



Phase Transitions in the Kinetic Ising Model on the Temporal Directed Random Regular Graph

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We investigate a kinetic Ising model with the Metropolis algorithm on the temporal directed random q -regular graph. We introduce a rewiring time τ as a parameter of the model and show that there is a critical $\tau = \tau^*$ above which the model exhibits continuous order-disorder phase transition and below which the transition is discontinuous. We discuss our findings in a light of the very recent results obtained for a generalized kinetic Ising model on quenched and annealed networks.

Keywords: kinetic Ising model, Metropolis algorithm, phase transitions, complex networks, temporal networks, opinion dynamics

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1. INTRODUCTION

Extensive studies have shown that order-disorder phase transitions in the equilibrium Ising model with the nearest-neighbors interactions are very robust to the type of a network. This means that observed transitions are not only continuous but also characterized with the same set of bulk critical exponents on quenched and annealed networks [1]. Moreover, it has been shown recently that even for the non-equilibrium Ising model with the heat-bath dynamics on the directed random q -regular graph (q -RRG) continuous phase transitions are observed [2]. Therefore results obtained in Jędrzejewski et al. [3], showing that kinetic Ising model with Metropolis dynamics can also display discontinuous phase transitions, were puzzling and confusing. To solve this puzzle a new generalized model with two heat baths—one for the Ising spins at temperature T_s and second for links of the graph at temperature T_L , has been recently proposed [4]. It has been shown that on contrary to the models investigated previously by Lee et al. [1], the kinetic model studied in Jędrzejewski et al. [3] is a non-equilibrium one and corresponds to the infinite temperature T_L of links. Furthermore, it has occurred that there is a critical temperature T_L^* at which switch from discontinuous to continuous phase transition is observed (so called tricritical point) [4]. However, the question arose if such a tricriticality could be exclusively explained on the basis of non-equilibrium driving. In other words, would it be possible to observe such a behavior also for other single-spin-flip dynamics (e.g., heat-bath)?

The above question has been asked and answered in Jędrzejewski et al. [5]. It has been shown that on the annealed non-equilibrium network, which within the generalized model by Park and Noh [4] corresponds to the infinite temperature of links ($T_L = \infty$), different dynamics leads to completely different results. For example, the model with heat-bath algorithm at $T_L = \infty$ displays a continuous phase transition, and the critical temperature only slightly deviates from the equilibrium one. On the other hand, the generalized Metropolis algorithm gives qualitatively different behavior, which depends on the model's parameter. It has been also observed that generally

higher flipping probabilities lead to discontinuous phase transitions. Furthermore, based on results obtained within generalized model with two heat-baths [4] and modified Metropolis algorithm studied in Jędrzejewski et al. [5], it has been suggested that tricriticality, i.e., switch from continuous to discontinuous phase transitions is driven by noise.

There are many possibilities to incorporate noise into the system. In Park and Noh [4] it has been introduced by the additional heat-bath for links, whereas in Jędrzejewski et al. [5] it has been introduced by the probability of flip in the absence of the energy changes W_0 : in the original Metropolis algorithm $W_0 = 1$, whereas for the heat-bath dynamics $W_0 = 1/2$. Within the generalized zero-temperature Glauber dynamics, W_0 can change continuously [6], which leads to very interesting results [7–9]. The idea of W_0 as a parameter has been used in Jędrzejewski et al. [5] and it has been shown that there is a tricritical point $W_0 = W_0^*$: for $W_0 < W_0^*$ the transition is continuous and for $W_0 > W_0^*$ it is discontinuous.

Here we propose another possibility to introduce noise into the system. We investigate the model on a temporal network and introduce rewiring time τ , as a parameter. Because $\tau = \infty$ corresponds to quenched network and $\tau = 1$ to $T_L = \infty$ (i.e., annealed network), one could expect that there is a critical time scale τ^* that separates these two regimes. Following very recent studies [4, 5], we analyze a kinetic Ising model with Metropolis dynamics on the temporal directed random regular graph to check whether this type of noise will also lead to tricriticality.

