



# Strong Isomorphism in Marinatto–Weber Type Quantum Games

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Our purpose is to focus attention on a new criterion for quantum schemes by bringing together the notions of quantum game and game isomorphism. A quantum game scheme is required to generate the classical game as a special case. Now, given a quantum game scheme and two isomorphic classical games, we additionally require the resulting quantum games to be isomorphic as well. We show how this isomorphism condition influences the players' strategy sets. We are concerned with the Marinatto–Weber type quantum game scheme and the strong isomorphism between games in strategic form.

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## 1. INTRODUCTION

The Marinatto–Weber (MW) scheme introduced in Marinatto and Weber (2000) is a straightforward way to apply the power of quantum mechanics to classical game theory. In the simplest case of  $2 \times 2$  games, the players manipulate their own qubits of a two-qubit state either with the identity  $\mathbb{1}$  or the Pauli operator  $\sigma_x$ . Therefore, it has found application in many other branches of game theory: from evolutionary game theory (Iqbal and Toor, 2004; Nawaz and Toor, 2010) to extensive-form games (Fraćkiewicz, 2014) and duopoly examples (Iqbal and Toor, 2002a; Khan et al., 2010). In Fraćkiewicz (2013a), we pointed out a few undesirable properties of the MW scheme and introduced a refined quantum game model.

Though it is possible to extend both the MW scheme and our refinement to consider more complex games than  $2 \times 2$ , possible generalizations can be defined in many different ways. A result concerning  $3 \times 3$  games can be found in Iqbal and Toor (2002b, 2004). The authors proposed suitable three-element sets of players' strategies to obtain a generalized  $3 \times 3$  game. On the other hand, our work (Fraćkiewicz, 2013b) provides another way to define players' strategy sets that remains valid for any finite  $n \times m$  games.

Certainly, one can find yet other ways to generalize the MW scheme. Hence, it would be interesting to place additional restrictions on a quantum game scheme and examine how they refine the quantum model. In this paper, we formulate a criterion in terms of isomorphic games. Given two isomorphic games, we require the corresponding quantum games to be isomorphic as well. If, for example, two bimatrix games differ only in the order of players' strategies, they describe the same problem from a game-theoretical point of view. Given a quantum scheme, it appears reasonable to assume that the resulting quantum game will not depend on the numbering of players strategies in the classical game.

## 2. PRELIMINARIES

### 2.1. Marinatto–Weber Type Quantum Game Scheme

In Frąckiewicz (2013a, 2015), we presented a refinement of the Marinatto–Weber scheme (Marinatto and Weber, 2000). The motivation of constructing our scheme was twofold. Our model enables the players to choose between playing a fixed quantum strategy and classical strategies. The second aim was to construct the scheme that generates the classical game by manipulating the players' strategies rather than the initial quantum state. In what follows, we recall the scheme for the case of  $2 \times 2$  bimatrix game,

$$t \begin{matrix} l & r \\ \left( \begin{matrix} (a_{00}, b_{00}) & (a_{01}, b_{01}) \\ (a_{10}, b_{10}) & (a_{11}, b_{11}) \end{matrix} \right), \text{ where } (a_{ij}, b_{ij}) \in \mathbb{R}. \end{matrix} \quad (1)$$

**Definition 1.** The quantum scheme for game (1) is defined on an inner product space  $(\mathbb{C}^2)^{\otimes 4}$  by the triple

$$\Gamma_Q = (H, (S_1, S_2), (M_1, M_2)), \quad (2)$$

where

- $H$  is a positive operator,

$$H = (\mathbb{1} \otimes \mathbb{1} - |11\rangle\langle 11|) \otimes |00\rangle\langle 00| + |11\rangle\langle 11| \otimes |\Psi\rangle\langle \Psi|, \quad (3)$$

and

$$|\Psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \quad (4)$$

such that  $\| |\Psi\rangle \| = 1$ ,

- $S_1 = \{P_i^{(1)} \otimes U_j^{(3)}, i, j = 0, 1\}$ ,  $S_2 = \{P_k^{(2)} \otimes U_l^{(4)}, k, l = 0, 1\}$  are the players' strategy sets, and the upper indices identify the subspace  $\mathbb{C}^2$  of  $(\mathbb{C}^2)^{\otimes 4}$  on which the operators

$$P_0 = |0\rangle\langle 0|, P_1 = |1\rangle\langle 1|, U_0 = \mathbb{1}, U_1 = \sigma_x, \quad (5)$$

are defined,

- $M_1$  and  $M_2$  are the measurement operators

$$M_{1(2)} = \mathbb{1} \otimes \mathbb{1} \otimes \left( \sum_{x,y=0,1} a_{xy}(b_{xy})|xy\rangle\langle xy| \right) \quad (6)$$

that depend on the payoffs  $a_{xy}$  and  $b_{xy}$  from equation (1).

The scheme proceeds in the similar way as the MW scheme – the players determine the final state by choosing their strategies and acting on operator  $H$ . As a result, they determine the following density operator:

$$\rho_f = \left( P_i^{(1)} \otimes P_k^{(2)} \otimes U_j^{(3)} \otimes U_l^{(4)} \right) H \left( P_i^{(1)} \otimes P_k^{(2)} \otimes U_j^{(3)} \otimes U_l^{(4)} \right) \\ = \begin{cases} |11\rangle\langle 11| \\ \otimes \left( U_j^{(3)} \otimes U_l^{(4)} |\Psi\rangle\langle \Psi| U_j^{(3)} \otimes U_l^{(4)} \right) & \text{if } i = j = 1, \\ |ij\rangle\langle ij| \\ \otimes \left( U_j^{(3)} \otimes U_l^{(4)} |00\rangle\langle 00| U_j^{(3)} \otimes U_l^{(4)} \right) & \text{if otherwise.} \end{cases} \quad (7)$$

