [19–21].

In the present paper we give first a general overview of the case of non-abelian dualities in various gravity theories developed by the authors in Garcia-Compean et al. [22–27]. These works were done motivated from Ganor and Sonnenschein [28], Mohammadi [29], Lozano [30], and Kehagias [31], which were based on the Rocek and Verlinde duality given in Rocek and Verlinde [32], and in the Buscher duality algorithm [33, 34].

A version of linearized gravitational duality was proposed in Nieto [35] based on our results mentioned in the previous paragraph. Recently this subject has been intensively studied with very interesting results, see for instance [36, 37], and references therein. It is not our purpose to overview this subject here.

Finally, as a new development in this direction, we study the gravitational duality of Hitchin's functional in seven dimensions and its relationship to the gravitational duality algorithm in the corresponding six-dimensional theories.

In section 2 we give a brief review on the work performed by our group regarding non-abelian gravitational duality. In section 3 we focus on the 3 dimensional Chern-Simons gravity where we describe in detail the duality algorithm. Section 4 is devoted to study the gravitational duality in the topological M-theory [10], which is described by using the Hitchin's functionals [17] in seven-dimensional manifold with G_2 -holonomy. Further, the six-dimensional Hitchin's functionals [18] are discussed. We find the corresponding dual actions by using the duality algorithm. Moreover, using a relation of Hitchin's functionals in six and seven dimensions, given via the Hamiltonian flow, we prove that the duality in the six-dimensional model is obtained from the duality in the seven dimensional theory. Thus the parent action in the six-dimensional theory follows from the corresponding action in the seven-dimensional case. Finally in section 5 we give our final remarks.

2. GRAVITATIONAL DUALITY IN FORM THEORIES OF GRAVITY

In this section we overview the idea of gravitational duality. There are many conceptions of this duality. We are going to give a path to our work. We will not intend to give a historical and detailed view on the subject here. For a review about work done on the different visions of gravitational duality and the development in early stages, see [2, 3].

First of all, we review some aspects from Garcia-Compean et al. [22]. This work was inspired by the work in Ganor and Sonnenschein [28], where a dual action for non-supersymmetric pure Yang-Mills theories is given. We start from the partition function

$$Z = \int \mathcal{D}A \mathcal{D}G e^{i \int L_p d^D x} = \int \mathcal{D}A \mathcal{D}G e^{i \int [g G_a^{\mu\nu} G_{\mu\nu}^a + G_a^{\mu\nu} F_{\mu\nu}^a(A)] d^D x}, \quad (1)$$

where $F_{\mu\nu}^a(A) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{bc}^a A_\mu^b A_\nu^c$. After integration of the auxiliary field G one get the original Yang-Mills Lagrangian $L_{YM} = -\frac{1}{4g} F_a^{\mu\nu}(A) F_{\mu\nu}^a(A)$. In order to get the dual action, one

can integrate with respect to the gauge field A , from which follows [28]

$$Z = \int \mathcal{D}G \sqrt{\det(2gM)^{-1}} e^{i \int \left(\frac{2\pi}{g} M_{\mu\nu}^{-1ab} \partial_\rho G_a^{\rho\mu} \partial_\sigma G_b^{\sigma\nu} + g G_a^{\mu\nu} G_{\mu\nu}^a \right) d^D x}, \tag{2}$$

where $M_{ab}^{\mu\nu} = f_{ab}{}^c G_c^{\mu\nu}$. This is an action of the Freedman-Townsend type [38]. Note that in these computations it was not necessary to relate $G_a^{\mu\nu}$ to $G_{\mu\nu}^a$ by any specific metric, the only condition is invariance under the required symmetries. This formulation can be generalized for a Lagrangian $L(\mathcal{F})$, which could also depend on other fields. Consider the associated partition function

$$Z = \int \mathcal{D}A \mathcal{D}G \mathcal{D}\mathcal{F} e^{i \int \left\{ L(\mathcal{F}) + 2\pi G_a^{\mu\nu} \left[\mathcal{F}_{\mu\nu}^a - F_{\mu\nu}^a(A) \right] \right\} d^D x}. \tag{3}$$

Proceeding as in the case of (1), after integration of A the partition function becomes

$$Z = \int \mathcal{D}G \mathcal{D}\mathcal{F} \sqrt{\det(2M)^{-1}} e^{i \int \left[2\pi M_{\mu\nu}^{-1ab} \partial_\rho G_a^{\rho\mu} \partial_\sigma G_b^{\sigma\nu} + G_a^{\mu\nu} \mathcal{F}_{\mu\nu}^a + L(\mathcal{F}) \right] d^D x}. \tag{4}$$

For example, if $L(\mathcal{F}) = \frac{\theta}{2\pi} F_a^{\mu\nu} F_{\mu\nu}^a$, where $F_a^{\mu\nu} = \mathfrak{g}_{ab}^{\mu\nu\rho\sigma} F_{\rho\sigma}^b$, is defined through some metric \mathfrak{g} . After integration of \mathcal{F} the partition function becomes

$$Z = \int \mathcal{D}G e^{i \int \left(\frac{2\pi}{g} M_{\mu\nu}^{-1ab} \partial_\rho G_a^{\rho\mu} \partial_\sigma G_b^{\sigma\nu} - \frac{\pi}{2\theta} G_a^{\mu\nu} G_{\mu\nu}^a \right) d^D x}. \tag{5}$$