Temporal networks are particularly useful to model social systems [10]. However, if the dynamics on the network is very rapid compared to the dynamics of the contacts then there is no need to model the system as a temporal network [10]. The model we investigate here is not a social model, but it could be treated as such Nyczka and Sznajd-Weron [11]. In fact, the q -neighbor Ising model Jędrzejewski et al. [3], reformulated later in terms of the kinetic Ising model on random regular graph, has been inspired by the q -voter model of opinion dynamics [12–15]. This is not entirely clear what is the scale of opinion changes and definitely exploring how opinion dynamics will be influenced by the dynamics of a network is an interesting task. However, this is not a paper about networks. Time-varying graph is for us merely an excuse to incorporate noise into the system and check how it will influence observed phase transitions. Nevertheless, we hope that our results may be also inspiring for people dealing with opinion dynamics on networks and also for this reason we use a graphical representation of so called spinson (spin+person) introduced by Nyczka and Sznajd-Weron [11] to illustrate the model in **Figure 1**.

2. MODEL

We investigate a kinetic Ising model with the Metropolis algorithm on the temporal directed random q -regular graph (q -RRG) of size N . We describe such a graph at time t by a non-symmetric adjacency matrix $A(t)$ with elements $A_{ij}(t) \neq A_{ji}(t)$, $i, j = 1, \dots, N$. If node i can be influenced by node j at time

t then $A_{ij}(t) = 1$, otherwise $A_{ij}(t) = 0$ and therefore in our case:

$$\forall i \sum_j A_{ij}(t) = q. \tag{1}$$

Above relation means that each node i can be influenced by q nodes but not necessary influences q other nodes. As usually, we set $A_{ii} = 0$ to avoid self-loops.

Each node of a graph is occupied by exactly one spin described by dynamic variable $S_i(t) = \pm 1, i = 1, 2, \dots, N$. The state of spins at time $s(t) = (S_1(t), S_2(t), \dots, S_N(t))$ can be changed at each elementary time step due to the interactions between spins and thermal fluctuations. Additionally, in Jędrzejewski et al. [3, 5] the adjacency matrix $A(t)$ could change at each elementary time step, i.e., in general $A(t + \Delta t) \neq A(t)$. In this paper we introduce a new parameter τ , which denotes a number of elementary steps during which a graph (and simultaneously adjacency matrix) does not change. It means that for a period of time $\tau \Delta t$ a graph is fixed. In **Figure 1** we present an example of a system of $N = 3$ nodes with degree $q = 1$ at three time steps: $t, t + \Delta t, t + \tau \Delta t$.

At time t a state of a system is given by:

$$s(t) = (+1, -1, -1), \quad A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2}$$

In the next step $t + \Delta t$ the state of spins changes $s(t + \Delta t) = (+1, +1, -1) \neq s(t)$ but adjacency matrix does not change, i.e., $A(t + \Delta t) = A(t)$. After τ steps spins and links are changed.

The network itself evolves completely randomly, i.e., after each τ steps we choose a new adjacency matrix at random with the constrain given by Equation (1), whereas spin variables change according to Metropolis algorithm, which means that each elementary time step consists of three consecutive steps:

