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On the Design of Superstable Prestressed Frameworks

Scott D. Kelly [1], Andrea Micheletti [2] and Alessandro Tiero [2]*

[1] Department of Mechanical Engineering and Engineering Sciences, University of North Carolina at Charlotte, Charlotte, NC, United States, [2] Department of Civil and Computer Science Engineering, University of Rome "Tor Vergata," Rome, Italy

The strength and stiffness of prestressed lattices, and their mechanical behavior, depend strongly on the underlying graph and the nodal conformation geometry. A special class of structures is that of superstable frameworks, that is, prestressed frameworks which are stable independently of material properties and level of prestress. After reviewing the main related notions and results in rigidity theory, we exploit the characterization of superstability for generic configurations to establish a construction for superstable systems on a given number of nodes generically placed in two or three dimensions.

Keywords: universal rigidity, superstability, generic configuration, Grünbaum polygon, tensegrity frameworks

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****Correspondence:***
Andrea Micheletti
micheletti@ing.uniroma2.it

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1. INTRODUCTION

The mechanical behavior of latticed structures is greatly affected by the properties of the underlying graph and the geometrical conformation of nodal positions. The simplest and most commonly adopted model is that of bar-and-joint framework, i.e., a graph together with a spatial placement of its vertices, or nodes. In **Figure 1** different types of framework are shown (cf. e.g., Gortler et al., 2010; Connelly, 2013). If a set of edge lengths is compatible with a finite motion, then the framework is *flexible* (**Figure 1A**). When a framework is *rigid* at a given configuration, there is no other configuration close to it with same edge lengths (**Figure 1B**). If the edge lengths are compatible with just one configuration in d dimensions, then the framework is *globally rigid* (**Figure 1C**). If that configuration is unique even if embedded in a higher dimensional space, then the framework is *universally rigid* (**Figure 1D**).

A distinction can be made between generic configurations, those for which the nodal coordinates are algebraically independent, and nongeneric ones, which do not satisfy this condition (cf. e.g., Gortler et al., 2010; Connelly, 2013). As we will see in the next section, in case of a generic configuration on n nodes, the minimum number of edges required for universal rigidity is equal to $e = dn - d(d+1)/2 + 1$, that is, $e = 2n - 2$ for $d = 2$ and $e = 3n - 5$ for $d = 3$.

Universal rigidity is strictly connected to superstability, the property of prestressed frameworks for which they are stable independently of the constitutive material properties and level of prestress. (Connelly, 1982; Zhang and Ohsaki, 2007; Micheletti, 2013). In particular, in the generic case, universal rigidity and superstability are equivalent (Connelly, 1982; Gortler and Thurston, 2014).

After reviewing the relevant notions and results in rigidity theory, we provide a specific construction to determine minimal generically universally rigid frameworks, i.e., generically universally rigid frameworks with the minimum number of edges, for an assigned set of nodal positions in two or three dimensions. The frameworks obtained with such a construction can be seen as a generalization of Grünbaum polygons (**Figure 2**) (Grünbaum and Shephard, 1978; Roth and Whiteley, 1981). In case of nongeneric configurations, it is possible to decrease the number of

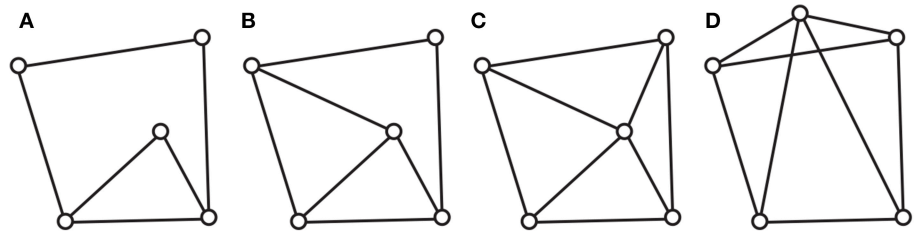

FIGURE 1 | Frameworks in two-dimensions belonging to different rigidity classes: flexible **(A)**, rigid **(B)**, globally rigid **(C)**, and universally rigid **(D)**.

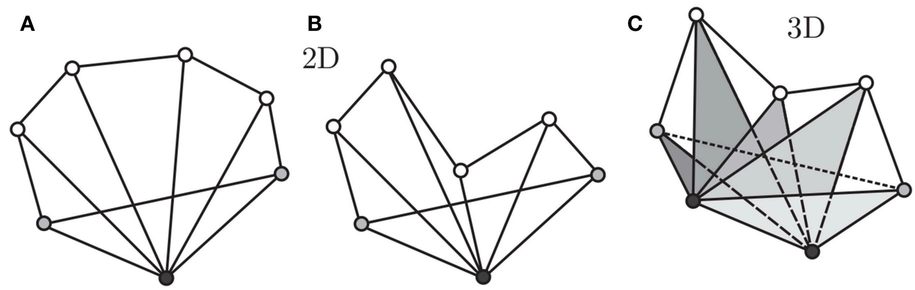

FIGURE 2 | A Grünbaum polygon **(A)**, and its nonconvex generalizations in two **(B)** and three dimensions **(C)**. Center nodes and neighboring nodes (as defined in Section 3.1) are depicted in black and gray, respectively.

edges further (**Figure 3**); however, limited theoretical results are available for nongeneric systems, and we plan to consider this case in a future study. Our main results (Theorems 9 and 10) are presented in Section 3. Basic definitions and supporting results are reviewed in Section 2.