This formulation has been applied in Garcia-Compean et al. [22] to topological gravity, considering the Pontryagin and Euler topological invariants $L_P = \frac{\theta_P}{2\pi} \int \varepsilon^{\mu\nu\rho\sigma} \delta_{ab}^{cd} R_{\mu\nu}^{ab}(\omega) R_{\rho\sigma}^{cd}(\omega)$ and $L_E = \frac{\theta_E}{2\pi} \int \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} R_{\mu\nu}^{ab}(\omega) R_{\rho\sigma}^{cd}(\omega)$, where a, b, c, d are Minkowski indices, ω_μ^{ab} is the spin connection and $\delta_{ab}^{cd} = \frac{1}{2}(\delta_a^c \delta_b^d - \delta_a^d \delta_b^c)$. Thus the gauge group is $SO(3, 1)$. In this case in (5) we take $G_{ab}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \eta_{ab,cd} G_{\rho\sigma}^{cd}$, respectively $G_{ab}^{\mu\nu} = \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} G_{\rho\sigma}^{cd}$, where $\eta_{ab,cd} = \frac{1}{2}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})$. In fact, the equivalence of $SO(3, 1)$ with $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ leads to the decomposition into self-dual and anti-self-dual metrics $\delta_{ab}^{cd} = \Pi_{ab}^{(+cd)} + \Pi_{ab}^{(-cd)}$ and $\varepsilon_{ab}^{cd} = 2i(\Pi_{ab}^{(+cd)} - \Pi_{ab}^{(-cd)})$. Hence it is enough to consider $L^{(\pm)} = \frac{\theta_G}{2\pi} \int \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^{(\pm)ab}(\omega) R_{\rho\sigma}^{(\pm)ab}(\omega)$, where $G_{\mu\nu}^{(\pm)ab} = \Pi^{(\pm)ab}{}_{cd} G_{\mu\nu}^{cd}$ and $G_{ab}^{(\pm)\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \eta_{ac}\eta_{bd} G_{\rho\sigma}^{(\pm)cd}$.

For the MacDowell-Mansouri theory worked out in Garcia-Compean et al. [23] we have considered the classically equivalent formulation $S = \int d^4x \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} \left(\tau^+ F_{\mu\nu}^{+ab} F_{\rho\sigma}^{+cd} - \tau^- F_{\mu\nu}^{-ab} F_{\rho\sigma}^{-cd} \right)$, hence it is sufficient to consider the self-dual action

$$\begin{aligned} S &= \int d^4x \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} F_{\mu\nu}^{+ab}(\omega, e) F_{\rho\sigma}^{+cd}(\omega, e) \\ &= 2i \int d^4x \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^{+ab}(\omega, e) F_{\rho\sigma ab}^+(\omega, e), \end{aligned} \tag{6}$$

where $F_{\mu\nu}^{ab}(\omega, e) = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_{\mu a}{}^c \omega_{\nu c}{}^b - \lambda^2 (e_\mu^a e_\nu^b - e_\nu^b e_\mu^a)$ is the $ISO(1, 3)$ field strength. In this case the partition function is given by

$$\begin{aligned} Z &= \int \mathcal{D}\omega^+ \mathcal{D}e \mathcal{D}G^+ \mathcal{D}\mathcal{F}^+ e^{-2 \int \left\{ \frac{1}{\kappa} * \mathcal{F}^{+\mu\nu} \mathcal{F}^{+ab}{}_{\mu\nu} + 2\pi * G^{+\mu\nu}{}_{ab} \left[\mathcal{F}^{+ab}{}_{\mu\nu} - F^{+ab}{}_{\mu\nu}(\omega, e) \right] \right\} d^4x}, \end{aligned} \tag{7}$$

where the $*$ corresponds to the Hodge dual. In this expression, the integration must be made considering the redundancy of the self-dual integration variables, as can be seen from the following identities: $u^{+ab} v_{ab}^+ = -4u_{0i}^+ v_{0i}^+$, $*G_{ab}^{+\mu\nu} \omega_\mu^{+ac} \omega_\nu^{+b} = 4 * G_{ij}^{+\mu\nu} \omega_{0i\mu}^+ \omega_{0j\nu}^+$ and $*G_{ab}^{+\mu\nu} e_\mu^a e_\nu^b = *G_{ij}^{+\mu\nu} e_\mu^i e_\nu^j - i\varepsilon_{ijk} *G_{ij}^{+\mu\nu} e_\mu^0 e_\nu^k$. Thus, integrating successively the quadratic terms in $\mathcal{F}_{\mu\nu}^{+0i}$, ω_μ^{+0i} and e_μ^i , we get

$$\begin{aligned} Z &= \int \mathcal{D}G^+ \mathcal{D}e \det(G^{+-1}) e^{-8 \int \left\{ -\pi^2 \kappa * G_{0i}^{+\mu\nu} G_{\mu\nu 0i}^+ - \frac{\pi}{4} G^{+-1ij} \partial_\rho * G_{0i}^{\sigma\mu} \partial_\sigma * G_{0j}^{\rho\nu} + \frac{1}{16\pi\lambda^2} G^{(0)\mu\nu} e_{\mu 0} e_{\nu 0} \right\} d^4x}, \end{aligned} \tag{8}$$

where the integration over $\mathcal{D}e$ stands for integration over e_μ^0 , and $G^{(0)\mu\nu} = G_{\rho\sigma}^{+-1ij} \varepsilon_{ikl} \varepsilon_{jmn} * G_{kl}^{+\rho\mu} * G_{mn}^{+\sigma\nu}$. After integration of e_μ^0 , it follows the dual partition function of (7)

$$\begin{aligned} Z &= \int \mathcal{D}G^+ \det(G^{+-1}) \sqrt{\det(G^{(0)-1})} e^{-2\pi \int \left\{ G^{+-1ij} \partial_\rho * G_{0i}^{\sigma\mu} \partial_\sigma * G_{0j}^{\rho\nu} + 4\pi\kappa * G_{0i}^{+\mu\nu} G_{\mu\nu 0i}^+ \right\} d^4x}. \end{aligned} \tag{9}$$

The MacDowell-Mansouri gravity can be generalized to supergravity. This is done simply by promoting the gauge group $SO(3,2)$ to the supergroup $OSp(1|4)$. This theory was also studied in the context of gravitational duality and a parent action was constructed and its dual action was given in Garcia-Compean et al. [24].

