A Fractional Time-Derivative Model for Severe Wear: Hypothesis and Implications

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Based on the example of wear of polymers, which exhibit a power-law time variation of the wear loss under constant loading conditions, a fractional time-derivative wear equation has been introduced. The wear contact problem with a fixed contact zone is solved using the known method of separation of spatial and time variables. It is shown that during the wear process, the contact pressure approaches a uniform distribution over the contact area, which is termed as a quasi-steady-state solution, since the mean volumetric wear rate does not tend to become constant. It is of interest that the contact pressure variation displays a decaying oscillatory nature in the case of severe wear, when the mean volumetric wear rate increases with time.

Keywords: severe wear, fractional time derivative, wearing-in, quasi-steady state, wear equation

1 INTRODUCTION

Wear is a tribological phenomenon that accompanies contact interaction of solids with interfacial sliding between their surfaces and manifests itself primarily in gradual loss of material due to the subsurface damage accumulation and surface degradation (Zmitrowicz, 2006). In wear material testing, the wear loss is usually measured in terms of the volume of lost material, V, which yields the wear depth, w, by relating to the area of contact, A, as

$$w = \frac{V}{A}. \tag{1}$$

Eq. 1 tentatively assumes that the wear loss is the same at each point of the contact zone, and this condition is characteristic of the steady-state wear process (Dundurs and Comninou, 1980; Páczelt and Mróz, 2007).

Usually, sliding wear tests are performed under constant normal loading conditions, which are characterized by either contact load, P, or mean contact pressure,

$$\bar{p} = \frac{P}{A}. \tag{2}$$

In the cases of mild or moderate wear, after some initial running-in (Wright and Kukureka, 2001; Khonsari et al., 2021) or wearing-in (Blau, 2005) period of time, the condition of steady state $p(x, t) \equiv \bar{p}$ is realized at the contact interface and maintained for a long time period, during which the wear material characteristics have been measured.

In the steady-state wear process, the linear wear rate, defined as the time derivative $\dot{w} = dw/dt$, is constant in time, and moreover, the contact pressure p achieves a constant value over the entire contact zone. So, in the steady-state wear regime, the volumetric wear loss V will be a linear function of the time variable t, and thus, the linear wear rate can be evaluated as
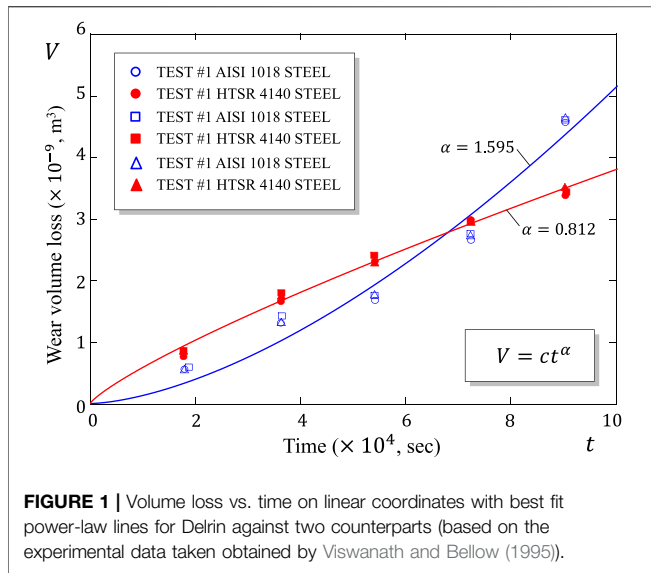
FIGURE 1 | Volume loss vs. time on linear coordinates with best fit power-law lines for Delrin against two counterparts (based on the experimental data taken obtained by Viswanath and Bellow (1995)).

$$\dot{w} = \frac{V(t_2) - V(t_1)}{A(t_2 - t_1)}, \qquad (3)$$

where t_1 and t_2 are any two different moments of time taken during the steady-state period.

In many cases, the wear equation can be written in the following form (Kragelsky, 1965):

$$V = k_w P v t. \qquad (4)$$

Here, v is the sliding velocity, and k_w is the wear coefficient. In view of **Eqs 1**, **2**, **Eq. 4** can be represented as

$$\dot{w} = k_w \bar{p} v, \qquad (5)$$

which is known as Archard's equation (Meng and Ludema, 1995), though similar wear equations were introduced earlier by Reye (1860), Khrushchov and Babichev (1941), and Holm (1946). The wear coefficient k_w is determined during the steady-state conditions by using **Eqs 3**, **5**.

A generalization of the wear **Eq. 4**, which was suggested by Rhee (1970) for polymer-bonded friction materials, takes the form

$$V = K_w P^\gamma v^\beta t^\alpha, \qquad (6)$$

where K_w is the wear factor, and α, β, γ are parameters. However, while **Eq. 6** has been successfully used to predict the wear resistance of polymer composite materials designed for extreme environmental conditions (Gardos, 1982; Sedakova and Kozyrev, 2021), there is a problem with applying the wear **Eq. 6** for solving the wear contact problems with the spatial-temporal variation of the contact pressures (Grzelczyk and Awrejcewicz, 2015; Ciavarella et al., 2020).

In recent years, there has been a growing interest in modeling severe wear (Nguyen et al., 2018; Popov and Pohrt, 2018; Li et al., 2020). In a broad sense, severe wear is defined as a form of wear characterized by a rapid increase in the amount and size of wear particles (Blau, 1992). In many situations, the regime of severe

wear can be characterized by the absence of any steady-state regime under constant conditions. This is so, for example, in wear of polymers (Viswanath and Bellow, 1995), when under constant normal load and sliding speed (see **Figure 1**), the wear volume loss V varies proportionally to some power of time, that is t^α, and therefore, the wear rate $\dot{w}$ will be a function of time (proportional to $t^{\alpha-1}$) during the entire testing period.

It is clear that in experiments like that, whose results are shown in **Figure 1**, **Eq. 3** is not applicable. Instead we can consider the ratio $V(t)/t$ that defines the mean volumetric wear rate. The two cases $\alpha = 0.812$ and $\alpha = 1.595$ differ by the decreasing/increasing trend of this quantity. The wear process with increasing in time mean volumetric wear rate (that is when $\alpha > 1$) will be termed as severe wear in a narrow sense. At the same time, the case $\alpha < 1$ may called mild wear, as the wear rate decreases towards zero during the wear process.

Further, when comparing the Archard **Eq. 4** with the Rhee **Eq. 6**, we see that for polymers the Archard wear coefficient k_w, which is evaluated as the ratio V/Pvt, is found to depend on time, that is $k_w \sim t\alpha - 1$. This means that for polymer-bonded friction materials, for which the Rhee wear equation applies, the concept of wear coefficient as a constant characteristic of tribo-pair is not applicable. However, the two cases $\alpha < 1$ and $\alpha > 1$ drastically differ in the time variation of the linear wear rate. In what follows, to avoid misinterpretations of mild and severe wear regimes, these two cases will be termed as the sub-Archard and super-Archard regimes of wear.

However, a natural concern arises about the effect of the wearing-in period, when the initial contact pressure evolves due to the contact geometry adaptation (Argatov I and Chai, 2020). In a pin-on-disk sliding wear tests, when a wearable pin is put on an abrasive disk, there exists a small initial gap between the surfaces brought into contact (see **Figure 2A**), which influences the initial contact pressure, $p(x, t)$ at $t = 0$, in the loaded state (see **Figure 2B**).