1. We choose randomly one spin i from the uniform distribution $U[0, N]$
2. We calculate the change of the energy, that would occur after a spin flip:

$$\Delta E = E(-S_i) - E(S_i) = 2S_i \sum_j A_{ij} S_j \tag{3}$$

3. We flip a spin ($S_i(t + \Delta t) = -S_i(t)$) with probability:

$$P(\Delta E) = \min(1, \exp(-\Delta E/T)). \tag{4}$$

As usual one Monte Carlo step (MCS) consists of N elementary steps, which means that $\Delta t = 1/N$. As we have mentioned, network changes after each τ steps completely randomly. Another approach has been used very recently in Park and Noh [4]. They have proposed to consider a system in contact with two heat baths: the Ising spins are in thermal contact with the heat bath at temperature T_S , whereas the links are in thermal contact with the heat bath at temperature T_L . The Hamiltonian for the whole system including the spins and the links was given by:

$$H(A, s) = -\frac{J}{2} \sum_{ij} A_{ij} S_i S_j \tag{5}$$

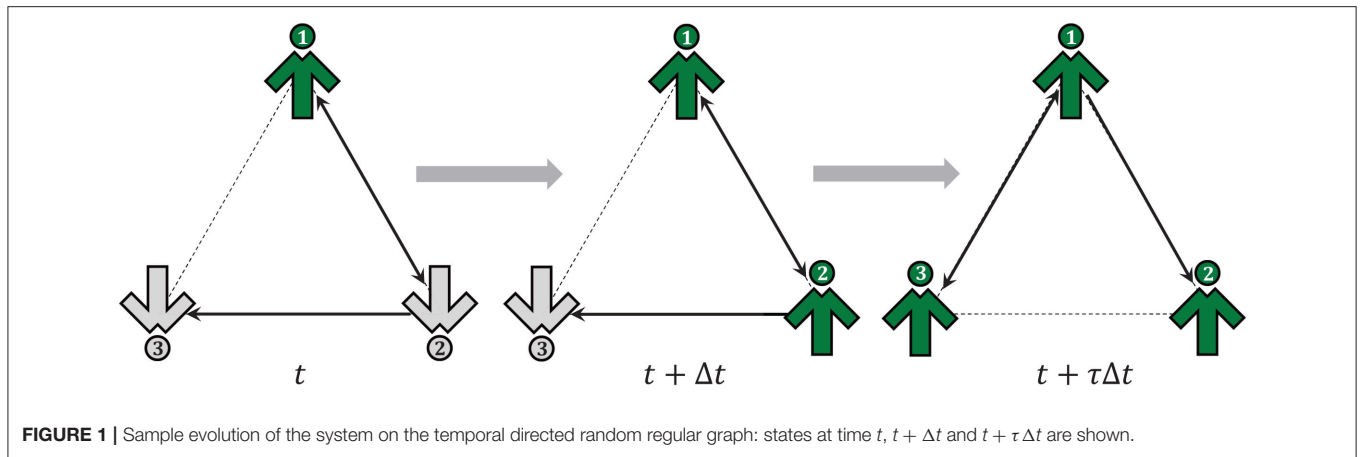


FIGURE 1 | Sample evolution of the system on the temporal directed random regular graph: states at time t , $t + \Delta t$ and $t + \tau \Delta t$ are shown.

and not only spin flips but also rewiring was carried out with a certain probability depending on the energy difference:

$$\Delta E = H(A', s') - H(A, s), \tag{6}$$

where for brevity $A' = A(t + \Delta t)$, $A = A(t)$, $s' = s(t + \Delta t)$, $s = s(t)$. This means that the probability of flipping the spin was given by:

$$P_S(\Delta E) = \min(1, \exp(-\Delta E/T_S)) \tag{7}$$

and the probability to accept a new configuration of links by:

$$P_L(\Delta E) = \min(1, \exp(-\Delta E/T_L)). \tag{8}$$

The original q -neighbor Ising model, introduced in Jędrzejewski et al. [3], is a limiting case of the above model (i.e., corresponds to $T_L = \infty$). In this paper we propose completely different approach—we do not claim that the network itself is in contact with a heat-bath. It changes randomly at regular time intervals. Yet, we could propose a generalized model based on the model with two heat-baths but additionally with two time scales: t_S for spins and $t_L = \tau t_S$ for links. In such a case the model we consider here would correspond again to $T_L = \infty$. This might be an interesting idea for the future.