For 3D Chern-Simons theories this approach simplifies considerably. In this case we have $L_{CS} = Tr(AH(A))$, where $H(A)$ is a $(D - 1)$ form. We exemplify it for $D = 3$, and following [25], consider the parent action $I_P = i Tr \int (\alpha A\mathcal{F} + \beta G\mathcal{F} + \gamma GdA + \eta GA^2)$, i.e.

$$I_P = i \int \varepsilon^{ijk} \left(\alpha A_i^a \mathcal{F}_{jka} + \beta G_i^a \mathcal{F}_{jka} + \gamma G_i^a \partial_j A_{ka} + \frac{\eta}{2} G_i^a f_{bca} A_j^b A_k^c \right), \tag{10}$$

where α, β, γ and η are constants. We show that independently of the integration order in the partition function, the resulting dual action is the same (self-dual). In the next section we make it explicitly since it is an important example in order to compare with the dual actions of Hitchin's functionals in seven and six dimensions (see section 4).

If we integrate first over G in the partition function

$$Z = \int \mathcal{D}A \mathcal{D}G \mathcal{D}\mathcal{F} \exp(iI_P), \tag{11}$$

we get

$$Z = \int \mathcal{D}A \mathcal{D}\mathcal{F} \delta \left[\beta \mathcal{F}_{ij}^a + \frac{\gamma}{2} (\partial_j A_i^a - \partial_j A_i^a) + \frac{\eta}{2} f_{bca} A_i^b A_j^c \right] e^{i \int \varepsilon^{ijk} \alpha A_i^a \mathcal{F}_{jka} d^3x}, \tag{12}$$

from which, after integration over \mathcal{F} gives

$$Z = \int \mathcal{D}A \det \left(\frac{2\pi}{\beta} \right) e^{-i \frac{\alpha\gamma}{\beta} \int \varepsilon^{ijk} A_i^a (\partial_j A_{ka} + \frac{\eta}{2\gamma} f_{bca} A_j^b A_k^c) d^3x}. \tag{13}$$

Now, if we integrate over \mathcal{F} in (11), we get

$$Z = \int \mathcal{D}A \mathcal{D}G \delta(\alpha A_i^a + \beta G_i^a) e^{i \int \varepsilon^{ijk} (\gamma G_i^a \partial_j A_{ka} + \frac{\eta}{2} G_i^a f_{bca} A_j^b A_k^c) d^3x}. \tag{14}$$

Integrating now over A and after the change $G_i^a \rightarrow -G_i^a$ it yields

$$Z = \int \mathcal{D}G \det \left(\frac{2\pi}{\alpha} \right) e^{-i \frac{\beta\gamma}{\alpha} \int \varepsilon^{ijk} G_i^a (\partial_j G_{ka} + \frac{\beta\eta}{2\alpha\gamma} f_{bca} G_j^b G_k^c) d^3x}, \tag{15}$$

and, otherwise, integrating over G gives (13).

Finally, the integration over A and then over \mathcal{F} can be performed if we observe that the parent action (10) can be rewritten, after partial integration for the third term and simple algebraic manipulations, as

$$I_P = \int \left[\frac{\eta}{2} M_{ab}^{ij} \tilde{A}_i^a \tilde{A}_j^b - \frac{2\alpha^2}{\eta} M_{ij}^{-1ab} \tilde{\mathcal{F}}_a^i \tilde{\mathcal{F}}_b^j - \frac{\beta\gamma}{\alpha} \varepsilon^{ijk} G_i^a \left(\partial_j G_{ka} - \frac{\beta\eta}{2\alpha\gamma} f_{bca} G_j^b G_k^c \right) \right] d^3x, \tag{16}$$

where $M_{ab}^{ij} = \varepsilon^{ijk} f_{abc} G_k^c$, $\tilde{A}_i^a = A_i^a + \frac{2\alpha}{\eta} M_{ij}^{-1ab} (\mathcal{F}_b^j + \frac{\gamma}{2\alpha} \varepsilon^{jkl} \partial_k G_{lb})$, and $\tilde{\mathcal{F}}_a^i = F_a^i + \frac{\gamma}{2\alpha} \varepsilon^{ijk} (\partial_j G_{ka} + \frac{\beta\eta}{\alpha\gamma} f_{abc} G_j^b G_k^c)$. Thus, the integration over A and \mathcal{F} in the partition function (11), leads to the integration of the first two gaussian terms in the partition function of (16), whose contributions cancel up to a factor, following (17).

Therefore, the parent action (10) leads to the dual partition functions (13) and (17). The dependence of the coupling constants in the determinants in these partition functions can be eliminated by rescalings, after which both actions coincide with

$$Z = \int \mathcal{D}A e^{-i\alpha\beta\gamma \int \varepsilon^{ijk} A_i^a (\partial_j A_{ka} + \frac{\beta\eta}{2\gamma} f_{bca} A_j^b A_k^c) d^3x}, \tag{17}$$

where constant factors have been discarded and for (17) a parity transformation $G \rightarrow -G$ must be made. Note that apparently the partition functions (13) and (17) have inverted coupling constants, but after the preceding rescalings, they have the same dependence on the coupling constants. This is a characteristic of 3 dimensional Chern-Simons.

Gravity in three dimensions is also described as a gauge theory. This is a Chern-Simons theory with gauge group being $SO(2,2)$, $ISO(1,3)$ and $SO(1,3)$ according if the cosmological constant takes negative, zero or positive values. These theories also

do admit a dual gravitational description and this was described in Garcia-Compean et al. [25]. This will be reviewed in the next section. The generalization to Chern-Simons supergravity is also obtained by promoting for instance the group $SO(2,2)$ to the supergroup $Osp(2,2|1)$. The analysis of the gravitational duality was performed in Garcia-Compean et al. [26].