During the wear process, because of the contact geometry adaptation, the contact pressure $p(x, t)$ evolves in time and approaches a steady-state pattern of uniform distribution over the contact zone (see **Figure 3A**). The corresponding steady-state shape of the pin surface facing the disk is characterized by the limiting gap function, $\Delta_\infty(x)$, which does not depend on the initial gap (see **Figure 3B**).
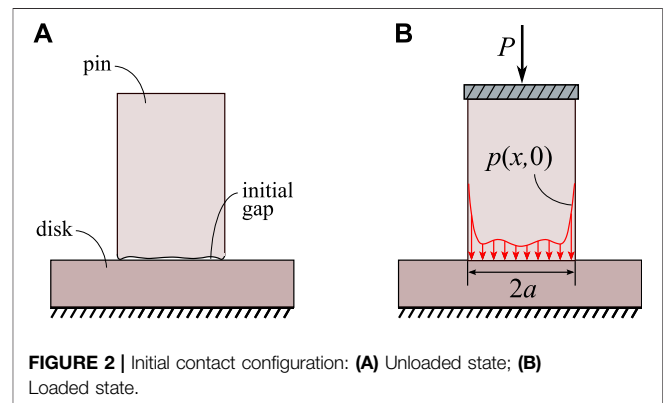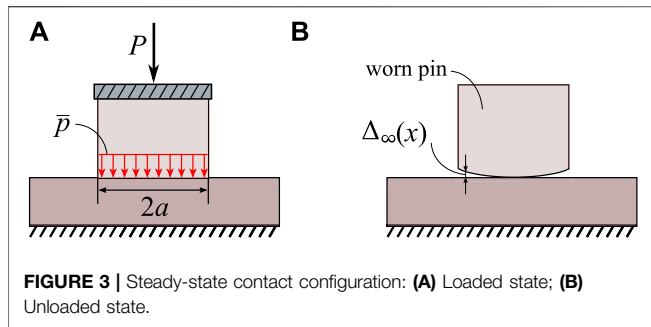

FIGURE 2 | Initial contact configuration: **(A)** Unloaded state; **(B)** Loaded state.

FIGURE 3 | Steady-state contact configuration: **(A)** Loaded state; **(B)** Unloaded state.

When the theoretical modeling framework of the contact geometry adaptation (**Figures 2**, **3**) is applied to the case of super-Archard wear, the main problem to solve in this way is a generalization of the experimentally observed non-stationary relation between $\dot{w}$ and $\bar{p} = \mathrm{const}$ for the practically important case $p \neq \mathrm{const}$. To date there are no publications on generalizations of the Rhee model (6) similar to the Archard wear equation, which follows from **Eq. 5** by replacing the mean contact pressure $\bar{p}$ with the contact pressure p. It should be stressed that the same approach does not work for the Rhee wear **Eq. 6**. To this extent, the concept of fractional time-derivative can be efficiently applied to gain insight into the contact pressure evolution in severe wear.

In recent years, the apparatus of fractional differentiation has been used in mechanics, in particular, to generalize models of diffusion (Mainardi, 1996), rough contact (Argatov, 2003), and viscoelasticity (Su et al., 2021). In the present study, we use the classical Riemann–Liouville derivative. It should be noted that a number of different approaches exist to introduce a more general notion of the derivative (Ortigueira and Machado, 2015). In particular, a so-called fractal fractional derivative (Chen et al., 2010; He, 2011) does not involve the integral convolution and represents a local operator. However, whereas fractals have emerged as a useful mathematical tool in tribological research (Ling, 1990; Borodich and Onishchenko, 1993; Borodich, 1999), the present study pioneers the use of concept of the fractional derivative in modeling wear processes.

Since the theory of wear contact problem is well established (Aleksandrov and Kovalenko, 1980; Kovalenko, 1985), we adopt the approach suggested in the constructive review (Argatov I and Chai Y. S, 2020) and take a general point of view on the description of the pin/disk contact interaction, which is applicable both in the three- and two-dimensional settings. To illustrate the contact pressure evolution, we consider a two-dimensional wear contact problem (Galin, 1976), for which a detailed analysis has been available in the literature (Aleksandrov et al., 1978; Argatov and Fadin, 2011). It is anticipated that with certain modifications the fractional time-derivative approach developed below can be applied to model other temporary-spatial severe damage processes like mechanochemical corrosion (Sedova and Pronina, 2022), wear/scratch damage (Dasari et al., 2009), and wear of metamaterials (Garland et al., 2020).

The rest of the paper is organized as follows. In **Section 2**, we introduce a fractional time-derivative (FTD) wear equation as a straightforward generalization of the Archard wear equation by replacing the time derivative in the definition of the linear wear rate with appropriate fractional derivative. The FTD wear equation when applied under constant load conditions predicts a power-law time variation for the volumetric wear similar to the Rhee wear equation. This observation explains the main goal of the present study and that is to generalize the Rhee wear equation to the case of non-constant (spatial-temporal) variations of the contact pressure. It is hypothesized that the FTD wear equation provides such a generalization. The main body of the paper is devoted to the analysis of implications drawn from the solution of the model wear contact problem formulated in **Section 3**. In particular, the existence of a quasi-steady state is identified in **Section 4** and, using the method of variables separation (**Section 5**), the evaluation of the contact pressure towards the quasi-steady state is considered in detail in **Section 6**. A particularly novel aspect—oscillatory nature—of the contact pressure variations is discussed in **Section 7**. The spatial-temporal variation of the worn profile is presented in **Section 8**. Finally, in **Section 9**, we discuss the obtained results and further generalizations and formulate the conclusions.

2 FRACTIONAL TIME-DERIVATIVE WEAR EQUATION

In its basic version due to Archard (1953), the wear equation linearly relates the wear rate at a point with coordinate x and at time t to the contact pressure as

$$\dot{w}(x,t) = \kappa p(x,t) \tag{7}$$

with some constant κ, which is related to the wear coefficient, k_w, as $\kappa = k_w v$, where v is the speed of relative sliding at the contact interface.

Under the assumption that the worn material is absent at the initial moment $t = 0$, that is $w(x, 0) = 0$, **Eq. 7** is equivalent to the relation

$$w(x,t) = \kappa \int_0^t p(x,\tau)\,\mathrm{d}\tau. \tag{8}$$

The integral form **Eq. 8** of the wear **Eq. 7** is used in formulating the wear contact problems (Galin, 1976; Argatov and Tato, 2012), as the wear depth $w(x, t)$ directly describes the evolution of the contact geometry in the direction normal to the contact interface.

Let us replace the differentiation with respect to time on the left-hand side of **Eq. 7** with a more general notion of the derivative. In particular, we make use of the fractional Riemann–Liouville derivative of order α, which will be denoted by D_t^α. In this way, we straightforwardly arrive at the equation

$$\mathrm{D}_t^\alpha w(x,t) = \kappa p(x,t), \tag{9}$$

which contains two parameters, namely, κ and α.

In the framework of fractional calculus, from **Eq. 9**, it follows that

$$w(x,t) = \frac{\kappa}{\Gamma(\alpha)} \int_0^t \frac{p(x,\tau)}{(t-\tau)^{1-\alpha}} \, d\tau, \tag{10}$$

where $\Gamma(x)$ is the gamma function.

The right-hand side of **Eq. 10** contains the Riemann–Liouville integral of order α which is consistent with the inversion of the respective fractional derivative appearing in **Eq. 9**.

It is to emphasize that by taking $\alpha = 1$ in **Eq. 10**, in view of the normalization condition $\Gamma(1) = 1$, we recover the integral form **Eq. 8** of the non-fractional Archard wear equation. Therefore, it makes sense to assume that $0 < \alpha < 2$, since **Eq. 9** generalizes Archard's **Eq. 7**, which exactly corresponds to the basic case $\alpha = 1$.