We are aware that the complete rewiring at every τ steps is perhaps not the most appropriate in the context of e.g., changing friends in social systems. Probably a continuous rewiring (at every step but only with a certain probability) would be more reliable. Indeed we could generalize our model by introducing a probability of rewiring p_{rew} in a single time step. Within such a generalization the model described above would correspond to $p_{rew} = 1$. However, as we have checked, such a generalization would only rescale results, since they depend only on τ/p_{rew} , not separately on τ and p_{rew} . For example that behavior of the model with $\tau = 100$ and $p_{rew} = 0.01$ is exactly the same as for $\tau = 10^4$ and $p_{rew} = 1$.

3. RESULTS

Our generalized model reduces to the q -neighbor Ising model for $\tau = 1$ and therefore should display a discontinuous phase

transition for $q \geq 4$. On the other hand for $\tau = \infty$ it reduces to the equilibrium Ising model on q -RRG, which means that should always display a continuous phase transition. This means that somewhere in between, for a critical τ^* we should observe a switch from discontinuous to continuous phase transition. The maximum jump of the order parameter:

$$m = \frac{1}{N} \sum_{i=1}^N S_i \tag{9}$$

and simultaneously the maximum hysteresis has been observed for $q = 4$. Moreover, also recent papers focused on $q = 4$ [4, 5] and therefore we will focus here on this value.

We investigate the model by Monte Carlo simulations and estimate the magnetization of the system as an ensemble average over M samples:

$$m = \frac{1}{M} \sum_{j=1}^M m_j, \tag{10}$$

where m_j is the stationary magnetization in j -th sample, defined as:

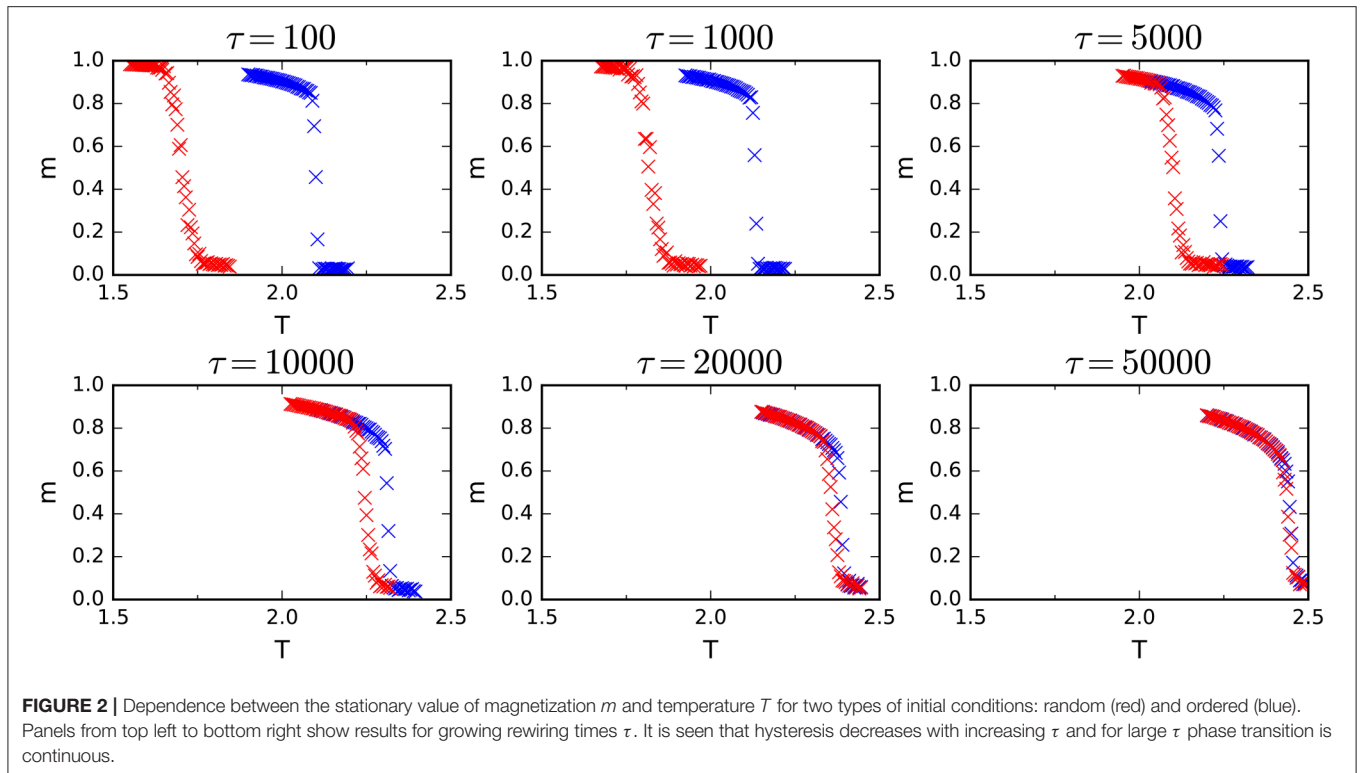
$$m_j = \frac{1}{N} \sum_{i=1}^N S_i^j, \tag{11}$$