Finally for the Plebański formulation, which is also known as of the BF-type, it is somewhat different. In order to analyze it, we consider the original complex version with a $SU(2)$ symmetry and an action [13] $I = \int \varepsilon^{\mu\nu\rho\sigma} \left[\frac{1}{k} \Sigma_{\mu\nu}^i F_{\rho\sigma i}(\omega) + \phi_{ij} \Sigma_{\mu\nu}^i \Sigma_{\rho\sigma}^j \right] d^4x$, where ϕ is a traceless Lagrange multiplier matrix. The solution of the constraints $\varepsilon^{\mu\nu\rho\sigma} (\Sigma_{\mu\nu}^i \Sigma_{\rho\sigma}^j - \frac{1}{3} \delta^{ij} \Sigma_{\mu\nu}^k \Sigma_{\rho\sigma k}) = 0$ is $\Sigma_{\mu\nu}^i = \frac{1}{2} (e_\mu^0 e_\nu^i + \frac{i}{2} e_{jk}^i e_\mu^j e_\nu^k)$, which substituted into the action gives the Palatini action. Following the previous analysis, we consider the parent action

$$I = \int \varepsilon^{\mu\nu\rho\sigma} \left(\alpha \Sigma_{\mu\nu}^i \mathcal{F}_{\rho\sigma i} + \beta G_{\mu\nu}^i \mathcal{F}_{\rho\sigma i} + \gamma G_{\mu\nu}^i \partial_\rho \omega_{\sigma i} + \eta G_{\mu\nu}^i \varepsilon_{ijk} \omega_\rho^j \omega_\sigma^k + \phi_{ij} \Sigma_{\mu\nu}^i \Sigma_{\rho\sigma}^j \right). \tag{18}$$

First we observe that the quadratic form in Σ , the last term, cannot be integrated in combination with the first term to give a \mathcal{F}^2 term, because ϕ is traceless, and the first term is contracted with δ_{ij} . Furthermore, similar to the Chern-Simons case, the integrations over \mathcal{F} and G , and over \mathcal{F} and Σ , give all of them the Plebański action

$$Z = \int \mathcal{D}\Sigma \mathcal{D}\omega \mathcal{D}\phi e^{i \int \varepsilon^{\mu\nu\rho\sigma} \left[-\frac{\alpha\gamma}{\beta} \Sigma_{\mu\nu}^i (\partial_\rho \omega_{\sigma i} + \frac{\eta}{\gamma} \varepsilon_{ijk} \omega_\rho^j \omega_\sigma^k) + \phi_{ij} \Sigma_{\mu\nu}^i \Sigma_{\rho\sigma}^j \right]}. \tag{19}$$

However, the integration over ω , \mathcal{F} and G in (18) gives

$$Z = \int \mathcal{D}\Sigma \mathcal{D}\phi \sqrt{\det(M^{-1})} e^{i \int \left(\frac{\alpha\gamma^2}{\beta\eta} M_{\mu\nu}^{-1ij} \partial_\rho * \Sigma_i^{\rho\mu} \partial_\sigma * \Sigma_j^{\rho\nu} + \varepsilon^{\mu\nu\rho\sigma} \phi_{ij} \Sigma_{\mu\nu}^i \Sigma_{\rho\sigma}^j \right)}, \tag{20}$$

where $M_{ij}^{\mu\nu} = \varepsilon_{ij}^k * \Sigma_k^{\mu\nu}$.

3. GRAVITATIONAL DUALITY IN 3D CHERN-SIMONS GRAVITY

In the present section we explain the details of gravitational duality for the Chern-Simons theory in three dimensions. This subject has been discussed in the previous section for a general Chern-Simons theory. However here we intend to exhibit the details corresponding to gravitational Chern-Simons.

2 + 1 gravity dimensions is a theory that has played a very important role as a toy model of four dimensional general relativity at the classical and quantum levels. In Witten [15] 2 + 1 gravity was described in terms of the standard and exotic actions. In Garcia-Compean et al. [26] we showed that both actions correspond to the self-dual and anti-self-dual of the Chern-Simons actions with respect the Lorentz gauge group. The given Lie algebras \mathfrak{g} of the gauge groups have three generators constructed with four capital Latin letters $A, B, C, D = 0, \dots, 3$

and correspond to $\mathfrak{g} = \mathfrak{so}(3, 1)$ and $\mathfrak{g} = \mathfrak{so}(2, 2)$ for $\lambda > 0$ and $\lambda < 0$ respectively. The Lie algebra \mathfrak{g} is generated by M_{AB} , which satisfy $[M_{AB}, M_{CD}] = if_{ABCD}^{EF} M_{EF}$, where f_{ABCD}^{EF} are its corresponding structure constants.

Consider the non-abelian Chern-Simons action

$$I(A) = \int_X \frac{g}{4\pi} \text{Tr}(A \wedge H), \tag{21}$$

where g is the Chern-Simons coupling, $H = dA + \frac{2}{3}A \wedge A$ and Tr is a quadratic form of \mathfrak{g} such that it satisfies $\text{Tr}(M_{AB}M_{CD}) = \eta_{(AC}\eta_{BD)}$. The gauge field $A = A_i^{AB}M_{AB}dx^i$ and the H -field $H = H_{ij}^{AB}M_{AB}dx^i \wedge dx^j$. In local coordinates we have

$$I(A) = \int_X d^3x \frac{g}{4\pi} \varepsilon^{ijk} A_i^{AB} \left(\partial_j A_{kAB} + \frac{1}{3} f_{ABCDEF} A_j^{CD} A_k^{EF} \right). \tag{22}$$

The duality algorithm require to propose a *parent action*, which in [26] has been proposed as

$$I_P(A, B, G) = \int_X d^3x \varepsilon^{ijk} \left(a B_i^{AB} H_{jkAB} + b A_i^{AB} G_{jkAB} + c B_i^{AB} G_{jkAB} \right), \tag{23}$$

where

$$H_{jkAB} = \partial_j A_{kAB} + \frac{1}{3} f_{ABCDEF} A_j^{CD} A_k^{EF}. \tag{24}$$