Remark 1. Observe that in the case of steady state, when $p(x,t) \equiv \bar{p}$, **Eq. 10** yields

$$w = \frac{\kappa \bar{p}}{\alpha \Gamma(\alpha)} t^\alpha, \tag{11}$$

and thus, the mean linear wear rate (evaluated from the very onset of the wear process) will be non-constant

$$\frac{w}{t} = \frac{\kappa \bar{p}}{\alpha \Gamma(\alpha)} t^{\alpha-1}.$$

In view of **Eq. 1**, from **Eq. 11**, it follows that

$$V = \frac{\kappa P}{\alpha \Gamma(\alpha)} t^\alpha. \tag{12}$$

Eq. 12 can be fitted to the experimental data shown in **Figure 1** to uniquely determine both parameters κ and α.

3 WEAR CONTACT PROBLEM FORMULATION

Following Komogortsev (1985) and Argatov and Fadin (2011), we consider a wear contact problem for an elastic solid (pin) in contact with a rigid base (disk) with a fixed zone of contact, $x \in \omega$, under a constant normal load, P. In this case, the resultant of the contact pressure $p(x,t)$ satisfies the equilibrium equation

$$\int_\omega p(x,t) \, dx = P, \quad t \geq 0. \tag{13}$$

The wear contact problem can be reduced to the following governing integral equation in the domain $x \in \omega$, $t \geq 0$, which involves two unknowns, namely, the contact pressure $p(x,t)$ and the contact displacement $\delta_0(t)$:

$$\int_\omega K(x,\xi) p(\xi,t) \, d\xi = \delta_0(t) - [\Delta_0(x) + w(x,t)]. \tag{14}$$

Here, $K(x,\xi)$ is a given surface-influence function (with x and ξ being the points of observation and integration), $\Delta_0(x)$ is a known

function of initial gap between the contacting surfaces, and $w(x,t)$ is the wear depth which is related to the contact pressure $p(x,t)$ by **Eq. 10**. It is to note that the gap function is usually subject to the centering condition $\Delta_0(0) = 0$.

We recall that the surface-influence function $K(x,\xi)$ is defined as the normal component of the corresponding vector Green's function restricted to the surface of the elastic solid, and thus, the equilibrium equations inside the elastic solid (Shillor et al., 2004) are naturally satisfied by the construction of the function $K(x,\xi)$. The reduction of the wear contact problem to the corresponding governing integral equation is well known (Argatov I and Chai Y. S, 2020) and allows applying the boundary element method (Sfantos and Aliabadi, 2006) for the direct evaluation of the contact pressures.

4 UNIFORM CONTACT PRESSURE AS A QUASI-STEADY STATE

By integrating **Eq. 10** over the contact interval and taking into account the equilibrium **Eq. 13**, we derive the following the relations for the volumetric wear loss:

$$\begin{aligned} V(t) &= \int_\omega w(x,t) \, dx \\ &= \frac{\kappa}{\Gamma(\alpha)} \int_0^t \frac{d\tau}{(t-\tau)^{1-\alpha}} \int_\omega p(x,\tau) \, dx = \frac{\kappa P t^\alpha}{\alpha \Gamma(\alpha)}. \end{aligned} \tag{15}$$

Thus, making use of **Eq. 15** and following the previously introduced method (Komogortsev, 1985), we can reduce the governing integral **Eq. 14** to the equation

$$\int_\omega \left(K(x,\xi) - \frac{1}{A} K_1(\xi) \right) [p(\xi,t) - p(\xi,0)] \, d\xi + w(x,t)$$
$$= \frac{\kappa P}{A} \frac{t^\alpha}{\alpha \Gamma(\alpha)}, \tag{16}$$

where $x \in \omega$ and $t > 0$. Moreover, A denotes the measure of the contact area defined as

$$A = \int_\omega dx, \tag{17}$$

and we have introduced the notation

$$K_1(\xi) = \int_\omega K(x,\xi) \, dx. \tag{18}$$

The form of **Eq. 16** is preferable in wear contact problems (Komogortsev, 1985), as the effect of the initial contact configuration, which is associated with the gap function $\Delta_0(x)$, is now incorporated into the initial contact pressure density $p(x,0)$.

Now, let us introduce the notation for the mean contact pressure and the point-wise deviation of the contact pressure from its mean value

$$\bar{p} = \frac{P}{A}, \qquad q(x,t) = p(x,t) - \bar{p}. \tag{19}$$

Thus, in view of (**Eqs 15**, **19**), **Eq. 16** can be rewritten in the form

$$\int_\omega K_2(x,\xi)\big[q(\xi,t) - q(\xi,0)\big]\,\mathrm{d}\xi + \frac{\kappa}{\Gamma(\alpha)}\int_0^t \frac{q(x,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau = 0,$$
(20)

where $x \in \omega$ and $t > 0$, and we have introduced the notation

$$K_2(x,\xi) = K(x,\xi) - \frac{1}{A}K_1(\xi) - \frac{1}{A}K_1(x).$$
(21)

We note that in regard to the kernel of the integral **Eq. 16**, the third term on the right-hand side of **Eq. 21** has been introduced for symmetrization (Argatov and Chai, 2019). This is convenient but does not affect the result because of the zero-mean property

$$\int_\omega q(\xi,t)\,\mathrm{d}\xi = 0, \quad t \ge 0,$$
(22)

which follows from the definition $(19)_2$ of the function $q(x,t)$ and the equation of equilibrium **Eq. 13**.

To this end, the behavior of the residual function $q(x,t)$ as time progresses is shown to be described by **Eq. 20**. By analogy with the wear contact problems based on the Archard wear equation, it can be anticipated that the deviation of the contact pressure from its mean value diminishes with time. In the case under consideration, we also find that $q(x, t) \to 0$ as $t \to \infty$, and hence the mean contact pressure $\bar{p}$ will be termed as a quasi-steady-state solution.

5 NON-DIMENSIONALIZATION AND SEPARATION OF VARIABLES

For the sake of simplicity, we assume that the pin material is isotropic and can be characterized by Young's modulus, E, and Poisson's ratio, ν. In contact mechanics, an important role is played by the reduced elastic modulus $E^* = E/(1 - \nu^2)$. Also, let a denote a characteristic size of the contact domain ω, which can be taken to be equal to the half-diameter of ω.

Following Komogortsev (1985), we take advantage of the fact that **Eq. 20**, in view of **Eq. 10**, is in a separable form (the kernel function $K_2(x, \xi)$ does not depend on the time variable t) that makes it possible to construct its solution in the form

$$q(x,t) = \sum_{n=1}^\infty b_n(t)\phi_n(x),$$
(23)

where $\phi_n(x)$ is the nth eigenfunction of the integral operator with the kernel $K_2(x, \xi)$ given by **Eq. 21**, i.e.,

$$\frac{E^*}{a}\int_\omega K_2(x,\xi)\phi_n(\xi)\,\mathrm{d}\xi = \lambda_n\phi_n(x), \quad x \in \omega.$$
(24)

We note that the non-dimensionalising factor E^*/a has been introduced on the left-hand side of **Eq. 24** to ensure that the eigenvalues $\lambda_1, \lambda_2, \ldots$ are dimensionless. It is pertinent to note

here that, in view of **Eqs 21**, **22**, the eigenfunctions possess the zero-mean property

$$\int_\omega \phi_n(\xi)\,\mathrm{d}\xi = 0 \quad (n = 1, 2, \ldots).$$
(25)

Without loss of generality, we assume that the solutions of the eigenvalue problem **Eq. 24** are normalized as

$$\frac{1}{A}\int_\omega \big[\phi_n(\xi)\big]^2\mathrm{d}\xi = 1 \quad (n = 1, 2, \ldots).$$
(26)

Thus, taking into account relations $(19)_2$ and **Eq. 23**, we can represent the contact pressure in the form

$$p(x,t) = \bar{p} + \sum_{n=1}^\infty b_n(t)\phi_n(x),$$
(27)

where $\bar{p}$ is given by $(19)_1$, and the set of coordinate functions $\phi_n(x)$ is uniquely defined by **Eqs 24**, **26**).