Next, the payoffs for player 1 and 2 are

$$\text{tr}(\rho_f M_1) \text{ and } \text{tr}(\rho_f M_2). \quad (8)$$

As it was shown in Frąckiewicz (2015), scheme (2) can be summarized by the following matrix game

$$\begin{matrix} P_0^{(1)} \otimes \mathbb{1}^{(3)} \\ P_0^{(1)} \otimes \sigma_x^{(3)} \\ P_1^{(1)} \otimes \mathbb{1}^{(4)} \\ P_1^{(1)} \otimes \sigma_x^{(4)} \end{matrix} \begin{pmatrix} P_0^{(2)} \otimes \mathbb{1}^{(4)} & P_0^{(2)} \otimes \sigma_x^{(4)} & P_1^{(2)} \otimes \mathbb{1}^{(4)} & P_1^{(2)} \otimes \sigma_x^{(4)} \\ X_{00} & X_{01} & X_{00} & X_{01} \\ X_{10} & X_{11} & X_{10} & X_{11} \\ X_{00} & X_{01} & \Delta_{00} & \Delta_{01} \\ X_{10} & X_{11} & \Delta_{10} & \Delta_{11} \end{pmatrix}, \quad (9)$$

where

$$X_{ij} = (a_{ij}, b_{ij}), \quad \text{for } i, j = 0, 1 \\ \Delta_{00} = |\alpha|^2 X_{00} + |\beta|^2 X_{01} + |\gamma|^2 X_{10} + |\delta|^2 X_{11}, \\ \Delta_{01} = |\alpha|^2 X_{01} + |\beta|^2 X_{00} + |\gamma|^2 X_{11} + |\delta|^2 X_{10}, \quad (10) \\ \Delta_{10} = |\alpha|^2 X_{10} + |\beta|^2 X_{11} + |\gamma|^2 X_{00} + |\delta|^2 X_{01}, \\ \Delta_{11} = |\alpha|^2 X_{11} + |\beta|^2 X_{10} + |\gamma|^2 X_{01} + |\delta|^2 X_{00}.$$

### 2.2. Strong Isomorphism

The notion of strong isomorphism defines classes of games that are the same up to the numbering of the players and the order of players' strategies. The following definitions are taken from Gabarró et al. (2011) [see also Nash (1951), Peleg et al. (1999), and Sudhölter et al. (2000)]. The first one defines a mapping that associates players and their actions in one game with players and their actions in the other game.

**Definition 2.** Let  $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$  and  $\Gamma' = (N, (S'_i)_{i \in N}, (u'_i)_{i \in N})$  be games in strategic form. A game mapping  $f$  from  $\Gamma$  to  $\Gamma'$  is a tuple,  $f = (\eta, (\varphi_i)_{i \in N})$ , where  $\eta$  is a bijection from  $N$  to  $N$  and for any  $i \in N$ ,  $\varphi_i$  is a bijection from  $S_i$  to  $S'_{\eta(i)}$ .

In general case, the mapping  $f$  from  $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$  to  $(N, (S'_i)_{i \in N}, (u'_i)_{i \in N})$  identifies player  $i \in N$  with player  $\eta(i)$  and maps  $S_i$  to  $S_{\eta(i)}$ . This means that a strategy profile  $(s_1, \dots, s_n) \in S_1 \times \dots \times S_n$  is mapped into profile  $(s'_1, \dots, s'_n)$  that satisfies equation  $s'_{\eta(i)} = \varphi_i(s_i)$  for  $i \in N$ .

The notion of game mapping is a basis for the definition of game isomorphism. Depending on how rich structure of the game is to be preserved, we can distinguish various types of game isomorphism. One that preserves the players' payoff functions is called a strong isomorphism. The formal definition is as follows:

**Definition 3.** Given two strategic games  $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$  and  $\Gamma' = (N, (S'_i)_{i \in N}, (u'_i)_{i \in N})$ , a game mapping  $f = (\eta, (\varphi_i)_{i \in N})$  is called a strong isomorphism if relation  $u_i(s) = u'_{\eta(i)}(f(s))$  holds for each  $i \in N$  and each strategy profile  $s \in S_1 \times \dots \times S_n$ .

From the above definition, it may be concluded that if there is a strong isomorphism between games  $\Gamma$  and  $\Gamma'$ , they may differ merely by the numbering of players and the order of their strategies.

The following lemma shows that relabeling players and their strategies do not affect the game with regard to Nash equilibria.

If  $f$  is a strong isomorphism between games  $\Gamma$  and  $\Gamma'$ , one may expect that the Nash equilibria in  $\Gamma$  map to ones in  $\Gamma'$  under  $f$ .

**Lemma 1.** Let  $f$  be a strong isomorphism between games  $\Gamma$  and  $\Gamma'$ . Strategy profile  $s^* = (s_1^*, \dots, s_n^*) \in S_1 \times \dots \times S_n$  is a Nash equilibrium in game  $\Gamma$  if and only if  $f(s^*) \in S'_1 \times \dots \times S'_n$  is a Nash equilibrium in  $\Gamma'$ .

### 3. APPLICATION OF GAME ISOMORPHISM TO MARINATTO–WEBER TYPE QUANTUM GAME SCHEMES

It is not hard to see that we can define various schemes based on the MW approach. We can modify operator (3) and the players' strategies to construct another scheme still satisfying the requirement about generalization of the input game. The following example of such a scheme is particularly interesting. Let us consider a triple

$$\Gamma'_Q = (H', (S'_1, S'_2), (M_1, M_2)) \quad (11)$$

with the components defined as follows:

- $H'$  is a positive operator,

$$H' = |00\rangle\langle 00| \otimes |00\rangle\langle 00| + |01\rangle\langle 01| \otimes |0\rangle\langle 0| \otimes \rho_2 + |10\rangle\langle 10| \otimes \rho_1 \otimes |0\rangle\langle 0| + |11\rangle\langle 11| \otimes |\Psi\rangle\langle \Psi|, \quad (12)$$

where  $|\Psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$  such that  $\| |\Psi\rangle \| = 1$ ,  $\rho_1$  and  $\rho_2$  are the reduced density operators of  $|\Psi\rangle\langle \Psi|$ , i.e.,  $\rho_1 = \text{tr}_2(|\Psi\rangle\langle \Psi|)$  and  $\rho_2 = \text{tr}_1(|\Psi\rangle\langle \Psi|)$ ,

- $S'_1 = \{P_0^{(1)} \otimes \mathbb{1}^{(3)}, P_0^{(1)} \otimes \sigma_x^{(3)}, P_1^{(1)} \otimes \mathbb{1}^{(3)}\}$  and  $S'_2 = \{P_0^{(2)} \otimes \mathbb{1}^{(4)}, P_0^{(2)} \otimes \sigma_x^{(4)}, P_1^{(2)} \otimes \mathbb{1}^{(4)}\}$  are the players' strategy sets,
- $M_1$  and  $M_2$  are the measurement operators defined by equation (6).