The duality algorithm allows to recover the original action (22) after an integration with respect the Lagrange multipliers B and G . In fact, in the preceding section we have presented a somewhat more general parent action, although the details of the computations in [25] are the same, in such a way that the resulting actions, with $a = -g/4\pi$, $b = 1$ and $c = 1$, are

$$I(A) = \int_X d^3x \frac{g}{4\pi} \varepsilon^{ijk} A_i^{AB} \left(\partial_j A_{kAB} + \frac{1}{3} f_{ABCDEF} A_j^{CD} A_k^{EF} \right). \tag{25}$$

Whereas the dual one is

$$I_D(B) = \int_X d^3x \frac{g}{4\pi} \varepsilon^{ijk} B_i^{AB} \left(\partial_j B_{kAB} + \frac{1}{3} f_{ABCDEF} B_j^{CD} B_k^{EF} \right). \tag{26}$$

In Chern-Simons gravity Newton's gravitational constant G_N is related to the Chern-Simons coupling constant g [16] i.e.

$$g = -\frac{1}{4G_N}. \tag{27}$$

In the context of Abelian Chern-Simons theory, this duality was previously working out by Balchandran [39]. In particular, it was shown that in the Abelian case, the consistency of the dual theory requires from periodic boundary conditions for the dual fields B_i^{AB} . More recently this duality symmetry has been explored in the context of supersymmetric Chern-Simons QED [40]. Finally a more exhaustive analysis was carried out in Witten [41].

4. GRAVITATIONAL DUALITY IN HITCHIN'S GRAVITY THEORIES

In this section we will discuss the so called topological M-theory [10]. This is a theory in a seven-dimensional manifold X with G_2 -holonomy and stable real 3 and 4-forms Φ and $G = \star\Phi$. The Hitchin's functionals [17, 18] describing the volume of the seven-manifold X in terms of the stable forms are given by

$$V_7(\Phi) = \int_X \Phi \wedge \star\Phi, \tag{28}$$

and

$$V_7(G) = \int_X G \wedge \star G. \tag{29}$$

The critical points of these actions determine special geometric structures on X . For instance the variation of Equation (28) determines a metric with G_2 -holonomy which can be constructed through solutions of equations of motion

$$d\Phi = 0, \quad d\star\Phi = 0. \tag{30}$$

Another form we can see that this is true is as follows. If one takes Φ to be an exact form: $\Phi = dB$ with B being a two form on X . Then immediately one has $d\Phi = 0$. Then the variation of the volume (28) is given by

$$\delta V_7(\Phi) = 2 \int_X \delta\Phi \wedge \star\Phi. \tag{31}$$

Taking into account that $\Phi = dB$ we have

$$\delta V_7(\Phi) = -2 \int_X \delta B \wedge d\star\Phi = 0. \tag{32}$$

Then we have that Equation (30) are fulfilled. Similarly this procedure can be carried out for Equation (29).

A similar procedure can be implemented for Equation (29). It is easy to see that the corresponding equations of motion are

$$dG = 0, \quad d\star G = 0. \tag{33}$$

For seven manifolds which are the global product $X = M \times I$, where I is the finite interval. The stable forms in the 7-dimensional theory induce stable forms in the 6-dimensional theory through a Hamiltonian flow. This determines the form of Φ and $\star\Phi$ in terms of the 6-dimensional stable real 3-form ρ and 2-form k [10]. This is given by

$$\Phi = \rho(t) + k(t) \wedge dt, \quad \star\Phi = \sigma + \widehat{\rho} \wedge dt, \tag{34}$$

where $\sigma = \frac{1}{2}k \wedge k$. In terms of stable forms ρ and k the Hitchin's action is written as [10]

$$\begin{aligned} V_7(\Phi) &= \int_M \rho \wedge \widehat{\rho} + \frac{1}{2} \int_M k \wedge k \wedge k \\ &= 2V_H(\rho) + 3V_S(\sigma), \end{aligned} \tag{35}$$

where

$$V_H(\rho) = \frac{1}{2} \int_M \rho \wedge \widehat{\rho} \tag{36}$$

and

$$V_S(\sigma) = \frac{1}{6} \int_M k \wedge k \wedge k. \tag{37}$$

Variations of (36) and (37) with respect to ρ and k respectively lead to the following equations of motion

$$d\rho = 0, \quad dk = 0. \tag{38}$$

The first equation implies the existence of a closed holomorphic and invariant $(0, 3)$ form Ω on M with $d\Omega = 0$ and the existence of a Kähler form k on M . That means that M is a Calabi-Yau manifold.

4.1. Gravitational Duality in Topological M-Theory

In this subsection we will find the dual action to the Hitchin's actions (28) and (29). In order to do that we consider the following parent action

$$I_P(\Phi, B, \Lambda) = \int_X \left(aB \wedge \star\Phi + b\Phi \wedge \star\Lambda + cB \wedge \star\Lambda \right), \tag{39}$$

where Φ, B, Λ belong to $\Omega^3(X)$ and a, b and c are undetermined constants. Integrating out with respect to the Lagrange multipliers Λ and B we will regain the original action (28). Thus in Euclidean signature

$$\exp \{ -I(\Phi) \} = \int DBD\Lambda \exp \left(-I_P(\Phi, B, \Lambda) \right), \tag{40}$$

after integration we have

$$I(\Phi) = -\frac{ab}{c} \int_X \Phi \wedge \star\Phi. \tag{41}$$

If we select the constants, $b = 1$ and $c = -1$, then

$$I(\Phi) = a \int_X \Phi \wedge \star\Phi. \tag{42}$$

If we take $a = 1$ then

$$I(\Phi) = V_7(\Phi). \tag{43}$$

Now we can obtain the dual action by integrating out (40) with respect to B and Φ

$$\exp \{ -I_D(\Lambda) \} = \int D\Phi DB \exp \left(-I_P(\Phi, B, \Lambda) \right). \tag{44}$$