By setting $t = 0$, from **Eq. 27**, it follows that

$$p(x,0) = \bar{p} + \sum_{n=1}^\infty b_n(0)\phi_n(x),$$
(28)

where $p(x, 0)$ solves the integral equation of initial contact, which is obtained from **Eq. 14** by setting $t = 0$, whereas, in view of **Eq. 26**, the coefficients of the infinite sum are given by

$$b_n(0) = \frac{1}{A}\int_\omega \big[p(x,0) - \bar{p}\big]\phi_n(x)\,\mathrm{d}x.$$
(29)

In the next section, we consider the evolution of the functions $b_n(t)$, which satisfy the initial conditions (29). The corresponding equation is simply obtained by substituting the expansion (**Eq. 23**) into **Eq. 20** and utilizing **Eq. 24**.

6 EVOLUTION OF THE CONTACT PRESSURE TOWARDS THE QUASI-STEADY STATE

In view of **Eq. 24**, the series solution **Eq. 23** satisfies **Eq. 20**, if and only if

$$b_n(t) + \frac{E^*\kappa}{\lambda_n a\Gamma(\alpha)}\int_0^t \frac{b_n(\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau = b_n(0).$$
(30)

Equation 30 is classified as Abel's integral equation of the second kind (see, e.g., (Gorenflo and Mainardi, 1997; Gorenflo et al., 2014)), and its solution $b_n(t)$ is given in the form

$$b_n(t) = b_n(0)E_\alpha\left(-\frac{E^*\kappa}{\lambda_n a}t^\alpha\right)$$
(31)

in terms of the single-parameter Mittag-Leffler function of order α, defined as

$$E_\alpha(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(\alpha k + 1)},$$
(32)

FIGURE 4 | Mittag-Leffler type function.
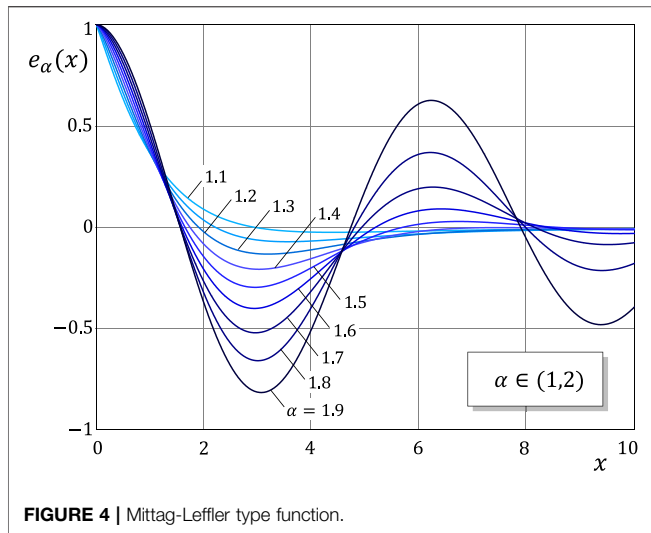
where $\Gamma(x)$ is the gamma function.

It is also convenient to consider the Mittag-Leffler type function

$$e_\alpha(x) = E_\alpha(-x^\alpha), \tag{33}$$

which for $\alpha \in (0, 1)$ is known (Mainardi, 2020) to be completely monotone on the positive real semi-axis. The variation of the function $e_\alpha(x)$ for different values of the parameter $\alpha \in (1, 2)$ in shown in **Figure 4**.

Moreover, let us introduce a characteristic time of the wearing-in process

$$\tau_{\text{in}} = \left(\frac{a}{E^\star \kappa} \right)^{1/\alpha}. \tag{34}$$

So, in view of **Eqs 33, 34**, **Eq. 31** can be represented as

$$b_n(t) = b_n(0)e_\alpha\left(\frac{t}{\lambda_n^{1/\alpha}\tau_{\text{in}}} \right) \tag{35}$$

and thus, using **Eqs 27, 35**, we find that

$$p(x, t) = \bar{p} + \sum_{n=1}^\infty b_n(0)e_\alpha\left(\frac{t}{\lambda_n^{1/\alpha}\tau_{\text{in}}} \right)\phi_n(x). \tag{36}$$

To characterize the evolution of the contact pressure $p(x, t)$ as the time variable t tends to infinity, we consider the following asymptotic formula for $e_\alpha(x)$, $\alpha \in (0, 2)$, $\alpha \neq 1$, is known for sufficiently large values of the argument (Mainardi, 2020):

$$e_\alpha(x) \sim \frac{\sin(\alpha\pi)}{\pi} \frac{\Gamma(\alpha)}{x^\alpha}, \quad x \to \infty. \tag{37}$$

Thus, in view of **Eqs 36, 37**, the contact pressure approaches the uniform distribution $\bar{p}$ at large times. It is to emphasize that this conclusion is valid for both cases $\alpha \in (0, 1)$ and $\alpha \in (1, 2)$. Finally, we note that when $\alpha = 1$, we have $E_\alpha(-x^\alpha) = \exp(-x)$, and thus we recover the known solution for the wear contact problem (Komogortsev, 1985; Argatov and Fadin, 2011), and the approach of $p(x, t)$ to $\bar{p}$ is exponentially fast.

Remark 2. It should be noted that the asymptotic **Eq. 37** reflects the decaying behavior of the Mittag-Leffler function $E_\alpha(x)$ on the negative real axis as $x \to -\infty$. This formula can be used to analyze the asymptotics of the contact pressure (**Eq. 36**), provided all eigenvalues λ_n are positive. To prove that this is the fact, we consider the elastic energy, $\mathcal{E}_0$, produced by the surface load $q(x, 0)$, which, in view of **Eqs 23, 26**, can be evaluated as follows:

$$\begin{aligned}
2\mathcal{E}_0 &= \int_\omega q(x, 0) \int_\omega K(x, \xi)q(\xi, 0)\,\mathrm{d}\xi\mathrm{d}x \\
&= \sum_{m=1}^\infty b_m(0) \int_\omega \phi_n(x)\left\{ \frac{E^\star}{a} \sum_{n=1}^\infty \lambda_n b_n(0)\phi_n(x) + \text{const} \right\}\mathrm{d}x \\
&= \frac{E^\star A}{a} \sum_{n=1}^\infty \lambda_n[b_n(0)]^2.
\end{aligned}$$
$$\tag{38}$$

Thus, since the elastic energy $\mathcal{E}_0$ is positive for any nontrivial set of coefficients $b_n(0)$, $n = 1, 2, \ldots$, from **Eq. 38** it follows that $\lambda_n > 0$, $n = 1, 2, \ldots$.

7 OSCILLATORY NATURE OF THE CONTACT PRESSURE VARIATION IN SUPER-ARCHARD WEAR

Equation 37 provides information about the limiting evolution of the contact pressure $p(x, t)$ as the time variable t tends to infinity. During the wearing-in period, the evolution of $p(x, t)$ strongly depends on the variation of the Mittag-Leffler type function $e_\alpha(x)$, which changes its behavior from completely monotonic for $\alpha \in (0, 1)$ to decaying oscillatory for $\alpha \in (1, 2)$ (see **Figure 4**).