It is immediate that the resulting final state  $\rho'_f$  is a density operator for each (pure or mixed) strategy profile. For example, player 1's strategy  $P_0^{(1)} \otimes \sigma_x^{(3)}$  and player 2's strategy  $P_1^{(2)} \otimes \mathbb{1}^{(4)}$  imply

$$\begin{aligned} \rho'_f &= \left( P_0^{(1)} \otimes P_1^{(2)} \otimes \sigma_x^{(3)} \otimes \mathbb{1}^{(4)} \right) H' \left( P_0^{(1)} \otimes P_1^{(2)} \otimes \sigma_x^{(3)} \otimes \mathbb{1}^{(4)} \right) \\ &= |01\rangle\langle 01| \otimes |1\rangle\langle 1| \otimes \rho_2. \end{aligned} \quad (13)$$

As a result, the players' payoff functions  $u'_1$  and  $u'_2$  given by  $\text{tr}(\rho'_f M_1)$  and  $\text{tr}(\rho'_f M_2)$ , respectively, are well defined. It is also clear that scheme (16) produces the classical game in a similar way to scheme (2). The players play the classical game as long as they choose the strategies  $P_0 \otimes \mathbb{1}$  and  $P_0 \otimes \sigma_x$ . This can be seen by determining  $\text{tr}(\rho'_f M_{1(2)})$  for each strategy profile and arranging the obtained values into a matrix. As an example, let us determine  $\text{tr}(\rho'_f M_{1(2)})$  for the final state  $\rho'_f$  given by equation (18). Let  $|\Psi\rangle$  represent a general two-qubit state,

$$|\Psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle. \quad (14)$$

Since

$$\begin{aligned} \rho_1 &= (|\alpha|^2 + |\beta|^2)|0\rangle\langle 0| + (\alpha\gamma^* + \beta\delta^*)|0\rangle\langle 1| \\ &\quad + (\gamma\alpha^* + \delta\beta^*)|1\rangle\langle 0| + (|\gamma|^2 + |\delta|^2)|1\rangle\langle 1|, \\ \rho_2 &= (|\alpha|^2 + |\gamma|^2)|0\rangle\langle 0| + (\alpha\beta^* + \gamma\delta^*)|0\rangle\langle 1| \\ &\quad + (\beta\alpha^* + \delta\gamma^*)|1\rangle\langle 0| + (|\beta|^2 + |\delta|^2)|1\rangle\langle 1|, \end{aligned} \quad (15)$$

the players' strategies  $P_0^{(1)} \otimes \sigma_x^{(3)}$  and  $P_1^{(2)} \otimes \mathbb{1}^{(4)}$  generate the following form of the final state:

$$\begin{aligned} \rho'_f &= |01\rangle\langle 01| \otimes ((|\alpha|^2 + |\gamma|^2)|10\rangle\langle 10| + (\alpha\beta^* + \gamma\delta^*)|10\rangle\langle 11| \\ &\quad + (\beta\alpha^* + \delta\gamma^*)|11\rangle\langle 10| + (|\beta|^2 + |\delta|^2)|11\rangle\langle 11|). \end{aligned}$$

Hence

$$\begin{aligned} (\text{tr}(\rho'_f M_1), \text{tr}(\rho'_f M_2)) &= (|\alpha|^2 + |\gamma|^2)(a_{10}, b_{10}) \\ &\quad + (|\beta|^2 + |\delta|^2)(a_{11}, b_{11}). \end{aligned} \quad (16)$$

The values  $(\text{tr}(\rho'_f M_1), \text{tr}(\rho'_f M_2))$  for all strategy combinations are given by the following matrix:

$$\begin{array}{ccc} & P_0^{(2)} \otimes \mathbb{1}^{(4)} & P_0^{(2)} \otimes \sigma_x^{(4)} & P_1^{(2)} \otimes \mathbb{1}^{(4)} \\ \begin{array}{l} P_0^{(1)} \otimes \mathbb{1}^{(3)} \\ P_0^{(1)} \otimes \sigma_x^{(3)} \\ P_1^{(1)} \otimes \mathbb{1}^{(3)} \end{array} & \begin{pmatrix} X_{00} & X_{01} & \Delta_{02} \\ X_{10} & X_{11} & \Delta_{12} \\ \Delta_{20} & \Delta_{21} & \Delta_{22} \end{pmatrix} & \end{array} \quad (17)$$

where

$$\begin{aligned} X_{ij} &= (a_{ij}, b_{ij}), \quad \text{for } i, j = 0, 1 \\ \Delta_{02} &= (|\alpha|^2 + |\gamma|^2)X_{00} + (|\beta|^2 + |\delta|^2)X_{01}; \\ \Delta_{12} &= (|\alpha|^2 + |\gamma|^2)X_{10} + (|\beta|^2 + |\delta|^2)X_{11}; \\ \Delta_{20} &= (|\alpha|^2 + |\beta|^2)X_{00} + (|\gamma|^2 + |\delta|^2)X_{10}; \\ \Delta_{21} &= (|\alpha|^2 + |\beta|^2)X_{01} + (|\gamma|^2 + |\delta|^2)X_{11}; \\ \Delta_{22} &= |\alpha|^2 X_{00} + |\beta|^2 X_{01} + |\gamma|^2 X_{10} + |\delta|^2 X_{11}. \end{aligned} \quad (18)$$

It follows easily that matrix game (17) is a genuine extension of equation (1). Although payoff profiles  $\Delta_{ij} \neq \Delta_{22}$  are also achievable in equation (1), the players, in general, are not able to obtain  $\Delta_{22}$  when choosing their (mixed) strategies.

To sum up, scheme (11) might seem to be acceptable as long as scheme (2) is acceptable. Matrix game (17) includes equation (1) and depending on the initial state  $|\Psi\rangle$  it may give extraordinary Nash equilibria. It is worth pointing out that the Nash equilibria in equation (17) correspond to correlated equilibria in equation (1) (Frąckiewicz, 2016). However, scheme (11) fails to imply the isomorphic games when the input games are isomorphic. We can make this clear with the following example.

**Example 1.** Let us consider the game of "Chicken"  $\Gamma_1$  and its (strongly) isomorphic counterpart  $\Gamma_2$ ,

$$\Gamma_1: \begin{array}{cc} & l & r \\ \begin{array}{c} t \\ b \end{array} & \begin{pmatrix} (6, 6) & (2, 7) \\ (7, 2) & (0, 0) \end{pmatrix}, & \Gamma_2: \begin{array}{cc} & l' & r' \\ \begin{array}{c} t' \\ b' \end{array} & \begin{pmatrix} (2, 7) & (6, 6) \\ (0, 0) & (7, 2) \end{pmatrix}. \end{array} \quad (19)$$

The corresponding isomorphism  $f = (\pi, \varphi_1, \varphi_2)$  is defined by components

$$\begin{aligned} \pi(i) &= i \text{ for } i = 1, 2, & \varphi_1 &= (t \rightarrow t', b \rightarrow b'), \\ \varphi_2 &= (l \rightarrow r', r \rightarrow l'). \end{aligned} \quad (20)$$