2. BASIC RIGIDITY DEFINITIONS AND RESULTS

Different classes of rigidity can be defined for frameworks, and in each of them the generic and nongeneric cases can be distinguished. Prestressed (or prestressable) systems naturally comes into play when characterizing globally rigid and universally rigid frameworks. In writing the present section, we borrowed some of the notation, terminology, and examples from the cited literature and in particular from Connelly (2013) and Gortler et al. (2010).

2.1. Rigidity Classes

Let E^d be the d-dimensional Euclidean space and V^d the associated vector space. A *graph* $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is given by a set $\mathcal{N}$ of *n nodes* together with a set $\mathcal{E}$ of *e edges* connecting pairs of‘ nodes. The edge connecting nodes $i, j \in \mathcal{N}$ is denoted by $ij \in \mathcal{E}$. Graphs are finite and undirected, without loops or multiple edges. A *configuration* in E^d for the graph $\mathcal{G}$ is an assignment of a position vector $p_i \in V^d$ to each node $i \in \mathcal{N}$, so that a corresponding point $P_i \in E^d$ is determined by its position with respect to a chosen origin $O \in E^d$. We denote by $p \in V^{nd}$ the vector grouping all nodal position vectors. A *framework* is given by a graph together with a configuration, that is, $\mathcal{F} = (\mathcal{G}, p)$ is a framework with graph $\mathcal{G}$ and configuration p.

Associated to a framework is the set $\mathcal{L}(\mathcal{G}, p)$ of the half-squared edge-lengths,

$$\mathcal{L}(\mathcal{G}, p) = \{\lambda_{ij} \in \mathbb{R}^+, ij \in \mathcal{E} : \lambda_{ij} = \frac{1}{2}|p_i - p_j|^2\}.$$

A configuration q is *admissible* for $(\mathcal{G}, p)$ if $\mathcal{L}(G, q) = \mathcal{L}(G, p)$. Two configurations p and q are *congruent*, and we write $p \equiv q$, if $|p_i - p_j| = |q_i - q_j|$ for every choice of i and j in $\mathcal{N}$. Equivalently, two configurations are congruent if they differ by an isometry of E^d, i.e., a composition of translations, rotations and reflections. A framework $(\mathcal{G}, p)$ is *rigid* if there is an $\varepsilon > 0$ such that any other admissible configuration q for which $|p-q| < \varepsilon$ is congruent to p.

The jacobian of $\mathcal{L}(\mathcal{G}, p)$, which is an e-by-dn matrix, is the *rigidity matrix*, R. A framework is *infinitesimally rigid* if the rank of R is equal to $nd - d(d+1)/2$, or equivalently, if the only solutions to the system of equations $R\dot{p} = 0$ are *rigid velocities*, i.e., nodal velocities in a rigid motion (**Figure 4**). For example, the equation corresponding to the edge ij in this system is given by

$$(p_i - p_j) \cdot (\dot{p}_i - \dot{p}_j) = 0$$

which is obtained by setting equal to zero the first derivative of λ_{ij}. The solutions of $R\dot{p} = 0$ which are not rigid velocities are called *mechanisms*.

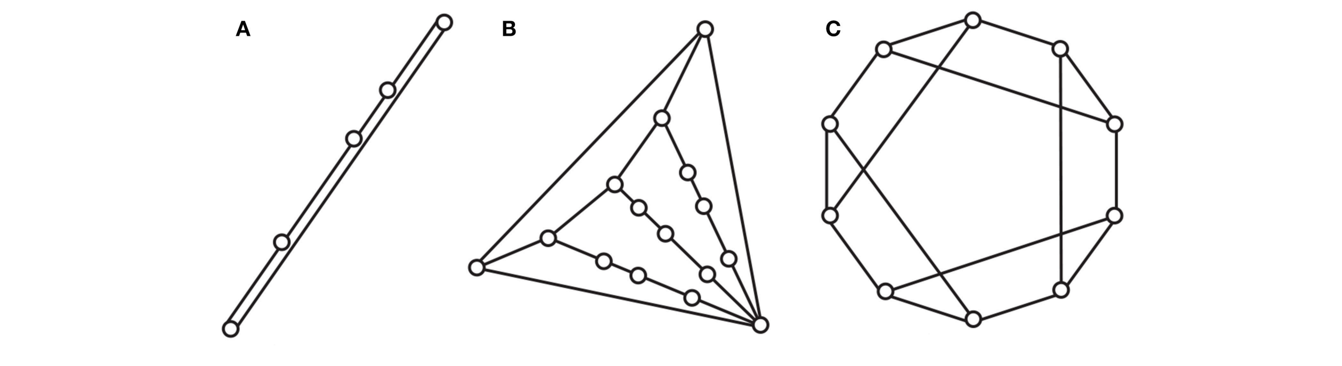

FIGURE 3 | Examples of nongeneric minimal systems. All these frameworks have less than $2n - 2$ edges and are universally rigid in two dimensions. In **(A)** all nodes need to stay aligned; **(B)** a *cable net* is attached to a *triangle*; **(C)** a system which we call *Snelson polygon*.