To illustrate the contact pressure variation during the wearing-in period, we consider the example of two-dimensional contact for an elastic layer and a wearable indenter, which establishes a fixed contact interval $x \in (-a, a)$ of a relatively small half-width, a, compared to the layer thickness. The corresponding wear contact problem in the case of the Archard type wear equation was studied by Aleksandrov et al. (1978), using the Galerkin method for evaluating the eigenfunctions. We refer to the paper by Argatov and Fadin (2011) for more details.

Figure 5 shows the time-variation of the relative contact pressure $p(x, t)/\bar{p}$ at the center of the contact zone, $x = 0$, for the cases of sub-Archard ($\alpha = 0.812$) and super-Archard ($\alpha = 1.595$) wear. It is clear that both curves start from the same point at $t = 0$ (because of the same initial condition) and tends to a horizontal line of unit value as $t \to \infty$ (because of the same quasi-steady state). A striking difference between them occurs during the wearing-in period that lasts about a few times as much as the characteristic time τ_{in}, where the super-Archard curve shows a marked decaying oscillatory behavior.

In the example of contact geometry under consideration, which is taken to be almost flat (with a small initial gap between the contacting surfaces), the initial contact pressure shows a strong stress concentration in a periphery region of contact (see curves $\tau = 0$ in **Figures 6, 7**). That is why, in the initial period of time, the wear rate $\mathrm{D}_t^\alpha w(x, t)$ will be much
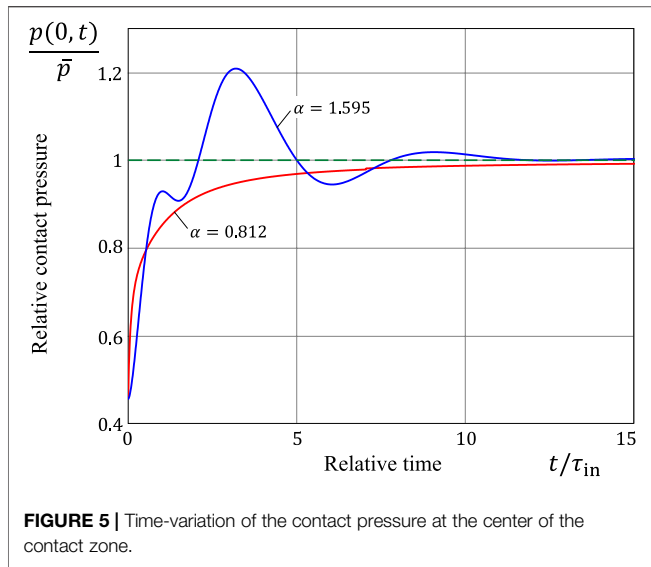
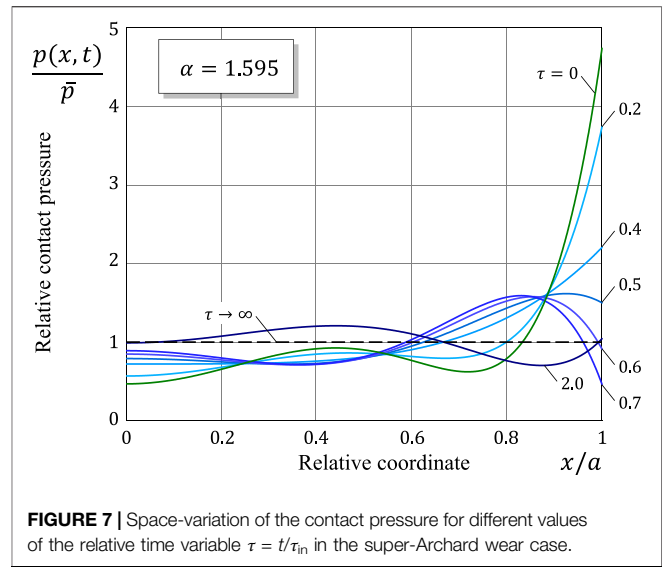**FIGURE 5 |** Time-variation of the contact pressure at the center of the contact zone.

FIGURE 7 | Space-variation of the contact pressure for different values of the relative time variable $\tau = t/\tau_{in}$ in the super-Archard wear case.
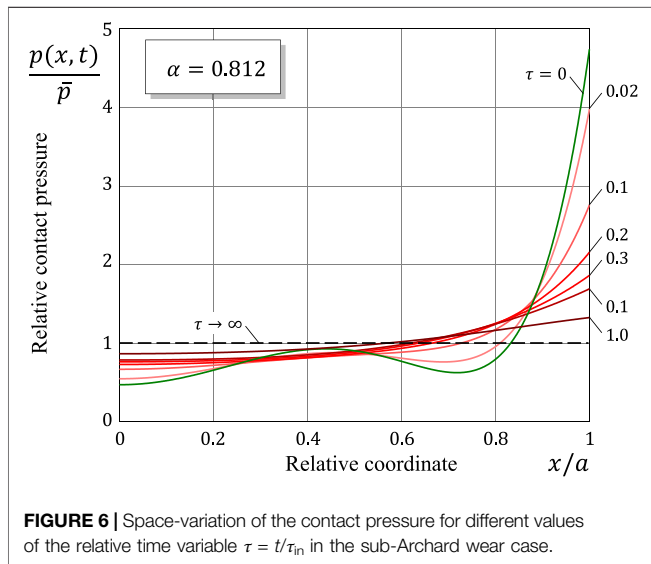

FIGURE 6 | Space-variation of the contact pressure for different values of the relative time variable $\tau = t/\tau_{in}$ in the sub-Archard wear case.
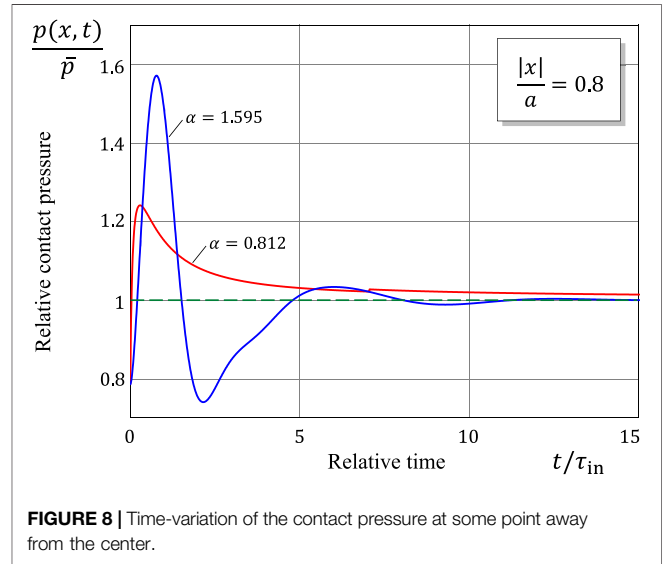

FIGURE 8 | Time-variation of the contact pressure at some point away from the center.

higher near the contact contour, and, as a result of the contact geometry adaptation, the evolution of the contact pressure in the periphery region may be non-monotonic even in the sub-Archard wear case (see **Figure 8**). However, as it is seen from both **Figures 5**, **8**, the oscillatory type of the contact pressure evolution is a characteristic feature of the case of super-Archard wear only.

The discrete-time progression of the spatial variation of the contact pressure is shown in **Figures 6**, **7** in the cases of sub-Archard and super-Archard wear, respectively. Observe that a drastic reduction in the contact stress concentration occurs during the initial time interval of duration comparable with the characteristic time τ_{in}. From the comparison of **Figures 6**, **7**, it can be concluded that the wearing-in period in super-Archard wear lasts much longer than in the case of sub-Archard wear.