here S_i^j denotes a value of an i -th spin in the j -th sample in the stationary state. All results presented in this paper has been averaged over $M = 10^2$ samples.

To check the type of the phase transition between ordered (ferromagnetic) and disordered (paramagnetic) phase we start with two types of initial conditions:

1. ordered, i.e., $S_i(0) = 1$ for all $i = 1, \dots, N$,
2. random, i.e., with probability 1/2 initial value of spin $S_i(0) = 1$ and with complementary probability $S_i(0) = -1$.

As long as the phase transition is discontinuous we should obtain two temperatures below which the system is ordered and above which $m = 0$: T_1 for the ordered initial conditions and T_2 for



the random initial conditions. For $\tau > \tau^*$ spinodal lines T_1, T_2 should collapse to a single line of a continuous phase transition $T^* = T_1 = T_2$.

In **Figure 2** we present dependence between the stationary value of magnetization m and temperature T for two types of initial conditions for the system size $N = 10^4$. It is seen that indeed for small τ results depend on initial conditions. With increasing τ hysteresis decreases and eventually the phase transition between ordered ferromagnetic and disordered paramagnetic phases becomes continuous. It is worth to recall here, that the real phase transitions and simultaneously real size of hysteresis are defined only in the thermodynamic limit $N \rightarrow \infty$. If simulated system is too small one can mistakenly draw a conclusion of the continuous phase transition because the hysteresis increases with the system size (see **Figure 3**). This phenomenon is probably well known to all researchers that are specialized in numerical analysis of the phase transitions but we hope that results presented in **Figure 3** and the above comment will be instructive for the newcomers in the field. From **Figure 2** one can determine spinodal lines $T_1(\tau)$ and $T_2(\tau)$ and this allows to estimate the critical value of $\tau = \tau^*$ above which the transition switches to continuous.