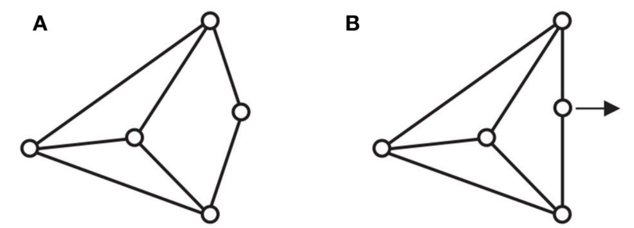

FIGURE 4 | An infinitesimally rigid framework **(A)** and a rigid framework which is not infinitesimally rigid (b). The horizontal nodal velocity vector shown in **(B)**, when all the other nodes are fixed, cause null changes in length but it is not a rigid velocity.

A framework $(\mathcal{G}, p)$ is *globally rigid* if any admissible configuration q is congruent to p (**Figure 5**). A framework is *universally rigid* if it is globally rigid in all dimensions (**Figure 6**). Universal rigidity implies global rigidity, which implies infinitesimal rigidity, which implies rigidity.

2.2. Generic Rigidity

A configuration is *generic* if the coordinates in p are algebraically independent over the integers, i.e., if the nodal coordinates do not satisfy any nontrivial polynomial equation with integer coefficients. Intuitively, if the configuration is nongeneric, then it is special in some way. For example the framework in **Figure 7A** is globally rigid, while the one in **Figure 7B**, where three nodes are aligned on a diagonal, is not. Another example is given in **Figure 4**, with configurations **Figures 4A,B** being respectively generic and nongeneric.

A framework $(\mathcal{G}, p)$ is *generically rigid* if it is rigid and p is generic. Rigidity is a generic property, i.e., it is a property of the graph, not the configuration: if a framework is rigid at a generic configuration then it is rigid at every other generic configuration. Moreover, at generic configurations, rigidity and infinitesimal rigidity are equivalent.

The minimum number of edges necessary for generic rigidity are $2n - 3$ in 2D and $3n - 6$ in 3D. Intuitively, in 2D, we can start with an edge connecting two nodes, then iteratively adding one node connected to the other nodes by two noncollinear edges. In 3D, we can start with a nondegenerate *triangle* (three vertices and three edges), then iteratively adding one *tripod*, i.e., a node connected to the other nodes by three noncoplanar edges. These constructions constitute particular *Henneberg sequences* (Eren et al., 2004b): sequences of operations which preserve minimal generic rigidity.

2.3. Stresses and Tensegrities

The characterization of global rigidity has been given in the literature in terms of *stress*. A stress ω is an assignment of a real number ω_{ij} to each edge ij of the framework. A *selfstress* for $(\mathcal{G}, p)$ is a stress satisfying at every node i the nodal equilibrium equation

$$\sum_j \omega_{ij}(p_j - p_i) = 0\,,$$

where the summation is extended to every node j connected to node i by an edge. The equilibrium equations can be written in matrix form as

$$A\omega = 0\,,$$

with A the dn-by-e *equilibrium matrix*. Selfstresses belong to the nullspace of the equilibrium matrix. One classic result is that $A = R^T$, so that the number of independent selfstresses s and mechanisms m are related to n and e by the following rule

$$dn - d(d+1)/2 - e = m - s\,, \tag{1}$$

where $d(d+1)/2$ is the number of independent rigid motions in E^d. This rule follows from the orthogonality of the fundamental subspaces (nullspace and image of the transposed) of R and A.

A fundamental object is the *stress matrix*, Ω, a n-by-n matrix whose entries are defined as follows:

$$\Omega_{ij} = \begin{cases} 0\,, & i \neq j, ij \notin \mathcal{E} \\ -\omega_{ij}\,, & i \neq j, ij \in \mathcal{E} \\ \sum_h \omega_{ih}\,, & i = j \end{cases}$$

where ω is a selfstress. The stress matrix is equal to the weighted Laplacian of the graph, with weights given by the selfstress values

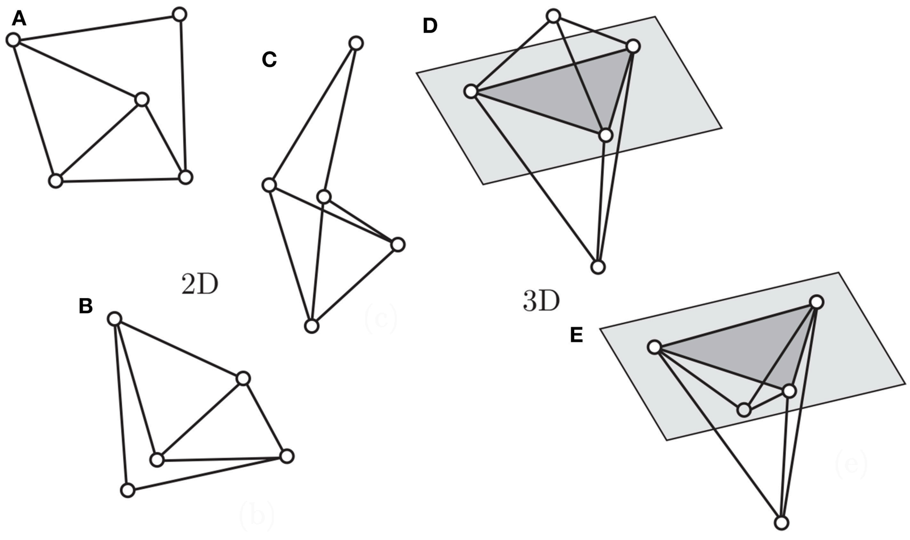

FIGURE 5 | The rigid frameworks **(A)**, in two dimensions, and **(D)**, in three dimensions, are not globally rigid: there exist more than one non-congruent admissible configurations, for example, **(B,C)** for **(A)**, and **(E)** for **(D)**.