8 EVOLUTION OF THE WORN PROFILE

Due to the wear process, the contact geometry gradually changes, while the contact pressure approaches a uniform distribution of the applied load over the contact area. The initial contact geometry is characterized by the initial gap function $\Delta_0(x)$, which is subject to the centering condition $\Delta_0(0) = 0$. The current gap function, $\Delta(x, t)$, is defined as follows (Argatov I. I and Chai Y. S, 2020):

$$\Delta(x,t) = \int_\omega [K(0,\xi) - K(x,\xi)] p(\xi,t) \, d\xi. \qquad (39)$$

When the contact pressure density $p(x, t)$ is given by the eigenfunction expansion method in the form **Eq. 36**, in view of **Eqs 18**, **24**, we will have
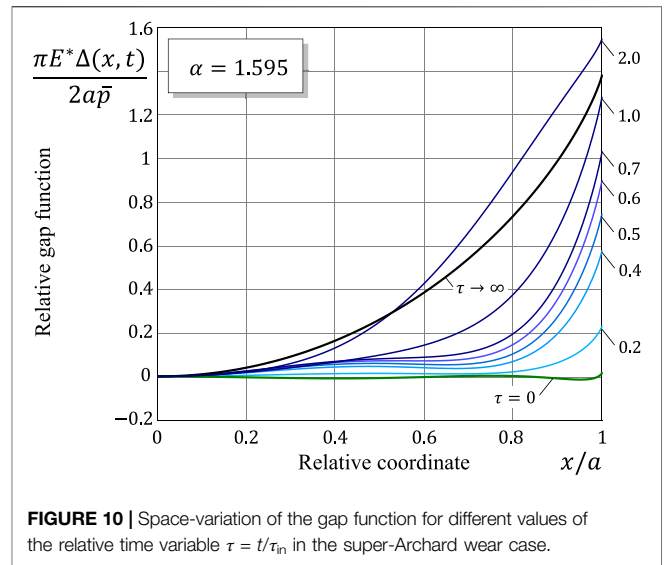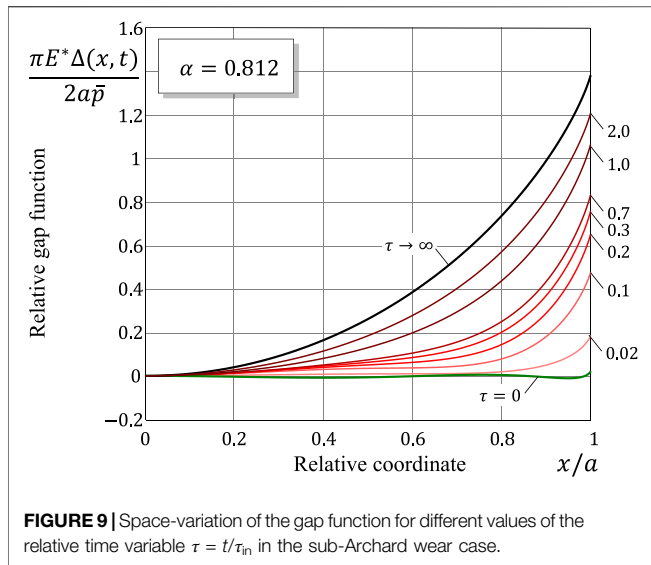
FIGURE 9 | Space-variation of the gap function for different values of the relative time variable $\tau = t/\tau_{\text{in}}$ in the sub-Archard wear case.

FIGURE 10 | Space-variation of the gap function for different values of the relative time variable $\tau = t/\tau_{\text{in}}$ in the super-Archard wear case.

$$\int_\omega K(x,\xi)p(\xi,t)\,\mathrm{d}\xi = \bar{p}K_1(x) + \frac{a}{E^\star}\sum_{n=1}^\infty \lambda_n b_n(t)\phi_n(x)$$
$$+ \sum_{n=1}^\infty \frac{b_n(t)}{A}\int_\omega K_1(\xi)\phi_n(\xi)\,\mathrm{d}\xi, \tag{40}$$

where we have used the zero-mean property (**Eq. 22**) of the eigenfunctions $\phi_n(x)$, $n = 1, 2, \ldots$, and the symmetry property of the surface influence function, that is $K(x, \xi) = K(\xi, x)$.

Since $b_n(t) \to 0$ as $t \to \infty$, the limiting gap function is defined only by the first term on the right-hand side of **Eq. 40**, and thus, we obtain

$$\Delta_\infty(x) = \bar{p}[K_1(0) - K_1(x)]. \tag{41}$$

So, from **Eqs 39–41**, it follows that

$$\Delta(x,t) = \Delta_\infty(x) + \frac{a}{E^\star}\sum_{n=1}^\infty \lambda_n b_n(t)\big[\phi_n(0) - \phi_n(x)\big]. \tag{42}$$

The evolution of the relative gap function is illustrated in **Figures 9**, **10**. It is of interest to observe a non-monotonic progression of the worn profile $\Delta(x, t)$ towards the limiting shape $\Delta_\infty(x)$ in the case of super-Archard wear.

Figure 11 presents the evolution of wear depth $w(0, t)$, which corresponds to the evolution of the contact pressure at the center of the contact zone shown in **Figure 5**. It is to note that, in view of **Eq. 34**, the wear equation transforms as

$$w(x,t) = \frac{a}{E^\star\Gamma(\alpha)}\int_0^{t/\tau_{\text{in}}}\left(\frac{t}{\tau_{\text{in}}} - \bar{\tau}\right)^{\alpha-1}p(x,\bar{\tau})\,\mathrm{d}\bar{\tau},$$

where $\bar{\tau}$ is a dimensionless integration time-variable. The above formula explains the normalization used in **Figures 11–13**.

Interpretation of the evolution of the relative gap function shown in **Figures 9**, **10** will be simplified if we consider the evolution of the pin shape described by the function $D(x, t) = \Delta_0(x) + w(x, t)$ (see **Figures 12**, **13**). It is of interest to observe that
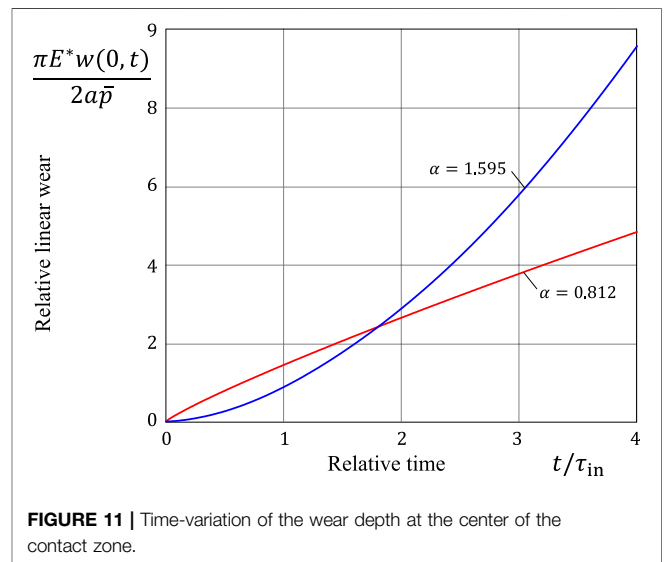

FIGURE 11 | Time-variation of the wear depth at the center of the contact zone.

in the case of super-Archard wear, the evolution of the worn profile is also found to be slightly oscillatory (see **Figures 10**, **13**), as it could be predicted based on **Eq. 39**, which defines the current gap function, and the oscillatory evolution of the contact pressure. It should be emphasized that, though the worn profile progresses non-monotonically in time, this does not mean that locally (in spatial sense) there could exist profile growth, because the fractional wear **Eq. 9** predicts a positive wear depth $w(x, t)$ for any positive contact pressure history $p(x, \tau)$, $\tau \in (0, t)$.