Set  $|\Psi\rangle = (|00\rangle + |01\rangle + |10\rangle)/\sqrt{3}$ . Using equation (9), we can write quantum approach (2) to games (19) as

$$\Gamma_{Q1}: \begin{matrix} P_0^{(1)} \otimes \mathbb{1}^{(3)} \\ P_0^{(1)} \otimes \sigma_x^{(3)} \\ P_1^{(1)} \otimes \mathbb{1}^{(4)} \\ P_1^{(1)} \otimes \sigma_x^{(4)} \end{matrix} \begin{pmatrix} P_0^{(2)} \otimes \mathbb{1}^{(4)} & P_0^{(2)} \otimes \sigma_x^{(4)} & P_1^{(2)} \otimes \mathbb{1}^{(4)} & P_1^{(2)} \otimes \sigma_x^{(4)} \\ (6, 6) & (2, 7) & (6, 6) & (2, 7) \\ (7, 2) & (0, 0) & (7, 2) & (0, 0) \\ (6, 6) & (2, 7) & (5, 5) & (2\frac{2}{3}, 4\frac{1}{3}) \\ (7, 2) & (0, 0) & (4\frac{1}{3}, 2\frac{2}{3}) & (3, 3) \end{pmatrix} \quad (21)$$

and

$$\Gamma_{Q2}: \begin{matrix} P_0^{(1)} \otimes \mathbb{1}^{(3)} \\ P_0^{(1)} \otimes \sigma_x^{(3)} \\ P_1^{(1)} \otimes \mathbb{1}^{(4)} \\ P_1^{(1)} \otimes \sigma_x^{(4)} \end{matrix} \begin{pmatrix} P_0^{(2)} \otimes \mathbb{1}^{(4)} & P_0^{(2)} \otimes \sigma_x^{(4)} & P_1^{(2)} \otimes \mathbb{1}^{(4)} & P_1^{(2)} \otimes \sigma_x^{(4)} \\ (2, 7) & (6, 6) & (2, 7) & (6, 6) \\ (0, 0) & (7, 2) & (0, 0) & (7, 2) \\ (2, 7) & (6, 6) & (2\frac{2}{3}, 4\frac{1}{3}) & (5, 5) \\ (0, 0) & (7, 2) & (3, 3) & (4\frac{1}{3}, 2\frac{2}{3}) \end{pmatrix} \quad (22)$$

It is fairly easy to see that games (21) and (22) differ in the order of the first two strategies and the second two strategies of player 2. Thus, the games are strongly isomorphic. More formally, one can check that a game mapping  $\tilde{f} = (\eta, \tilde{\varphi}_1, \tilde{\varphi}_2)$ , where

$$\begin{aligned} \tilde{\varphi}_1 &= (P_i^{(1)} \otimes \mathbb{1}^{(3)} \rightarrow P_i^{(1)} \otimes \mathbb{1}^{(3)}, P_i^{(1)} \otimes \sigma_x^{(3)} \rightarrow P_i^{(1)} \otimes \sigma_x^{(3)}), \\ \tilde{\varphi}_2 &= (P_k^{(2)} \otimes \mathbb{1}^{(4)} \rightarrow P_k^{(2)} \otimes \sigma_x^{(4)}, P_k^{(2)} \otimes \sigma_x^{(4)} \rightarrow P_k^{(2)} \otimes \mathbb{1}^{(4)}), \end{aligned} \quad (23)$$

for  $i, k = 0, 1$  is a strong isomorphism.

In the next section, we prove a more general result about scheme (2).

Let us now consider scheme (11). Matrix (17) in terms of input games (19) implies

$$\Gamma'_{Q1}: \begin{matrix} P_0^{(1)} \otimes \mathbb{1}^{(3)} \\ P_0^{(1)} \otimes \sigma_x^{(3)} \\ P_1^{(1)} \otimes \mathbb{1}^{(3)} \end{matrix} \begin{pmatrix} P_0^{(2)} \otimes \mathbb{1}^{(4)} & P_0^{(2)} \otimes \sigma_x^{(4)} & P_1^{(2)} \otimes \mathbb{1}^{(4)} \\ (6, 6) & (2, 7) & (4\frac{2}{3}, 6\frac{1}{3}) \\ (7, 2) & (0, 0) & (4\frac{2}{3}, 1\frac{1}{3}) \\ (6\frac{1}{3}, 4\frac{2}{3}) & (1\frac{1}{3}, 4\frac{2}{3}) & (5, 5) \end{pmatrix} \quad (24)$$

and

$$\Gamma'_{Q2}: \begin{matrix} P_0^{(1)} \otimes \mathbb{1}^{(3)} \\ P_0^{(1)} \otimes \sigma_x^{(3)} \\ P_1^{(1)} \otimes \mathbb{1}^{(3)} \end{matrix} \begin{pmatrix} P_0^{(2)} \otimes \mathbb{1}^{(4)} & P_0^{(2)} \otimes \sigma_x^{(4)} & P_1^{(2)} \otimes \mathbb{1}^{(4)} \\ (2, 7) & (6, 6) & (3\frac{1}{3}, 6\frac{2}{3}) \\ (0, 0) & (7, 2) & (2\frac{1}{3}, \frac{2}{3}) \\ (1\frac{1}{3}, 4\frac{2}{3}) & (6\frac{1}{3}, 4\frac{2}{3}) & (2\frac{2}{3}, 4\frac{1}{3}) \end{pmatrix}. \quad (25)$$

With Lemma 1, we can show that games (24) and (25) are not isomorphic. Comparing the sets of pure Nash equilibria in both games, we find the equilibrium profiles

$$\left\{ \left( P_0^{(1)} \otimes \mathbb{1}^{(3)}, P_0^{(2)} \otimes \sigma_x^{(4)} \right), \left( P_0^{(1)} \otimes \sigma_x^{(3)}, P_0^{(2)} \otimes \mathbb{1}^{(4)} \right), \left( P_1^{(1)} \otimes \mathbb{1}^{(3)}, P_1^{(2)} \otimes \mathbb{1}^{(4)} \right) \right\} \quad (26)$$

in the first game and

$$\left\{ \left( P_0^{(1)} \otimes \mathbb{1}^{(3)}, P_0^{(2)} \otimes \mathbb{1}^{(4)} \right), \left( P_0^{(1)} \otimes \sigma_x^{(3)}, P_0^{(2)} \otimes \sigma_x^{(4)} \right) \right\} \quad (27)$$

in the second one.