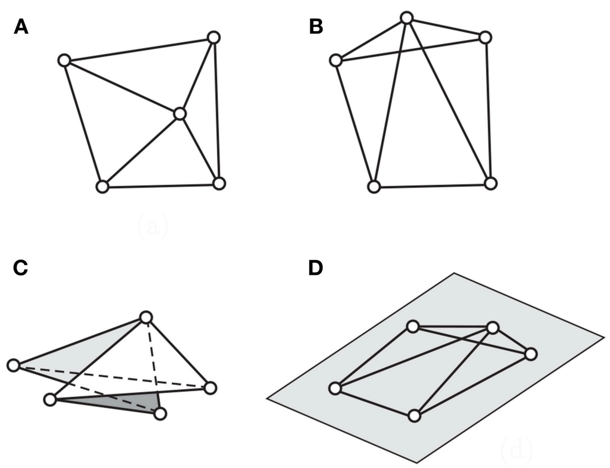

FIGURE 6 | Globally rigid **(A)** and universally rigid **(B)** frameworks. When these are considered in three dimensions, the framework **(A)** can flex out of plane, as in **(C)**, the one in **(B)** can only stay in a plane **(D)**.

on the edges. Notice that the weights can be either positive or negative, so that classic results on positively-weighted Laplacians do not apply.

A useful characterization has been given as follows (Connelly, 1982, 2013). A framework in E^d with the affine span of $p_1, \ldots, p_n$ being all E^d and a nonzero selfstress is *superstable* if the following conditions hold:

1. Ω is positive semidefinite;
2. Ω has rank $n - d - 1$;
3. there are no affine admissible motions.

Theorem 1. *Connelly (1982), see also Connelly (2013) A superstable framework is universally rigid.*

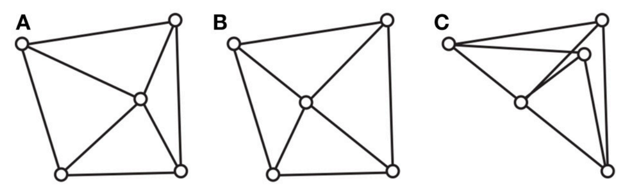

FIGURE 7 | globally rigid framework **(A)**, which loses this property in the nongeneric configuration **(B)**, where three nodes are aligned on a diagonal. For the latter framework the configuration **(C)** is also possible.

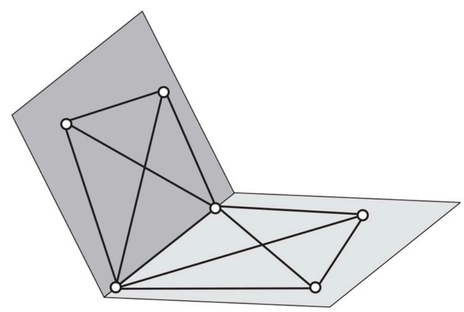

FIGURE 8 | This framework for $d = 3$ satisfies conditions (1) and (2), but not condition (3) of Theorem 1, there is an affine motion changing the angle between the highlighted planes, preserving all bar lengths.

Condition (1) implies that if there is another admissible configuration, then it has the same selfstress; condition (2) then implies that this other configuration is an affine image of the original one, and condition (3) implies that the affine image is actually congruent to the original configuration (cf. **Figure 8**). A particular class of superstable frameworks is that of cablenets, i.e., externally anchored frameworks where each edge has positive stress (**Figure 9A**).

Now we turn to generic configurations. A *simplex* in E^d is a framework on the complete graph on $d+1$ nodes, e.g., triangles in E^2 or tetrahedra in E^3. Simplices (and all frameworks on complete graphs) are universally rigid by definition, since admissible configurations must be congruent to each other. Every generic globally rigid framework in E^d which is not a simplex (i.e., it has at least $d+2$ nodes) admits at least one independent selfstress. This follows from the next theorem. A framework is *redundantly rigid* if it is rigid after the removal of an edge. A graph is c-connected if at least c nodes have to be removed from the graph to disconnect it.

Theorem 2. Hendrickson (1992). *If a framework with $n \geq d+2$ is generically globally rigid in E^d then it is redundantly rigid and $(d+1)$-connected.*

For $d = 2$ the theorem holds with an "if and only if" condition (Berg and Jordan, 2003, cf Connelly, 2013). Since generic redundant rigidity implies that there exist at least a selfstress, $s \geq 0$ and that there are no mechanisms, $m = 0$, it follows from (1) that in a generically globally rigid framework the number of edges is equal to or higher than

$$e = dn - d(d+1)/2 + 1\,, \tag{2}$$

e.g., $2n-2$ in 2D or $3n-5$ in 3D.

A complete characterization of generic global rigidity has been given in the following theorem.

Theorem 3. *A framework with $n \geq d+2$ is generically globally rigid in E^d if and only if there is a nonzero selfstress whose stress matrix has rank $n-d-1$.*

The "if" part is due to Connelly (1982), the "only if" part to Gortler et al. (2010).

The next theorem provide the converse of Theorem 1 in the generic case.