In the case of sub-Archard wear, when $0 < \alpha < 1$, the duration of wearing-in process, when the contact pressure approaches the mean value $\bar{p}$ over the entire contact area, is proportional to the characteristic time parameter τ_{in}, introduced by **Eq. 34**. In the case of super-Archard wear, as it is seen from **Figure 14**, the duration of the wearing-in period, T_{in}, strongly depends on the
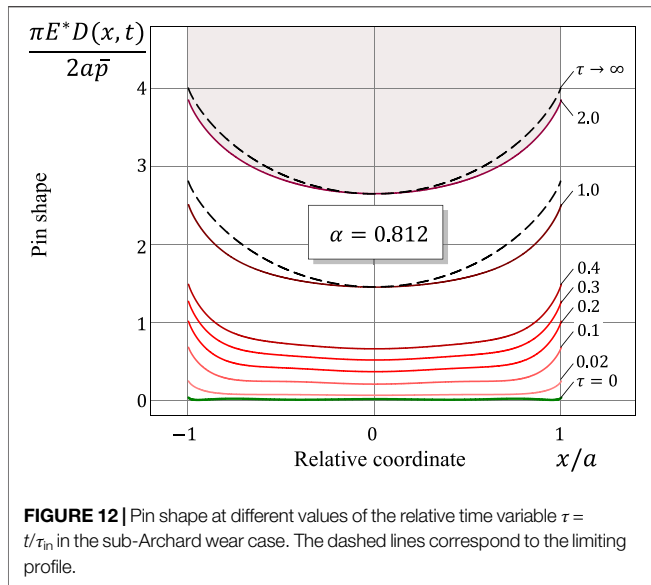
FIGURE 12 | Pin shape at different values of the relative time variable $\tau = t/\tau_{\mathrm{in}}$ in the sub-Archard wear case. The dashed lines correspond to the limiting profile.
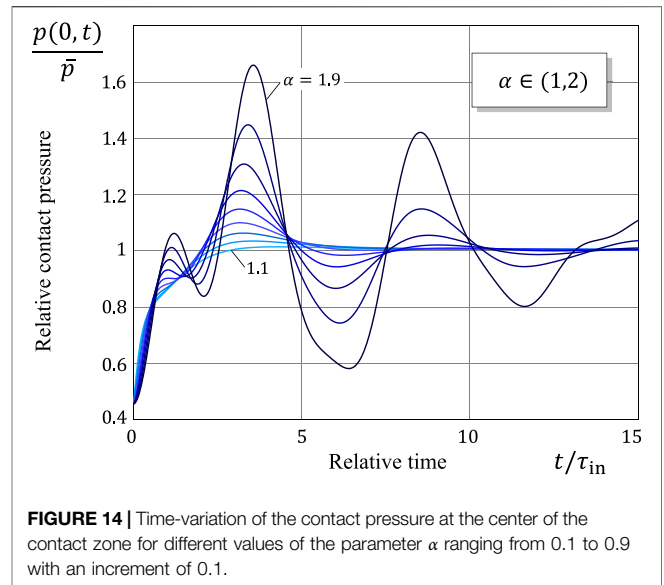

FIGURE 14 | Time-variation of the contact pressure at the center of the contact zone for different values of the parameter α ranging from 0.1 to 0.9 with an increment of 0.1.
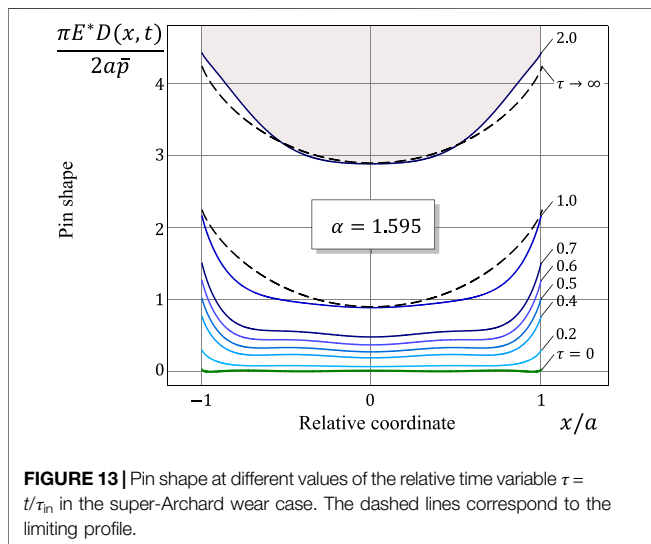

FIGURE 13 | Pin shape at different values of the relative time variable $\tau = t/\tau_{\mathrm{in}}$ in the super-Archard wear case. The dashed lines correspond to the limiting profile.

value of $\alpha \in (1, 2)$, because, if it increases, the amplitude of the decaying oscillations increases as well.

9 DISCUSSION AND CONCLUSION

First of all, we note that the example of two-dimensional contact has been used for illustrative purposes. A more realistic information about the eigenvalues λ_n, which depend on the global geometry of the wearable pin, can be obtained by means of finite element simulations (Liu et al., 2014).

We would like to emphasize that the numerical results presented in **Figures 5–13** were obtained based on the exact solution of the model two-dimensional wear contact problem for an elastic layer (Aleksandrov et al., 1978) with a controllable error determined by the approximate evaluation of the eigenvalues λ_n.

In the analysis of real pin-on-disk experiments, the application of FEM-based numerical methods is required (Liu et al., 2014). However, the oscillating nature of the contact pressure variation observed in **Figure 5** is caused by the properties of solutions (see **Figure 4**) of Abel's integral **Eq. 30**, which will remain the same in the three-dimensional contact problems. To be more precise, the contact geometry affects the eigenvalue problem **Eq. 24**, which depends on the surface-influence function (see **Eq. 21**), whereas the evolution **Eq. 30** is fully determined by the adopted fractional time-derivative (FTD) model for wear. This means that while FEM simulations cannot serve for the model verification, spatial-temporal variation of the contact geometry and the contact pressures, if observed experimentally, would support the introduced FDT wear equation. At the same time, though verification experimental studies fall outside the scope of this theoretical work, the simple analytical model presented above will definitely aid in planning such experiments.
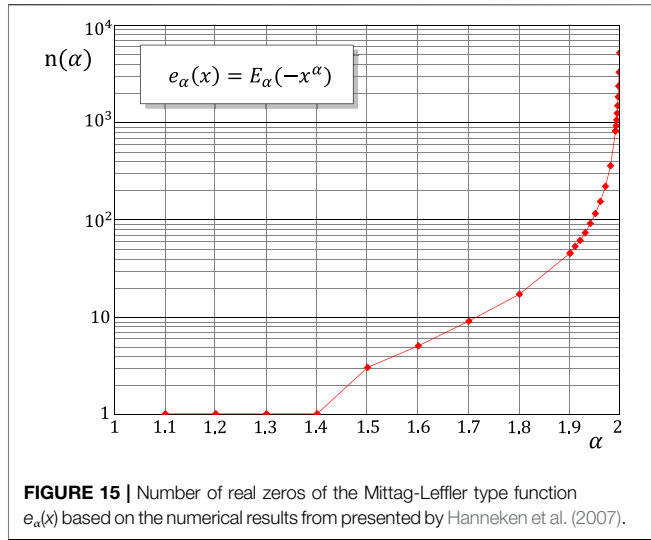
Based on the presented analysis, it becomes clear that the temporary evolution of wear process, which is governed by the fractional time-derivative model (10), is determined by the properties of the Mittag-Leffler type function $e_\alpha(x)$, which, in turn, strongly depend on the value of the index α. In particular, for $\alpha \in (1, 2)$, the number of zeros, $n(\alpha)$, of the function $e_\alpha(x)$, $x \in (0, \infty)$, which determines the number of oscillations, exponentially increases as α approaches 2 (see **Figure 15**).