## 4. APPLICATION OF GAME ISOMORPHISM TO GENERALIZED MARINATTO-WEBER QUANTUM GAME SCHEME

Additional criteria for a quantum game scheme may have a significant impact on the way how we generalize these schemes. It can be easily seen in the case of the MW scheme (Marinatto and Weber, 2000) [or the refined scheme (2)], where the sets of unitary strategies are finite. The MW scheme provides us with a quantum model, where the strategy sets consist of the identity operator  $\mathbb{1}$  and the Pauli operator  $\sigma_x$ . Under this description, what subsets of unitary operators would be suitable for general  $n \times m$  games? The case of a 3-element strategy set can be identified with unitary operators  $\mathbb{1}_3, C$  and  $D$  acting on,  $\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \in \mathbb{C}^3$  where

$$\begin{aligned} \mathbb{1}_3|0\rangle &= |0\rangle, & C|0\rangle &= |2\rangle, & D|0\rangle &= |1\rangle, \\ \mathbb{1}_3|1\rangle &= |1\rangle, & C|1\rangle &= |1\rangle, & D|1\rangle &= |0\rangle, \\ \mathbb{1}_3|2\rangle &= |2\rangle, & C|2\rangle &= |0\rangle, & D|2\rangle &= |2\rangle. \end{aligned} \quad (28)$$

This construction can be found in Iqbal and Toor (2002b, 2004). Another way to generalize the MW scheme was presented in Frąckiewicz (2013b). Having given a strategic-form game, we identify the players'  $n$  strategies with  $n$  unitary operators  $V_k$  for  $k = 0, 1, \dots, n-1$ . They act on states of the computational basis  $\{|0\rangle, |1\rangle, \dots, |n-1\rangle\}$  as follows:

$$\begin{aligned} V_0|i\rangle &= |i\rangle, \\ V_1|i\rangle &= |i + 1 \bmod n\rangle, \\ &\vdots \\ V_{n-1}|i\rangle &= |i + n - 1 \bmod n\rangle. \end{aligned} \quad (29)$$

Both ways to generalize the MW scheme enable us to obtain the classical game. So at this level, neither equation (28) nor equation (29) is questionable. If we seek other properties, we see that the MW scheme outputs the classical game (or its isomorphic counterpart) when the initial state is one of the computational basis states. Given equations (28) and (29), only the latter case satisfies this condition. Further analysis would show that the MW scheme is invariant with respect to strongly isomorphic input games. It turns out that neither equation (28) nor equation (29) satisfies the isomorphism property.

**Example 2.** Let us take a look at the following  $2 \times 3$  bimatrix games:

$$\begin{matrix} & l & m & r \\ \Gamma: & t & (4, 8) & (0, 0) & (8, 8) \\ & b & (0, 4) & (4, 0) & (8, 0) \\ & & l' & m' & r' \\ \Gamma': & t' & (0, 0) & (4, 8) & (8, 8) \\ & b' & (4, 0) & (0, 4) & (8, 0) \end{matrix} \quad (30)$$

Consider the MW-type approaches  $\Gamma_Q$  and  $\Gamma'_Q$  to games (30) according to the following assignments:

$$\begin{matrix} & l & m & r \\ t & (P_{00} & P_{01} & P_{02}) \\ b & (P_{10} & P_{11} & P_{12}) \end{matrix}, \quad \text{where } P_{j_1 j_2} = |j_1 j_2\rangle\langle j_1 j_2|. \quad (31)$$

Then,

$$\begin{aligned} \Gamma_Q &= (|\Psi\rangle, (D_1, D_2), (M_1, M_2)), \\ \Gamma'_Q &= (|\Psi\rangle, (D'_1, D'_2), (M'_1, M'_2)), \end{aligned} \quad (32)$$

where

$$\begin{aligned} M_1 &= 4P_{00} + 8P_{02} & M'_1 &= 4P_{01} + 8P_{02} \\ &+ 4P_{11} + 8P_{12}, & &+ 4P_{10} + 8P_{12}, \\ M_2 &= 8P_{00} + 8P_{02} + 4P_{10}, & M'_2 &= 8P_{01} + 8P_{02} + 4P_{11}. \end{aligned} \quad (33)$$

Set the initial state  $|\Psi\rangle = (1/2)|00\rangle + (\sqrt{3}/2)|12\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^3$  and assume first that  $D_1 = D'_1 = \{\mathbb{1}_2, \sigma_x\}$  and  $D_2 = D'_2 = \{\mathbb{1}_3, C, D\}$  as in equation (28). Determining  $\text{tr}((U_1 \otimes U_2)|\Psi\rangle\langle\Psi| (U_1^\dagger \otimes U_2^\dagger)M_i)$  for every  $U_1 \otimes U_2 \in \{\mathbb{1}, \sigma_x\} \otimes \{\mathbb{1}_3, C, D\}$  and  $i = 1, 2$ , and doing similar calculations in the case of  $M'_i$ , we obtain

$$\begin{matrix} & \mathbb{1}_3 & C & D \\ \Gamma_Q: & \mathbb{1}_2 & \begin{pmatrix} (7, 2) & (2, 5) & (6, 0) \\ (6, 7) & (5, 6) & (7, 6) \end{pmatrix} \\ & \sigma_x & \\ \Gamma'_Q: & \mathbb{1}_2 & \begin{pmatrix} (6, 0) & (5, 2) & (7, 2) \\ (7, 6) & (2, 0) & (6, 7) \end{pmatrix} \\ & \sigma_x & \end{matrix} \quad (34)$$

On the other hand, replacing equation (28) by equation (29) gives  $D_2 = D'_2 = \{\mathbb{1}_3, V_1, V_2\}$ , where

$$\mathbb{1}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (35)$$

Then, we have

$$\begin{matrix} & \mathbb{1}_3 & V_1 & V_2 \\ \Gamma_Q: & \mathbb{1}_2 & \begin{pmatrix} (7, 2) & (0, 3) & (5, 2) \\ (6, 7) & (4, 6) & (2, 0) \end{pmatrix} \\ & \sigma_x & \\ \Gamma'_Q: & \mathbb{1}_2 & \begin{pmatrix} (6, 0) & (4, 2) & (2, 5) \\ (7, 6) & (0, 1) & (5, 6) \end{pmatrix} \\ & \sigma_x & \end{matrix} \quad (36)$$

There is no pure Nash equilibrium in the first game of equations (34) and (36), whereas there are two Nash equilibria in the second games. As a result, each pair of the games does not determine a strong isomorphism.

Example 2 shows that players' strategy sets defined by equations (28) and (35) need to be revised in order to have a generalized MW scheme invariant with respect to the isomorphism. We shall stick for the moment to considering games (30). Let  $\{A_{012}, A_{102}, A_{021}, A_{120}, A_{201}, A_{210}\}$  be player 2's strategy set defined to be

$$\begin{aligned} A_{012}|0\rangle &= |0\rangle & A_{102}|0\rangle &= |1\rangle & A_{021}|0\rangle &= |0\rangle & A_{120}|0\rangle &= |1\rangle & A_{201}|0\rangle &= |2\rangle & A_{210}|0\rangle &= |2\rangle, \\ A_{012}|1\rangle &= |1\rangle & A_{102}|1\rangle &= |0\rangle & A_{021}|1\rangle &= |2\rangle & A_{120}|1\rangle &= |2\rangle & A_{201}|1\rangle &= |0\rangle & A_{210}|1\rangle &= |1\rangle, \\ A_{012}|2\rangle &= |2\rangle & A_{102}|2\rangle &= |2\rangle & A_{021}|2\rangle &= |1\rangle & A_{120}|2\rangle &= |0\rangle & A_{201}|2\rangle &= |1\rangle & A_{210}|2\rangle &= |0\rangle. \end{aligned} \quad (37)$$