Theorem 4. Gortler and Thurston (2014). *A universally rigid framework $(\mathcal{G}, p)$ with p generic and $n \geq d+2$ is superstable.*

It is worth noticing that while global rigidity is a generic property, universal rigidity is not: if framework is universally rigid in a certain generic configuration, it can lose this property in a different generic configuration (compare cases (c) and (d) in **Figure 1**).

A less strict condition consists in requiring a configuration to be *general*. A configuration in E^d is general if no $d+1$ nodes are affinely dependent, e.g., there are no three collinear nodes in $d = 2$, or there are no three collinear nodes and no four coplanar nodes in $d = 3$. In this case we have the following result.

Theorem 5. Alfakih and Ye (2013). *A framework $(\mathcal{G}, p)$ with p general and $n \geq d+2$ is universally rigid if there is a nonzero selfstress whose stress matrix is positive semi-definite with rank $n-d-1$.*

It has been shown in Alfakih et al. (2013) that the converse of this theorem holds for $(d+1)$-lateration graphs, i.e., graphs obtained from a simplex by applying a sequence of $(d+1)$-valent node additions, i.e., the addition of a node connected by $d+1$ edges to the other nodes. An analogous result regarding global rigidity has been obtained previously in Anderson et al.

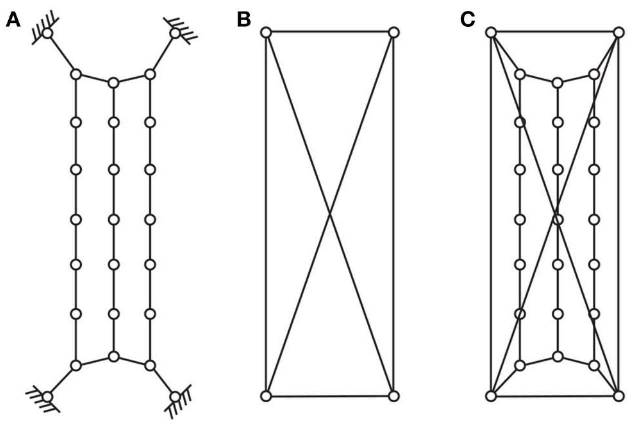

FIGURE 9 | A cablenet **(A)** can be anchored to a universally rigid framework **(B)** to obtain another universally rigid framework **(C)**.

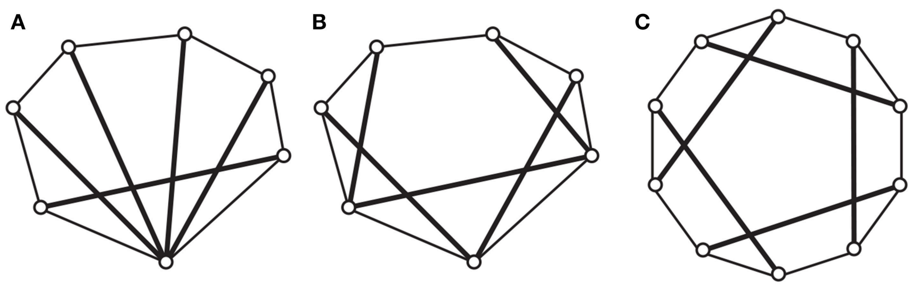

FIGURE 10 | Polygons satisfying Theorem 6: **(A)** a Grünbaum polygon; **(B)** a Cauchy polygon; **(C)** a Snelson polygon. Thin and thick lines corresponds respectively to cables and struts.

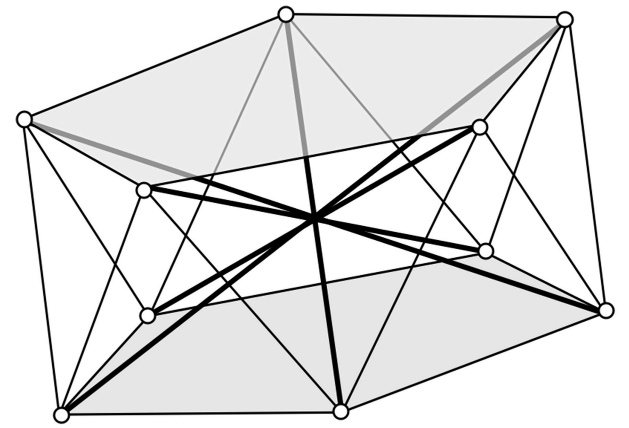

FIGURE 11 | A centrally symmetric framework with the shape of a pentagonal antiprism (with highlighted bases), see Theorem 7.

(2006). The number of edges of frameworks obtained in this way is

$$e = (d + 1)n - (d + 2)(d + 1)/2\,,$$

that is $e = 3n - 6$ for $d = 2$ and $e = 4n - 10$ for $d = 3$.

For n large, these values of e are 50% and 33% higher than the minimum value given by (2), respectively for $d = 2$ and $d = 3$.

By considering frameworks with a stress, the notion of tensegrity framework naturally comes into play. Indeed, many results have been first obtained for tensegrity frameworks, and then applied to the particular case of bar-frameworks.