Integrating out first with respect to B we get

$$\int D\Phi D\Lambda \delta[\star(a\Phi + c\Lambda)] \exp \left(-b \int_X \Phi \wedge \star\Lambda \right). \tag{45}$$

Further integration with respect to Φ leads to the dual action

$$I_D(\Lambda) = -\frac{bc}{a} \int_X \Lambda \wedge \star\Lambda. \tag{46}$$

As in the derivation of action (42) we have

$$I_D(\Lambda) = \frac{1}{a} \int_X \Lambda \wedge \star\Lambda. \tag{47}$$

Thus the dual action looks exactly of the same form as the Hitchin's action (42) but with the coupling constant inverted and interchanging the original degrees of freedom Φ by the dual variables Λ , which are the Lagrange multipliers.

Once again if $a = 1$ we have

$$I_D(\Lambda) = V_7(\Lambda), \tag{48}$$

where

$$V_7(\Lambda) = \int_X \Lambda \wedge \star\Lambda. \tag{49}$$

It is immediate to see that the equations of motion associated to the dual action (49) are

$$d\Lambda = 0, \quad d\star\Lambda = 0. \tag{50}$$

Then we conclude that the topological M-theory with action (28) is self-dual.

4.2. Derivation of the Parent Action in Six Dimensions From Topological M-Theory

As we mentioned before action (28) can be reduced to a theory in six dimensions which is the linear combination given by (35). Now we will show that under certain conditions the duality algorithm in the seven dimensional theory can be induced to a duality procedure in six dimensions from the action (39).

In addition to Equation (34) we have

$$B = b_3 + b_2 \wedge dt, \quad \star B = b_4 + \widehat{b}_3 \wedge dt, \tag{51}$$

and something similar happens for the Lagrange multiplier

$$\Lambda = \lambda_3 + \lambda_2 \wedge dt, \quad \star\Lambda = \lambda_4 + \widehat{\lambda}_3 \wedge dt. \tag{52}$$

We impose that all of them satisfy the Calabi-Yau condition

$$b_3 \wedge b_2 = 0, \quad \lambda_3 \wedge \lambda_2 = 0. \tag{53}$$

Moreover we assume the same dependence for $\widehat{\rho}$ and $\widehat{\lambda}_3$. That is, if we have $\widehat{\rho}(\rho) = \widehat{\lambda}_3(\lambda_3)$, this implies that $\rho = \lambda_3$. With these conditions it is possible to show that the parent action (39) can be reduced to the linear combination of two parent actions in six dimensions

$$I_P = I_P(\rho, b_3, \widehat{\lambda}_3) + I_P(\sigma, b_2, \widehat{\lambda}_4), \tag{54}$$

where

$$I_P(\rho, b_3, \widehat{\lambda}_3) = a \int_M b_3 \wedge \widehat{\rho} + b \int_M \widehat{\lambda}_3 \wedge \rho + c \int_M \widehat{\lambda}_3 \wedge b_3, \tag{55}$$

and

$$I_P(\sigma, b_2, \widehat{\lambda}_4) = a \int_M \sigma \wedge b_2 + b \int_M \widehat{\lambda}_4 \wedge k + c \int_M \widehat{\lambda}_4 \wedge b_2. \quad (56)$$

4.3. Gravitational Duality in Six Dimensions

Now we describe the duality in six dimensions. We start from the action (55). Integrating out with respect the Lagrange multiplier $\widehat{\lambda}_3$ we go back to the original action

$$I(\rho) = -\frac{ab}{c} \int_M \rho \wedge \widehat{\rho}. \quad (57)$$

If we take $a = b = 1$ and $c = -1$ as before. Then we get

$$I(\rho) = V_H(\rho). \quad (58)$$

Now the dual action can be obtained by calculating the effective action and integrating out with respect to b_3 and then with respect to $\widehat{\rho}$. This is given by

$$\exp \{ -I_D(\widehat{\lambda}_3) \} = \int D\widehat{\lambda}_3 D b_3 D \widehat{\rho} \exp \left(-I_P(\rho, b_3, \widehat{\lambda}_3) \right). \quad (59)$$

Integration with respect to b_3 yields

$$\int D\widehat{\lambda}_3 D \widehat{\rho} \delta(a\widehat{\rho} + c\widehat{\lambda}_3) \exp \left(-b \int_M \rho \wedge \widehat{\lambda}_3 \right). \quad (60)$$

Now we use the arguments given after Eq. (53), integration on $\widehat{\rho}$ and $\widehat{\lambda}_3$ can be expressed as integrations with respect to ρ and λ_3 , then

$$\int D\widehat{\lambda}_3 D \rho \delta(a\rho + c\lambda_3) \exp \left(-b \int_M \rho \wedge \widehat{\lambda}_3 \right). \quad (61)$$

Then integration with respect to ρ determines the dual theory

$$I_D(\rho) = -\frac{bc}{a} \int_M \lambda_3 \wedge \widehat{\lambda}_3. \quad (62)$$

Again, for $b = 1$ and $c = -1$ we have

$$I_D(\rho) = \frac{1}{a} \int_M \lambda_3 \wedge \widehat{\lambda}_3. \quad (63)$$

It is a self-dual theory that inverts the coupling a and interchanges the original degrees of freedom ρ by λ_3 .