Each  $A_{j_1 j_2 j_3}$  is a permutation matrix that corresponds to a specific permutation  $\pi = (0 \rightarrow j_1, 1 \rightarrow j_2, 2 \rightarrow j_3)$  of the set  $\{0, 1, 2\}$ . Note also that operators (28) and (38) are included in equation (37). Hence, the MW scheme with equation (37) implies, in particular, the classical game. We now check if it outputs the isomorphic games. Since there are now six operators available for player 2, the resulting game may be written as a  $2 \times 6$  bimatrix game with entries

$$\text{tr}((U_1 \otimes U_2)|\Psi\rangle\langle\Psi|(U_1^T \otimes U_2^T)M_i) \quad (38)$$

for  $U_1 \in \{\mathbb{1}_2, \sigma_x\}$  and  $U_2 \in \{A_\pi: \pi\text{-permutations of } \{0, 1, 2\}\}$ . As a result, we obtain

$$\Gamma_Q: \begin{matrix} & A_{012} & A_{102} & A_{021} & A_{120} & A_{201} & A_{210} \\ \mathbb{1}_2 & \begin{pmatrix} (7, 2) & (6, 0) & (4, 2) & (0, 3) & (5, 2) & (2, 5) \\ (6, 7) & (7, 6) & (0, 1) & (4, 6) & (2, 0) & (5, 6) \end{pmatrix} \\ \sigma_x & \end{matrix} \quad (39)$$

and

$$\Gamma'_Q: \begin{matrix} & A_{012} & A_{102} & A_{021} & A_{120} & A_{201} & A_{210} \\ \mathbb{1}_2 & \begin{pmatrix} (6, 0) & (7, 2) & (0, 3) & (4, 2) & (2, 5) & (5, 2) \\ (7, 6) & (6, 7) & (4, 6) & (0, 1) & (5, 6) & (2, 0) \end{pmatrix} \\ \sigma_x & \end{matrix} \quad (40)$$

The games determine the isomorphism  $\tilde{f} = (\text{id}_N, \tilde{\varphi}_1, \tilde{\varphi}_2)$ , where

$$\begin{aligned} \tilde{\varphi}_1 &= (\mathbb{1}_2 \rightarrow \mathbb{1}_2, \sigma_x \rightarrow \sigma_x), \\ \tilde{\varphi}_2 &= (A_{012} \rightarrow A_{102}, A_{102} \rightarrow A_{012}, A_{021} \rightarrow A_{120}, \\ & A_{120} \rightarrow A_{021}, A_{201} \rightarrow A_{210}, A_{210} \rightarrow A_{201}). \end{aligned} \quad (41)$$

Using permutation matrices leads us to formulate another generalized MW scheme. For simplicity, we confine attention to  $(n + 1) \times (m + 1)$  bimatrix games.

Let  $S_n$  be the set of all permutations  $\pi$  of  $\{0, 1, \dots, n\}$ . With each  $\pi$ , there is associated a permutation matrix  $A_\pi$ ,

$$A_\pi|i\rangle = |\pi(i)\rangle \text{ for } i = 0, 1, \dots, n. \quad (42)$$

We let  $B_\sigma$  denote the permutation matrix associated with a permutation  $\sigma \in S_m$ . Given  $(n + 1) \times (m + 1)$  bimatrix game  $\Gamma$ , we define

$$\Gamma_Q = (|\Psi\rangle, (D_1, D_2), (M_1, M_2)), \quad (43)$$

where

$$\begin{aligned} |\Psi\rangle &= \sum_{j_1=0}^n \sum_{j_2=0}^m \alpha_{j_1 j_2} |j_1 j_2\rangle \in \mathbb{C}^{n+1} \otimes \mathbb{C}^{m+1}, \\ D_1 &= \{A_\pi : \pi \in S_n\}, D_2 = \{B_\sigma : \sigma \in S_m\}, \\ (M_1, M_2) &= \sum_{j_1=0}^n \sum_{j_2=0}^m (a_{j_1 j_2}, b_{j_1 j_2}) P_{j_1 j_2}. \end{aligned} \quad (44)$$

Before stating the main result of this section, we start with the observation that the MW scheme remains invariant to numbering of the players. Consider two isomorphic bimatrix games:

$$\Gamma: \begin{matrix} & t_0 & t_1 & \cdots & t_m \\ s_0 & (a_{00}, b_{00}) & (a_{01}, b_{01}) & \cdots & (a_{0m}, b_{0m}) \\ s_1 & (a_{10}, b_{10}) & (a_{11}, b_{11}) & \cdots & (a_{1m}, b_{1m}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_n & (a_{n0}, b_{n0}) & (a_{n1}, b_{n1}) & \cdots & (a_{nm}, b_{nm}) \end{matrix} \quad (45)$$

and

$$\Gamma': \begin{matrix} & s'_0 & s'_1 & \cdots & s'_n \\ t'_0 & (b_{00}, a_{00}) & (b_{10}, a_{10}) & \cdots & (b_{n0}, a_{n0}) \\ t'_1 & (b_{01}, a_{01}) & (b_{11}, a_{11}) & \cdots & (b_{n1}, a_{n1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t'_m & (b_{0m}, a_{0m}) & (b_{1m}, a_{1m}) & \cdots & (b_{nm}, a_{nm}) \end{matrix}. \quad (46)$$

Clearly, the isomorphism is defined by a game mapping  $f = \{\pi, \varphi_1, \varphi_2\}$ , where

$$\pi = (1 \rightarrow 2, 2 \rightarrow 1), \quad \varphi_1(s_{j_1}) = s'_{j_1}, \quad \varphi_2(t_{j_2}) = t'_{j_2} \quad (47)$$

for  $j_1 = 0, 1, \dots, n, j_2 = 0, 1, \dots, m$ . The general MW scheme for equation (43) is simply given by equation (43). For  $\Gamma'$ , we can write

$$\Gamma'_Q = (|\Psi'\rangle, (D'_1, D'_2), (M'_1, M'_2)), \quad (48)$$

where

$$\begin{aligned} |\Psi'\rangle &= \sum_{j_1=0}^n \sum_{j_2=0}^m \alpha_{j_1 j_2} |j_2 j_1\rangle \in \mathbb{C}^{m+1} \otimes \mathbb{C}^{n+1}, \\ D'_1 &= \{B_\sigma, \sigma \in S_m\}, D'_2 = \{A_\pi, \pi \in S_n\}, \\ M'_1 &= \sum_{j_1=0}^n \sum_{j_2=0}^m b_{j_1 j_2} |j_2 j_1\rangle \langle j_2 j_1|, M'_2 = \sum_{j_1=0}^n \sum_{j_2=0}^m a_{j_1 j_2} |j_2 j_1\rangle \langle j_2 j_1|. \end{aligned} \quad (49)$$