A tensegrity framework is a framework where each edge can be labeled as a bar, a cable, or a strut: bars cannot change length, cables cannot increase in length, and struts cannot decrease in length. It turns out that a tensegrity framework is globally/universally rigid if the corresponding bar-framework is and the stress is *proper*, that is, cables have positive stress, and struts negative (Connelly, 2013). In other words, there is no difference between a bar-framework with a stress satisfying the theorems above and a tensegrity framework, with same graph and configuration, whose edges are labeled accordingly: cables if the stress is positive, struts if the stress is negative. Bars can be placed anywhere.

We conclude this section by reporting three results about known classes of frameworks. The first one is about convex polygons.

Theorem 6. *(Tensegrity polygons, Connelly, 1982). A tensegrity framework with the shape of a convex polygon, with cables on the outside, struts inside, and a proper a selfstress, is universally rigid (**Figure 10**).*

In the next section we will focus on the polygons like those in **Figures 2A**, **10A**, first described by Grünbaum and Shephard (1978).

The second result is about three-dimensional frameworks.

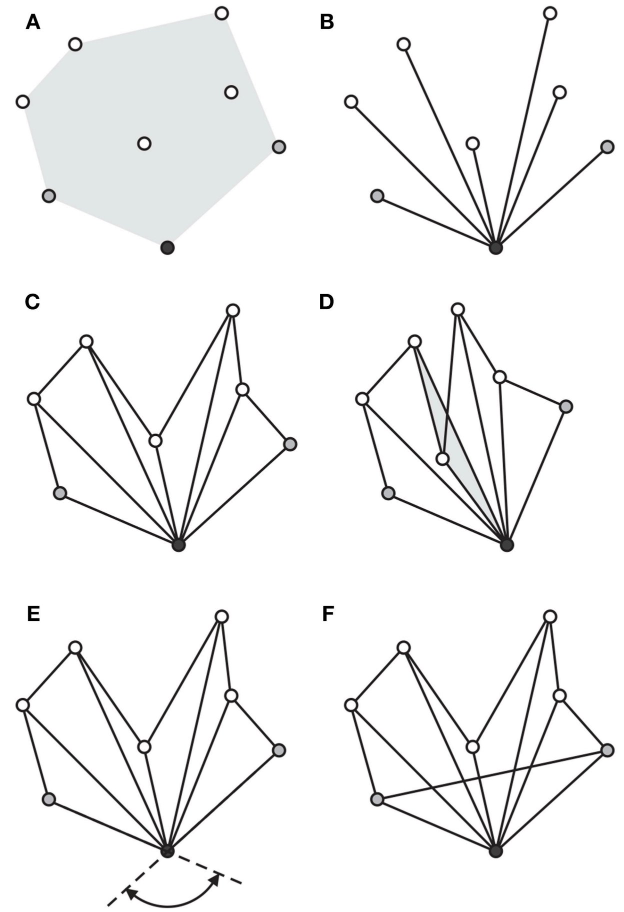

FIGURE 12 | Construction of a nonconvex Grünbaum polygon starting from a set of points in 2D: **(A)** three nodes corresponding to three consecutive vertices on the boundary of the convex hull of the set, the middle node is the *center* (in black), the other two nodes are its *neighbors* (in gray); **(B)** all the nodes are connected to the center by edges; **(C)** edges added to form adjacent triangles, so as to obtain a *fan*; **(D)** one of the admissible configurations of this fan, obtained by reversing the triangle shown (in light gray); **(E)** the completely unfolded configuration of the fan maximize the angle shown; **(F)** universal rigidity is obtained by adding the last edge between the two neighbors of the center.

Theorem 7. *(Central symmetric tensegrity polyhedra, Lovász, 2001; Bezdek and Connelly, 2006). Every tensegrity framework with the shape of a centrally symmetric polyhedron, with cables outside, bars connecting diametrically opposite pairs of vertices, and a proper selfstress, is universally rigid (**Figure 11**).*

The third result is about combining different frameworks together.

Theorem 8. *(Attachments, Ratmanski, 2010). Given two universally rigid frameworks in general position, it is possible to combine them into a universally rigid assembly if they have $d + 1$ nodes in common.*

Analogous results for globally rigid frameworks are presented in Eren et al. (2004a) and Connelly (2011).

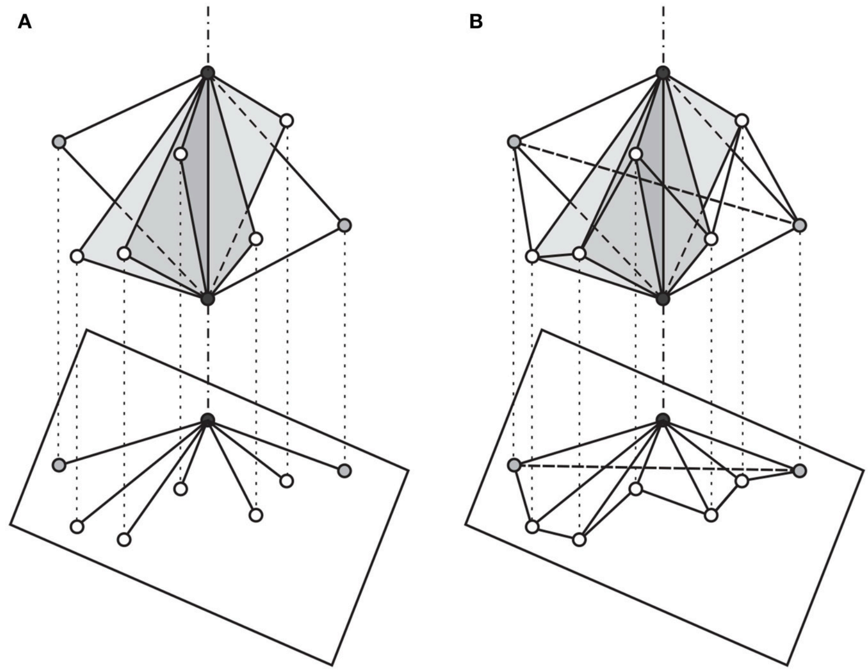

FIGURE 13 | Two phases, **(A)** and **(B)**, of the Construction of a three-dimensional Grünbaum framework, and the corresponding projection onto a nonconvex Grünbaum polygon (see description in Section 3.2).