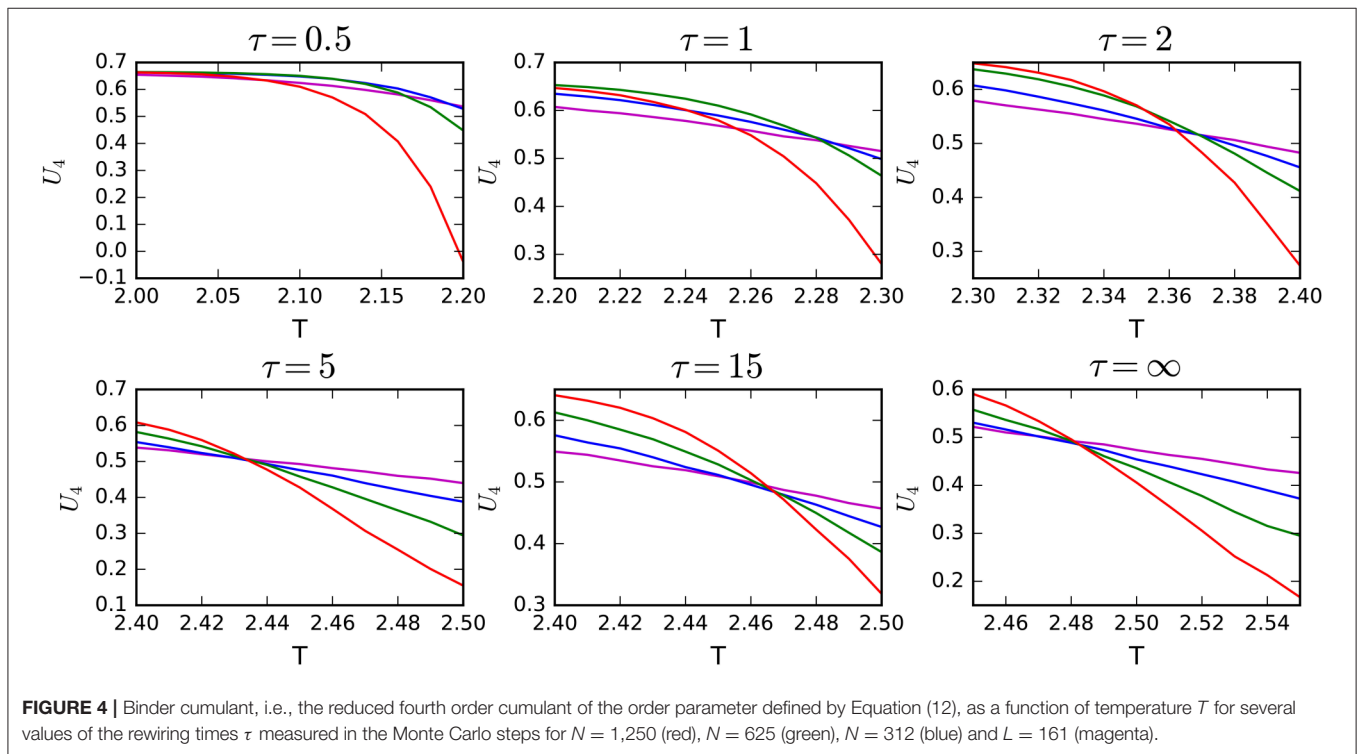
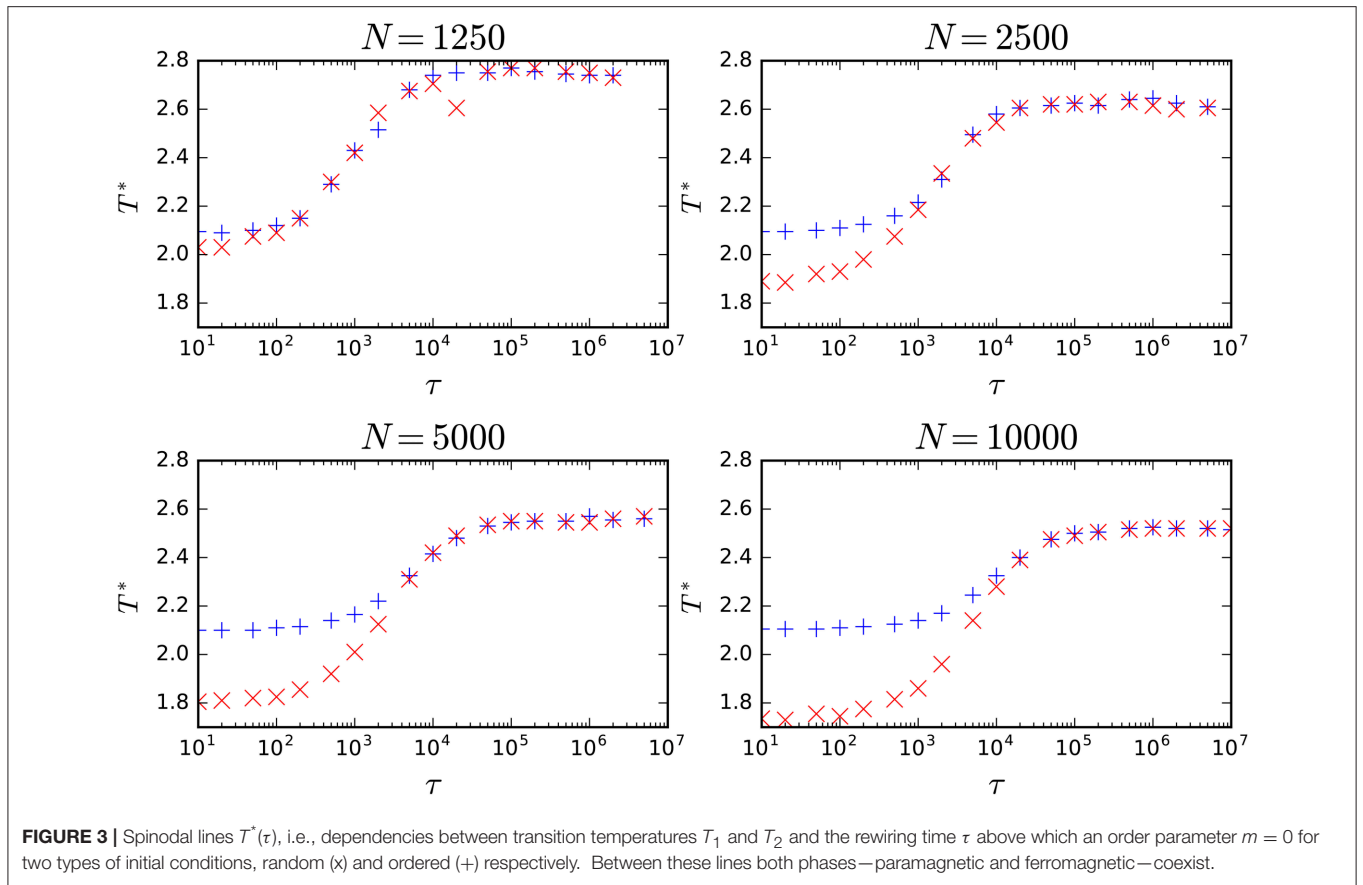
It is seen in **Figure 3** that with increasing τ spinodal lines are approaching each other and eventually tricritical point is reached at τ^* . Tricritical behavior for $q = 4$ has been also observed by Park and Noh [4] withing the Ising model on fluctuating network with two heat-baths and for the generalized Metropolis algorithm [5].