Observe that the FTD wear **Eq. 9** is linear, and under the constant load condition (**Eq. 13**) it implies the following relation between the contact load and the worn volume (see **Eq. 12**):

$$V = kP t^\alpha. \tag{43}$$

Here, $k = \kappa/\alpha\Gamma(\alpha)$ is a constant coefficient. It is to emphasize again that **Eq. 43** is in complete agreement with the Rhee **Eq. 6**, which in this special case ($\gamma = 1$) takes the form

$$V = K_{\mathrm{w}} P v^\beta t^\alpha. \tag{44}$$

FIGURE 15 | Number of real zeros of the Mittag-Leffler type function $e_\alpha(x)$ based on the numerical results from presented by Hanneken et al. (2007).

Thus, by comparing **Eqs 43**, **44**, we can write that $k = K_w v^\beta$, and therefore, we obtain $\kappa = \alpha\Gamma(\alpha)K_w v^\beta$. In other words, under the condition of constant sliding velocity, the wear coefficient κ in the FTD wear **Eq. 9** may be regarded to be parametrically dependent on the sliding velocity v. The effect of surface roughness of the hard counterpart (disk), along which a wearable solid (pin) slides, is incorporated in the value of the coefficient κ as well.

It is of practical interest to have a closer look at the quality of power-law approximations of the experimental data for the volumetric wear loss under constant load (Viswanath and Bellow, 1995), which are presented in **Figure 1**. In particular, it is instructive to consider the variation of the absolute error as shown in **Figure 16**, from where it is seen that in the case of super-Archard wear ($\alpha > 1$), the experimental data exhibits pronounced deviations from the power-law curve. This fact implies that the fractional wear rate $D_t^\alpha w(x, t)$ may depend nonlinearly on the contact pressure. In this way, we can generalize **Eq. 9** as

$$D_t^\alpha w(x, t) = \kappa\big[p(x, t)\big]^\gamma \qquad (45)$$

by introducing an additional fitting constant γ.

Correspondingly, in light of **Eq. 45**, **Eq. 10** can be generalized as follows:

$$w(x, t) = \frac{\kappa}{\Gamma(\alpha)} \int\limits_0^t \frac{\big[p(x, \tau)\big]^\gamma}{(t - \tau)^{1-\alpha}}\, d\tau. \qquad (46)$$

As a consequence of the nonlinear wear **Eq. 46**, the volumetric wear loss $V(t)$ will no longer be predicted by the power-law **Eq. 15**. Indeed, by integrating **Eq. 46**, we arrive at the equation

$$V(t) = \frac{\kappa}{\Gamma(\alpha)} \int\limits_0^t \frac{d\tau}{(t - \tau)^{1-\alpha}} \int\limits_\omega \big[p(x, \tau)\big]^\gamma dx, \qquad (47)$$

where the inner integral over the contact domain will no longer be constant under a constant load, but may become oscillatory during the wear process.

It is to note that the nonlinearity in the fractional wear **Eqs 45–47** has been introduced in the spirit of the Archard–Kragelsky model for the linear wear rate (Kragelsky, 1965), and thus, they can be applied under the assumption of constant speed of relative sliding at the contact interface with the wear coefficient κ being nonlinearly dependent on the sliding speed.
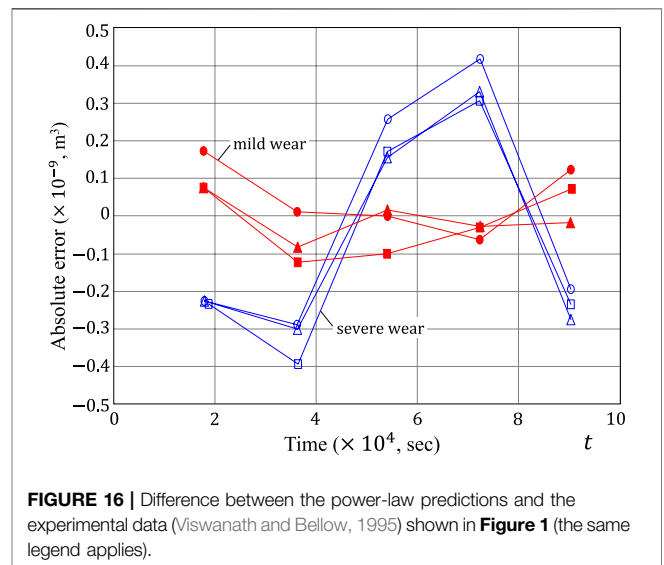
It should be emphasized that though the simple example of two-dimensional wear contact problem has been employed in the analysis above, the main findings about the decaying oscillatory nature of the approach to the quasi-steady state will hold true for the corresponding three-dimensional contact configurations, since this feature is rooted in the fractional time description of the wear process.

Observe (Soldatenkov, 2010) that the wear **Eq. 10** is merely a special case of the more general type

$$w(x, t) = \int\limits_0^t K_1(\tau)K_2(t - \tau)p(x, \tau)\, d\tau, \qquad (48)$$

where $K_1(t)$ and $K_2(t)$ are known functions. In particular, **Eq. 48** with $K_1(t) \equiv 1$ and $K_2(t) = (\kappa/\Gamma(\alpha))t^{\alpha-1}$ simply coincides with **Eq. 10**. On the other side, it is of interest to observe that **Eq. 48** with $K_1(t) = (\kappa/\Gamma(\alpha))t^{\alpha-1}$ and $K_2(t) \equiv 1$ in the case of constant loading also leads to **Eq. 11**. Moreover, it can be shown that **Eq. 9** with D_t^α being interpreted as the fractal derivative of the order α is also consistent with Rhee's **Eq. 6**, $\gamma = 1$, which served as a basis for introducing the fractional wear **Eq. 9**.

Thus, there are possible different ways of generalizing the Rhee wear **Eq. 6** to the case of non-constant (spatial-temporal) variations of the contact pressure, which have the same power-law time behavior in the steady state. While the present study gives no physical motivation for the fractional wear model (9), the hypotheses and implications formulated above can be verified experimentally without much difficulty. Also, a separate study is required to identify key differences and similarities between the wear equations discussed in the previous paragraph.

FIGURE 16 | Difference between the power-law predictions and the experimental data (Viswanath and Bellow, 1995) shown in **Figure 1** (the same legend applies).

The systematic analysis of the physical variables that influence the wear process of polymers (Viswanath and Bellow, 1995) is showing an intensive effect of temperature and thus deserves a comment that the thermo-elasticity framework (Yevtushenko and Pyryev, 1999) should be applied for modeling contact deformations once the temperature exceeds a certain threshold.

To conclude, the fractional time-derivative wear rate is introduced to describe a power-law time-growth of the volumetric wear loss under constant loading conditions, which is encountered in wear testing of polymers. The newly introduced fractional time-derivative wear equation allows to extend the experimentally observed dependence to the case of a variable contact pressure. The cases of sub-Archard and super-Archard wear are distinguished based on the value of the time exponent $\alpha \in (0, 2)$, which describes decreasing ($\alpha < 1$, "mild" wear) or increasing ($\alpha > 1$, "severe" wear) variation of the mean wear rate.

A striking implication of the developed wear contact model is the prediction of a decaying oscillatory behavior for the contact pressure variation in time during the wearing-in period, when it approaches a uniform distribution over the contact area. The main theoretical finding of the paper is supported by the analysis of the power-law approximations for the volumetric wear loss obtained under constant applied loads. The developed fractional time-derivative approach can be applied to modeling other temporary-spatial damage processes that maintain a power-law growth of the accumulated damage under constant loading conditions.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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