We now discuss the duality coming from the parent action (56). Once again integration with respect to the Lagrange multiplier $\widehat{\lambda}_4$ leads to the original action

$$I(\sigma) = a \int_M \sigma \wedge k. \quad (64)$$

Finally we will get the dual action. Before that we make the assumption that the Lagrange multiplier $\widehat{\lambda}_4$ can be rewritten as

$$\widehat{\lambda}_4 = \frac{1}{2} \widetilde{\lambda} \wedge \widetilde{\lambda}. \quad (65)$$

The dual action is then defined by

$$\exp \{ -I_D(\widetilde{\lambda}) \} = \int D b_2 D k \exp \{ -I_P(\sigma, b_2, \widetilde{\lambda}) \}. \quad (66)$$

Integration with respect to the field b_2 including condition (65) leads to

$$\int D \widetilde{\lambda} D k \delta(ak + c\widetilde{\lambda}) \exp \left(-\frac{b}{2} \int_M \widetilde{\lambda} \wedge \widetilde{\lambda} \wedge k \right). \quad (67)$$

Finally integration with respect to k determines the dual theory for $a = b = 1$ and $c = -1$

$$I_D(\widetilde{\lambda}) = \frac{1}{2} \int_M \widetilde{\lambda} \wedge \widetilde{\lambda} \wedge \widetilde{\lambda}. \quad (68)$$

5. FINAL REMARKS

In the present article we give an overview of some of our results regarding gravitational duality in some gravity theories [22–27]. In these papers we found some explicit dual actions to some specific theories of gravity. The duality procedure was implemented from the Rocek-Verlinde non-abelian duality algorithm applied to Yang-Mills theories [28–31].

In section 2, we reviewed the gravitation duality in a unified framework that contains the cases of topological gravity, MacDowell-Mansouri gravity, Chern-Simons gravity and BF-gravity. In order to present a complete case, in section 3 we overview in more detail the corresponding gravitational duality to Chern-Simons gravity in 2+1 dimensions.

In section 4 we give a new contribution to the subject. We apply for the first time the duality algorithm to the Hitchin’s volume functionals in seven and six dimensions. Hitchin’s functional in seven dimensions is the starting point to define topological M-theory [10]. In the present article we find the dual action which is written in terms of the dual degrees of freedom Λ and it is observed to be *self-dual* since it has the same form as the original action. Furthermore the dual action has inverted the coupling constant. Moreover the dual theory has the same equations of motion (50) than the original theory. Thus they have the same dynamics and both theories are classically equivalent.

The duality algorithm was also implemented for the Hitchin’s functionals in six dimensions. In this case we have two volume functionals $V_H(\rho)$ and $V_S(\sigma)$. We have found the dual actions for such functionals given by expressions (63) and (68). Moreover we showed that the corresponding parent giving rise to these actions given by expressions (55) and (56) follows, under certain sensible conditions, from the parent action of the topological M-theory (39). Thus we find that the duality algorithm in six dimensions come from the duality procedure of the underlying seven-dimensional theory. It is expected that this connection can be carried out to other duality procedures in form theories of gravity in lower dimensions. In a future work we expect to report our results in the search of this web of dualities from M-theories to lower dimensions including the four and three dimensions. It would be interesting to investigate if

the recent results [19–21], will be of some relevance in this analysis.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