We would like to stress here that determining the exact value of the critical temperature $T^*(\tau)$ for intermediates value of $1 < \tau < \infty$ and hence the critical rewiring time τ^* is far from being trivial. In equilibrium statistical mechanics, the standard procedure to identify the universality class and to estimate the critical temperature within the Monte Carlo simulations would be to calculate the reduced fourth order cumulant of the order parameter, widely known as the Binder cumulant, which for the Ising model reduces to Landau and Binder [16]:

$$U_4 = 1 - \frac{m^4}{3 < m^2 >^2}. \tag{12}$$

Drawing curves for U_4 as a function of a temperature for different sizes N one can determine the transition point from the common intersection point.

However, it is far from being obvious if the powerful cumulant intersection method can be applied in non-equilibrium systems. For example it cannot be used to absorbing phase transitions [6]. Nevertheless, we have decided to calculate U_4 for several values of τ to check if this method works for our non-equilibrium model. Until now we have measured the rewiring time τ in elementary steps. However, if we want to compare results for different sizes the rewiring time τ should be measured in Monte Carlo steps instead of elementary steps and therefore τ is given in MCS in **Figure 4**. It is seen in **Figure 4** that the method works pretty well for $\tau \geq 5\text{MCS}$. It works even for $\tau \approx 1\text{MCS}$. Although the intersection is not perfect it allows to estimate the critical temperatures, e.g., $T^*(\tau = 1\text{MCS}) \approx 2.3$ and $T^*(\tau = \infty) \approx 2.49$, which agrees pretty well with results presented in **Figures 2–4**



suggest that τ^* is of the order of 1MCS but to determine the precise value of the tricritical point τ^* is not so easy.

It should be noticed here that the critical temperature for large values of τ (i.e., almost quenched network) approaches $T^* \approx 2.49$, which is a significantly lower value than one given by the analytical formula derived by Dorogovtsev et al. for the equilibrium Ising model on networks with an arbitrary distribution of connections [17]:

$$\frac{J}{T^*} = \frac{1}{2} \ln \left(\frac{\langle k^2 \rangle}{\langle k \rangle^2 - 2\langle k \rangle} \right), \quad (13)$$

where J is a coupling constant. For q -RRG degrees k of all nodes are equal to q , i.e., $\langle k^2 \rangle = q^2$ and $\langle k \rangle = q$. Moreover, we have assumed that $J = 1$ and thus the critical temperature should be equal:

$$T^*(q) = 2 \left[\ln \left(\frac{q}{q-2} \right) \right]^{-1}, \quad (14)$$

which for $q = 4$ gives $T^*(4) = 2.88$. However, as noticed by Lipowski et al., an Ising model on the directed network is a non-equilibrium model [18] and therefore this is not surprising that our results differ from those obtained by Dorogovtsev et al. [17]. Moreover, our result is consistent with the results for the Ising model on undirected q -RRG, because it has been shown that for the Ising model with the Metropolis dynamics the critical temperature for the annealed network is significantly lower than for the quenched network [5]. Yet another result should be commented here, namely the critical temperature for the Ising model on directed q -RRG with the heat-bath dynamics [2]. In particular for $q = 4$ the critical temperature has been reported to be close to 3.089, which again is significantly larger value than T^* obtained here. Again, this is not very surprising in the light of recent findings by Jedrzejewski et al. because it has been shown numerically and analytically that the critical temperature for the Ising model on the annealed q -RRG with the heat bath dynamics is significantly larger than for the same model with the Metropolis algorithm.

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4. CONCLUSION

The aim of this paper was to check how the random rewiring of a network influences phase transitions in the kinetic Ising model with Metropolis algorithm on q -RRG. It has been shown in Jedrzejewski et al. [3] that if a network is rewired in every elementary time step then an order-disorder phase transition becomes discontinuous for $q \geq 4$, which has been a puzzling result. Therefore we decided to rewire the network not in every elementary step, but every τ steps and see how this would influence results. We have expected that for large τ the model will reproduce results of the equilibrium Ising model on the quenched q -RRG. However, we did not have an intuition if the quenched limit will be observed only for $\tau = \infty$ or maybe for any $\tau > 1$, or above some critical value τ^* . It has occurred that indeed τ^* exists but its value is comparable with a length of a single Monte Carlo step. This means that results of the kinetic Ising model on the temporal q -RRG, which changes randomly after each Monte Carlo step, are almost identical with results for the model on the static q -RRG. We have presented here results only for $q = 4$, but our conclusions are valid also for other values of $q > 3$. For $q = 3$ the phase transition is always continuous and there are no phase transitions for $q = 1$ and $q = 2$ [2, 3].

AUTHOR CONTRIBUTIONS

GM conducted all simulations and analyzed results. KS designed and supervised research and wrote the article.

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