Games determined by equations (43) and (48) are then isomorphic. To prove this, let  $\tilde{f} = (\pi, \tilde{\varphi}_1, \tilde{\varphi}_2)$  be a game mapping such that

$$\begin{aligned} \pi &= (1 \rightarrow 2, 2 \rightarrow 1), \quad \tilde{\varphi}_1 : D_1 \rightarrow D'_2, \quad \tilde{\varphi}_1(A_\pi) = A_\pi, \\ \tilde{\varphi}_2 : D_2 &\rightarrow D'_1, \quad \tilde{\varphi}_2(B_\sigma) = B_\sigma. \end{aligned} \quad (50)$$

On account of Definition 3, we have

$$\tilde{f}(A_\pi \otimes B_\sigma) = \varphi_2(B_\sigma) \otimes \varphi_1(A_\pi) = B_\sigma \otimes A_\pi. \quad (51)$$

As a result,

$$\begin{aligned} u'_{\pi(1)}(\tilde{f}(A_\pi \otimes B_\sigma)) &= u'_2(\tilde{f}(A_\pi \otimes B_\sigma)) \\ &= \text{tr} \left( \tilde{f}(A_\pi \otimes B_\sigma) |\Psi'\rangle \langle \Psi'| \tilde{f}(A_\pi \otimes B_\sigma)^T M'_2 \right) \\ &= \text{tr} \left( (\varphi_2(B_\sigma) \otimes \varphi_1(A_\pi)) |\Psi'\rangle \langle \Psi'| (\varphi_2(B_\sigma)^T \otimes \varphi_1(A_\pi)^T) M'_2 \right) \\ &= \text{tr} \left( (B_\sigma \otimes A_\pi) |\Psi'\rangle \langle \Psi'| (B_\sigma^T \otimes A_\pi^T) \sum_{j_1=0}^n \sum_{j_2=0}^m a_{j_1 j_2} |j_2 j_1\rangle \langle j_2 j_1| \right) \\ &= \text{tr} \left( (A_\pi \otimes B_\sigma) |\Psi\rangle \langle \Psi| (A_\pi^T \otimes B_\sigma^T) \sum_{j_1=0}^n \sum_{j_2=0}^m a_{j_1 j_2} |j_1 j_2\rangle \langle j_1 j_2| \right) \\ &= u_1(A_\pi \otimes B_\sigma). \end{aligned} \quad (52)$$

By a similar argument, we can show that  $u'_{\pi(2)}(\tilde{f}(A_\pi \otimes B_\sigma)) = u_2(A_\pi \otimes B_\sigma)$ . We can now formulate the following proposition:

**Proposition 1.** Assume that  $\Gamma$  and  $\Gamma'$  are strongly isomorphic bimatrix games and  $\Gamma_Q$  and  $\Gamma'_Q$  are the corresponding quantum games defined by equation (43). Then,  $\Gamma_Q$  and  $\Gamma'_Q$  are strongly isomorphic.

**Proof 1.** Let  $\Gamma$  and  $\Gamma'$  be bimatrix games of dimension  $n \times m$  and let  $f: \Gamma \rightarrow \Gamma', f = (\eta, \varphi_1, \varphi_2)$  be the strong isomorphism. Since the MW scheme is invariant to numbering of the players, there is no loss of generality in assuming  $\eta = \text{id}_N$ . Now, it follows from Definition 3 that games  $\Gamma$  and  $\Gamma'$  differ in the order of players' strategies. Let us identify players' strategies (i.e., rows and columns) in  $\Gamma$  with sequences  $(0, 1, \dots, n)$  and  $(0, 1, \dots, m)$ , respectively. Then, we denote by  $\pi^*$  and  $\sigma^*$  the permutations of the sets  $\{0, 1, \dots, n\}$  and  $\{0, 1, \dots, m\}$  associated with the order of strategies in game  $\Gamma'$ . A trivial verification shows that the payoff operator  $M'_i$  in  $\Gamma'_Q$  may be written as

$$M'_i = (A_{\pi^*} \otimes B_{\sigma^*}) M_i (A_{\pi^*} \otimes B_{\sigma^*})^T, \quad (53)$$

where  $A_{\pi^*}$  and  $B_{\sigma^*}$  are the permutation matrices corresponding to  $\pi^*$  and  $\sigma^*$ . Define a game mapping  $\tilde{f} = (\text{id}_N, \tilde{\varphi}_1, \tilde{\varphi}_2)$ , where

$$\begin{aligned} \tilde{\varphi}_1 : \{A_\pi : \pi \in S_n\} &\rightarrow \{A_\pi : \pi \in S_n\}, \quad \varphi_1(A_\pi) = A_{\pi^*} A_\pi; \\ \tilde{\varphi}_2 : \{B_\sigma : \sigma \in S_m\} &\rightarrow \{B_\sigma : \sigma \in S_m\}, \quad \varphi_2(B_\sigma) = B_{\sigma^*} B_\sigma. \end{aligned} \quad (54)$$

Hence,  $\tilde{f}$  maps  $A_\pi \otimes B_\sigma$  to  $A_{\pi^*} A_\pi \otimes B_{\sigma^*} B_\sigma$ . Thus, we obtain

$$\begin{aligned} u'_i(\tilde{f}(A_\pi \otimes B_\sigma)) &= \text{tr} \left( \tilde{f}(A_\pi \otimes B_\sigma) |\Psi\rangle \langle \Psi| (\tilde{f}(A_\pi \otimes B_\sigma))^T M'_i \right) \\ &= \text{tr} \left( (A_{\pi^*} A_\pi \otimes B_{\sigma^*} B_\sigma) |\Psi\rangle \langle \Psi| (A_{\pi^*} A_\pi \otimes B_{\sigma^*} B_\sigma)^T \right. \\ &\quad \left. \times (A_{\pi^*} \otimes B_{\sigma^*}) M_i (A_{\pi^*} \otimes B_{\sigma^*})^T \right) \\ &= \text{tr} \left( (A_{\pi^*} \otimes B_{\sigma^*})^T (A_{\pi^*} A_\pi \otimes B_{\sigma^*} B_\sigma) |\Psi\rangle \langle \Psi| \right. \\ &\quad \left. \times (A_{\pi^*} A_\pi \otimes B_{\sigma^*} B_\sigma)^T (A_{\pi^*} \otimes B_{\sigma^*}) M_i \right) \end{aligned}$$

$$\begin{aligned}
&= \text{tr} \left( \left( A_{\pi^*}^T A_{\pi^*} A_{\pi} \otimes B_{\sigma^*}^T B_{\sigma^*} B_{\sigma} \right) |\Psi\rangle\langle\Psi| \right. \\
&\quad \times \left. \left( A_{\pi}^T A_{\pi^*}^T A_{\pi^*} \otimes B_{\sigma}^T B_{\sigma^*}^T B_{\sigma^*} \right) M_i \right) \\
&= \text{tr} \left( (A_{\pi} \otimes B_{\sigma}) |\Psi\rangle\langle\Psi| (A_{\pi} \otimes B_{\sigma})^T M_i \right) = u_i(A_{\pi} \otimes B_{\sigma}).
\end{aligned} \tag{55}$$

This finishes the proof.