3. CONSTRUCTION OF MINIMAL GENERIC UNIVERSALLY RIGID FRAMEWORKS

In this section we show that it is always possible to construct frameworks on n given nodes in E^2 or E^3 with the minimum number of edges (2), irrespective of the generic/nongeneric property of the configuration. Such frameworks belong to a new class which generalizes that of Grünbaum polygons (Grünbaum and Shephard, 1978).

3.1. Generalized Grünbaum Polygons

Grünbaum polygons are frameworks obtained by placing nodes and edges at the vertices and the sides of a convex polygon, then by choosing one node, the *center node* (in black in **Figure 2A**), and by connecting all the other nodes to it with an edge, except the two *neighboring nodes* (in gray in **Figure 2A**). The construction is completed by adding one edge connecting the two neighboring nodes.

We provide here a similar construction to assign $(2n-2)$ edges to a given a set of nodes in E^2 in order to obtain a universally rigid framework. We will call the resulting framework a *nonconvex Grünbaum polygon* (**Figure 12**).

First, the convex hull of the nodes is constructed and three consecutive vertices on its boundary coinciding with three nodes are chosen (**Figure 12A**), the middle one becomes the center to which all the other nodes are connected (**Figure 12B**). Then, additional edges are added to form a contiguous sequence of triangles sharing the center as a vertex (**Figure 12C**), plus the last edge connecting the two neighboring nodes (**Figure 12F**).

Theorem 9. *Every nonconvex Grünbaum polygon is universally rigid.*

Proof. Up to the addition of the last edge, the framework can be viewed as forming a kind of *fan* shape which "unfold" from the center node (**Figure 12C**). This incomplete framework admits a number of configurations equal to 2^f, where f is the number of internal edges or *folds* of the fan (**Figure 12D**). The distance between the two neighboring nodes will reach a maximum only when the fan is completely unfolded. It follows that by adding the last edge between the two neighboring nodes, the unfolded configuration is unique.

By embedding this framework in a higher dimensional Euclidean space, the situation does not change. Since each triangle of a fan is universally rigid by itself and it can only rotate about a fold, relative to its neighboring triangles, the triangle inequality ensure that the distance between the two neighboring nodes has a global maximum when the fan is flat, therefore the Grümbaum polygon is universally rigid. □

Notice that this proof is valid for both convex and nonconvex Grünbaum polygons. Notice also that the construction works even if the center is aligned with its neighbors, or if two or more fold are collinear. The result holds even if the configuration is nongeneric, the main requirement being that the center and its neighbors are on the boundary of the convex hull.

3.2. Three-Dimensional Grünbaum Frameworks

In three dimensions we can obtain a perfectly analogous result for assigning $(3n-5)$ edges red to a given set of nodes in E^3. We construct the convex hull of this set. There will be at least four vertices of the hull forming two adjacent triangles, sharing one edge of the convex hull. The shared edge is the *central edge* of the framework, the two nodes on this edge are the *central nodes*, while the other two are the *neighboring nodes*. Now, we can add edges connecting each of the neighboring nodes to the central nodes. We do the same with the remaining nodes, by connecting them to the central nodes. In this way, we obtain a set of triangles in space, all sharing one edge (**Figure 13A**). Then, for each couple of neighboring triangles, we add an edge between the nodes so as to form a tetrahedron. Finally, the last edge of this construction is added between the two neighboring nodes (**Figure 13B**).

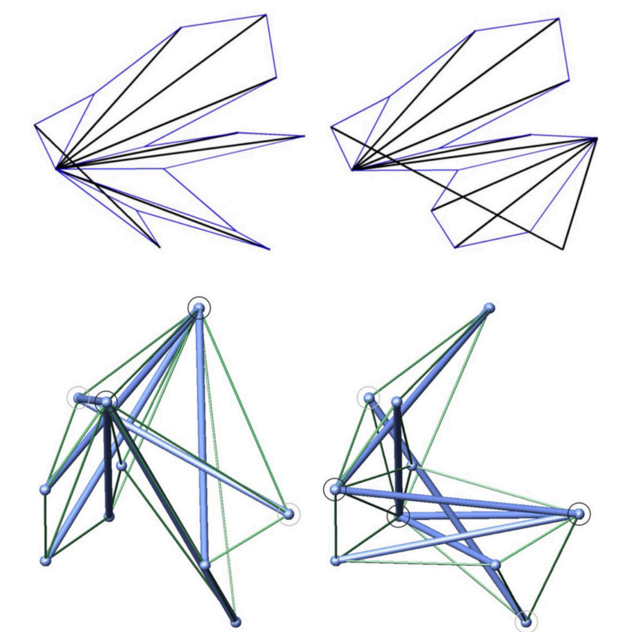

FIGURE 14 | (Top) Two universally rigid frameworks obtained from the same randomly generated nodal positions in E^2. The one on the top right is composed of two fans. Thin and thick edges correspond respectively to positive and negative stresses. **(Bottom)** Two constructions for the same randomly generated nodal positions in E^3. The one on the bottom right is composed by two fans. Black circles locate the central edges, gray circles locate the neighboring nodes. Thin and thick edges correspond respectively to positive and negative stresses.