REFERENCES

1. Giveon A, and Kutasov D. Brane dynamics and gauge theory. *Rev Mod Phys.* (1999) **71**:983. doi: 10.1103/RevModPhys.71.983
2. Nurmagambetov AJ. Duality-symmetric approach to general relativity and supergravity. *SIGMA* (2006) **2**:20.
3. Dehouck F. Gravitational duality in General Relativity and Supergravity theories. *Nucl Phys Proc Suppl* (2011) **216**:223. doi: 10.1016/j.nuclphysbps.2011.04.161
4. Olive DI. Introduction to electromagnetic duality. *Nucl Phys Proc Suppl.* (1997) **58**:43. doi: 10.1016/S0920-5632(97)00412-X
5. Vafa C, Witten E. A Strong coupling test of S duality. *Nucl Phys B.* (1994) **431**:3. doi: 10.1016/0550-3213(94)90097-3
6. Alvarez-Gaume L, Hassan SF. Introduction to S duality in N=2 supersymmetric gauge theories: a Pedagogical review of the work of Seiberg and Witten. *Fortsch Phys.* (1997) **45**:159. doi: 10.1002/prop.2190450302
7. Aharony O, Gubser, SS, Maldacena JM, Ooguri H, Oz Y. Large N field theories, string theory and gravity. *Phys Rept.* (2000) **323**:183. doi: 10.1016/S0370-1573(99)00083-6
8. Giveon A, Porrati M, Rabinovici E. Target space duality in string theory. *Phys Rept.* (1994) **244**:77. doi: 10.1016/0370-1573(94)90070-1
9. Choudhury S, Dey A, Halder I, Jain S, Janagal L, Minwalla S, et al. Bose-Fermi Chern-Simons dualities in the higgsed phase. (2018) arXiv:1804.08635 [hep-th].
10. Dijkgraaf R, Gukov S, Neitzke A, Vafa C. Topological M-theory as unification of form theories of gravity. *Adv Theor Math Phys.* (2005) **9**:603. doi: 10.4310/ATMP.2005.v9.n4.a5
11. Bershadsky M, Cecotti S, Ooguri H, Vafa C. Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes. *Commun Math Phys.* (1994) **165**:311. doi: 10.1007/BF02099774
12. Bershadsky M, Sadov V. Theory of Kahler gravity. *Int J Mod Phys A* (1996) **11**:4689. doi: 10.1142/S0217751X96002157
13. Plebański JF. On the separation of Einsteinian substructures. *J Math Phys.* (1977) **18**:2511. doi: 10.1063/1.523215
14. Achúcarro A, Townsend PK. A Chern-Simons action for three-dimensional anti-De Sitter supergravity theories. *Phys Lett B.* (1986) **180**:89. doi: 10.1016/0370-2693(86)90140-1
15. Witten E. (2+1)-Dimensional gravity as an exactly soluble system. *Nucl Phys B.* (1988) **311**:46. doi: 10.1016/0550-3213(88)90143-5
16. Carlip S. *Quantum Gravity in 2+1 Dimensions*. Davis: Cambridge University Press (1998). doi: 10.1017/CBO9780511564192
17. Hitchin NJ. *Stable Forms and Special Metrics*. (2001) math/0107101 [math-dg].
18. Hitchin NJ. The geometry of three-forms in six dimensions. *J Diff Geom.* (2000) **55**:547.
19. Herfray Y, Krasnov K, Scarinci C, Shtanov Y. A 4D gravity theory and G2-holonomy manifolds. (2016) arXiv:1602.03428 [hep-th].
20. Herfray Y, Krasnov K, Scarinci C. 6D Interpretation of 3D gravity. *Class Quant Grav.* (2017) **34**:45007. doi: 10.1088/1361-6382/aa5727
21. Krasnov K. General relativity from three-forms in seven dimensions. *Phys Lett B* (2017) **772**:300. doi: 10.1016/j.physletb.2017.06.025
22. García-Compeán H, Obregon O, Plebanski JF, Ramirez C. Towards a gravitational analog to S duality in nonAbelian gauge theories. *Phys Rev D.* (1998) **57**:7501. doi: 10.1103/PhysRevD.57.7501
23. García-Compeán H, Obregon O, Ramirez C. Gravitational duality in MacDowell-Mansouri gauge theory. *Phys Rev D.* (1998) **58**:104012. doi: 10.1103/PhysRevD.58.104012
24. García-Compeán H, Nieto JA, Obregon O, Ramirez C. Dual description of supergravity MacDowell-Mansouri theory. *Phys Rev D.* (1999) **59**:124003. doi: 10.1103/PhysRevD.59.124003
25. García-Compeán H, Obregon O, Ramirez C, Sabido M. Remarks on (2+1) selfdual Chern-Simons gravity. *Phys Rev D.* (2000) **61**:085022. doi: 10.1103/PhysRevD.61.085022
26. García-Compeán H, Obregon O, Ramirez C, Sabido M. On S duality in (2+1) Chern-Simons supergravity. *Phys Rev D.* (2001) **64**:024002. doi: 10.1103/PhysRevD.64.024002
27. García-Compeán H, Obregon O, Ramirez C. A Vector nonAbelian Chern-Simons duality. *Phys Rev D.* (2002) **65**:045006. doi: 10.1103/PhysRevD.65.045006
28. Ganor O, Sonnenschein J. The 'dual' variables of Yang-Mills theory and local gauge invariant variables. *Int J Mod Phys A.* (1996) **11**:5701. doi: 10.1142/S0217751X96002625
29. Mohammedi N. Classical duality in gauge theories. (1995) hep-th/9507040.
30. Lozano Y. S duality in gauge theories as a canonical transformation. *Phys Lett B.* (1995) **364**:19. doi: 10.1016/0370-2693(95)01081-1
31. Kehagias AA. A canonical approach to s duality in Abelian gauge theory. (1995) hep-th/9508159.
32. Roček M, Verlinde EP. Duality, quotients, and currents. *Nucl Phys B.* (1992) **373**:630. doi: 10.1016/0550-3213(92)90269-H
33. Buscher TH. A symmetry of the string background field equations. *Phys Lett B.* (1987) **194**:59. doi: 10.1016/0370-2693(87)90769-6
34. Buscher TH. Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models. *Phys Lett B.* (1988) **201**:466. doi: 10.1016/0370-2693(88)90602-8
35. Nieto JA. S duality for linearized gravity. *Phys Lett A.* (1999) **262**:274. doi: 10.1016/S0375-9601(99)00702-1
36. Bunster C, Henneaux M, Hortner S. Gravitational electric-magnetic duality, gauge invariance and twisted self-duality *J Phys A* (2013) **46**:214016. doi: 10.1088/1751-8113/46/26/269501 Erratum: [*J Phys A* (2013) **46**:269501] doi: 10.1088/1751-8113/46/21/214016
37. Bunster C, Henneaux M, Hörtnner S, Leonard A. Supersymmetric electric-magnetic duality of hypergravity *Phys Rev D* (2014) **90**:45029. doi: 10.1103/PhysRevD.90.069908 Erratum: [*Phys Rev D* (2017) **95**:69908] doi: 10.1103/PhysRevD.90.045029
38. Freedman DZ, Townsend PK. Antisymmetric tensor gauge theories and nonlinear sigma models. *Nucl Phys B.* (1981) **177**:282. doi: 10.1016/0550-3213(81)90392-8
39. Balachandran AP, Chandar L, Sathiapalan B. Chern-Simons duality and the quantum Hall effect. *Int J Mod Phys A.* (1996) **11**:3587. doi: 10.1142/S0217751X96001693
40. Kapustin A, Strassler MJ. On mirror symmetry in three-dimensional Abelian gauge theories. *JHEP* (1999) **9904**:21. doi: 10.1088/1126-6708/1999/04/021
41. Witten E. SL(2,Z) action on three-dimensional conformal field theories with Abelian symmetry. In: Shifman M, Vainshtein A, Wheeler J, editor *From Fields to Strings*, Vol. 2. Singapore: World Scientific Publishing Co. (2005), pp. 1173–200.

ACKNOWLEDGMENTS

HG-C thanks the Universidad de Guanajuato and OO for hospitality during the sabbatical stay. CR thanks VIEP-BUAP and PFCE-SEP for the financial support. OO was supported by CONACyT: Project 257919; Universidad de Guanajuato: Project CIIC130/2018 and PRODEP.

Conflict of Interest Statement: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Copyright © 2018 García-Compeán, Obregón and Ramirez. This is an open-access article distributed under the terms of the Creative Commons Attribution License (CC BY). The use, distribution or reproduction in other forums is permitted, provided the original author(s) and the copyright owner(s) are credited and that the original publication in this journal is cited, in accordance with accepted academic practice. No use, distribution or reproduction is permitted which does not comply with these terms.