An easy way of visualizing this framework is to project it along the direction of the central edge onto a plane, resulting in a fan-like framework, a nonconvex Grünbaum polygon. Similarly to what we have done before, we can consider the incomplete framework obtained by removing the last edge and argue that this admits a number of configuration equal to 2^f, with f defined for the projected framework as in the two-dimensional case. Among all these configurations, the one which is completely "unfolded" gives the maximum distance between the neighboring nodes, still using this term in analogy with the two-dimensional case. Once we add the last edge in this configuration, we obtain a globally rigid structure, which, by the triangle inequality is also universally rigid. We call frameworks obtained in this way *3D Grünbaum framework* and state the following theorem.

Theorem 10. *Every 3D Grünbaum framework is universally rigid.*

Notice that we can view these kind of frameworks, both in 2*D* and in 3*D*, as obtained by anchoring the nodes to a simplex, in the same way as we can anchor a cable-net to a (universally) rigid structure (**Figure 9**).

Notice also that we can find other generalized Grünbaum frameworks. For example, the one shown in **Figure 14** (top right) has two centers, corresponding to two fans with one side in common. It is easy to see that, in order

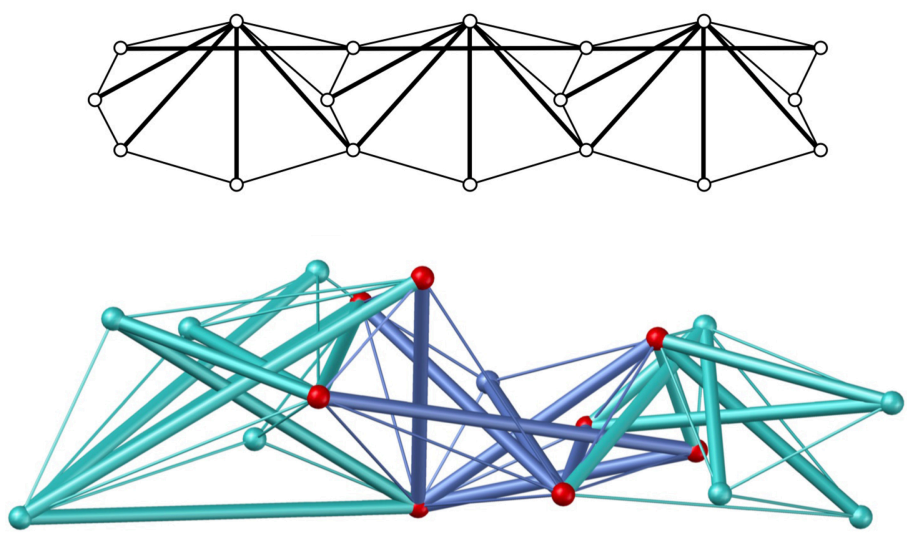

FIGURE 15 | (Top) A modular universally rigid framework in E^2 obtained by repetition of a universally rigid module. **(Bottom)** Construction of a modular universally rigid framework in E^3 for randomly generated nodal positions. The framework is composed by three universally rigid subframeworks (depicted in different color), each sharing four nodes with the adjacent one.

for multiple-fans frameworks to be universally rigid the centers should be on opposite sides of the edge connecting the neighbors. Analogous constructions exist also in three dimensions (**Figure 14**, bottom right).

In **Figure 15** we present two examples of application of Theorems 9, 10 in combination with Theorem 8. These examples shows how to avoid the occurrence of bars of excessive length by considering modular frameworks. In **Figure 15** (top), a universally rigid framework in E^2 is obtained by repetition of a universally rigid module, with adjacent modules having three nodes in common. In **Figure 15** (bottom), a universally rigid framework on randomly generated nodal positions in E^3 is composed by three universally rigid subframeworks, each sharing four nodes with the adjacent one. Space-filling universally rigid assemblies can be obtained in analogous fashion.

4. CONCLUDING REMARKS

After reviewing the main concepts and results in rigidity theory, we have given a construction for generic universally rigid frameworks in two and three dimensions with the minimum number of edges, with a significant improvement over existing methods. All together, these notions provide a set of useful tools that engineers can use to design superstable structural and mechanical systems, with guaranteed strength and stiffness properties. Additional applications include sensor networks, multi-agent systems, and protein conformation analysis. Our constructions relies on the computation of the convex hull of the set of nodes, a relatively quick operation, even if performed dynamically, i.e., if nodes are sequentially added and removed (see Hert and Schirra, 2018). For nongeneric configurations the number of edges can be further reduced, with the limitation that any change of configuration must happen on a lower-dimensional nongeneric manifold. The problem of constructing minimal nongeneric universally rigid frameworks will be the subject of future work.

AUTHOR CONTRIBUTIONS

SK and AM contributed conception and design of the study. All authors contributed to developing the study, wrote sections of the manuscript, revised it, read and approved the submitted version.

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