Note that operators (42) come down to  $\mathbb{1}$  and  $\sigma_x$  for  $n=1$ . Therefore, the original MW scheme preserves the isomorphism. The same conclusion can be drawn for the refined MW scheme (2).

**Corollary 1.** If  $\Gamma$  and  $\Gamma'$  are strongly isomorphic bimatrix games and  $\Gamma_Q$  and  $\Gamma'_Q$  are the corresponding games defined by equation (2). Then,  $\Gamma_Q$  and  $\Gamma'_Q$  are strongly isomorphic.

**Proof 2.** Let  $\Gamma$  and  $\Gamma'$  be strongly isomorphic  $2 \times 2$  bimatrix games. By Proposition 1, there exists a strong isomorphism  $\tilde{f} = (\text{id}_{\{1,2\}}, \tilde{\varphi}_1, \tilde{\varphi}_2)$  between the games  $\Gamma_Q$  and  $\Gamma'_Q$  played according to equations (43) and (44). Given the quantum approach (2) to  $\Gamma$  and  $\Gamma'$ , we define  $\tilde{g} = (\text{id}_{\{1,2\}}, \tilde{\xi}_1, \tilde{\xi}_2)$ , where

$$\begin{aligned}
\tilde{\xi}_1: S_1 &\rightarrow S_1, \quad \tilde{\xi}_1 \left( P_i^{(1)} \otimes U_j^{(3)} \right) = P_i^{(1)} \otimes \tilde{\varphi}_1 \left( U_j^{(3)} \right), \\
\tilde{\xi}_2: S_2 &\rightarrow S_2, \quad \tilde{\xi}_2 \left( P_k^{(2)} \otimes U_l^{(4)} \right) = P_k^{(2)} \otimes \tilde{\varphi}_2 \left( U_l^{(4)} \right).
\end{aligned} \tag{56}$$

Now, we have

$$\begin{aligned}
&u'_i \left( \tilde{g} \left( P_i^{(1)} \otimes P_k^{(2)} \otimes U_j^{(3)} \otimes U_l^{(4)} \right) \right) \\
&= \text{tr} \left( \tilde{g} \left( P_i^{(1)} \otimes P_k^{(2)} \otimes U_j^{(3)} \otimes U_l^{(4)} \right) \right. \\
&\quad \times H \tilde{g} \left( P_i^{(1)} \otimes P_k^{(2)} \otimes U_j^{(3)} \otimes U_l^{(4)} \right)^T M'_i \left. \right) \\
&= \text{tr} \left( P_i^{(1)} \otimes P_k^{(2)} \otimes \tilde{\varphi}_1 \left( U_j^{(3)} \right) \otimes \tilde{\varphi}_2 \left( U_l^{(4)} \right) \right. \\
&\quad \times H P_i^{(1)} \otimes P_k^{(2)} \otimes \tilde{\varphi}_1 \left( U_j^{(3)} \right)^T \otimes \tilde{\varphi}_2 \left( U_l^{(4)} \right)^T M'_i \left. \right) \\
&= \text{tr} \left( P_i^{(1)} \otimes P_k^{(2)} \otimes \tilde{f} \left( U_j^{(3)} \otimes U_l^{(4)} \right) \right. \\
&\quad \times H P_i^{(1)} \otimes P_k^{(2)} \otimes \tilde{f} \left( U_j^{(3)} \otimes U_l^{(4)} \right)^T M'_i \left. \right).
\end{aligned} \tag{57}$$

For fixed  $P_i \otimes P_k$ , we can write the right side of equation (57) in the form

$$\begin{aligned}
&\text{tr} \left( |ik\rangle\langle ik| \otimes \tilde{f} \left( U_j^{(3)} \otimes U_l^{(4)} \right) \rho \tilde{f} \left( U_j^{(3)} \otimes U_l^{(4)} \right)^T M'_i \right), \\
\rho &= \begin{cases} |\Psi\rangle\langle\Psi|, & (i,j) = (1,1); \\ |00\rangle\langle 00|, & (i,j) \neq (1,1). \end{cases}
\end{aligned} \tag{58}$$

By reasoning similar to equation (55), we conclude that

$$\begin{aligned}
&u'_i \left( \tilde{g} \left( P_i^{(1)} \otimes P_k^{(2)} \otimes U_j^{(3)} \otimes U_l^{(4)} \right) \right) \\
&= \text{tr} \left( |ik\rangle\langle ik| \otimes \tilde{f} \left( U_j^{(3)} \otimes U_l^{(4)} \right) \rho \tilde{f} \left( U_j^{(3)} \otimes U_l^{(4)} \right)^T M'_i \right)
\end{aligned}$$

$$\begin{aligned}
&= \text{tr} \left( |ik\rangle\langle ik| \otimes U_j^{(3)} \otimes U_l^{(4)} \rho \left( U_j^{(3)} \otimes U_l^{(4)} \right)^T M'_i \right) \\
&= u_i \left( P_i^{(1)} \otimes P_k^{(2)} \otimes U_j^{(3)} \otimes U_l^{(4)} \right).
\end{aligned} \tag{59}$$

We have thus proved that  $\Gamma_Q$  and  $\Gamma'_Q$  are isomorphic.

It is worth noting that the converse may not be true. Given isomorphic games  $\Gamma_Q$  and  $\Gamma'_Q$ , the input games  $\Gamma$  and  $\Gamma'$  may not determine the strong isomorphism. Indeed, bimatrix games

$$\begin{matrix} l & r \\ t & \begin{pmatrix} (3,1) & (0,0) \\ (0,0) & (1,3) \end{pmatrix} \end{matrix} \quad \text{and} \quad \begin{matrix} l' & r \\ t' & \begin{pmatrix} (4,0) & (0,0) \\ (0,0) & (0,4) \end{pmatrix} \end{matrix} \tag{60}$$

are not strongly isomorphic. However, the MW approach [with the initial state  $(|00\rangle + |11\rangle)/\sqrt{2}$ ] to each one of equation (60) implies the same output game given by

$$\begin{matrix} \mathbb{1} & \sigma_x \\ \mathbb{1} & \begin{pmatrix} (3,1) & (0,0) \\ (0,0) & (1,3) \end{pmatrix} \\ \sigma_x & \end{matrix} \tag{61}$$

## 5. CONCLUSION

The theory of quantum games does not provide us with clear definitions of how a quantum game should look like. In fact, only one condition is taken into consideration. A quantum game scheme is merely required to generalize the classical game. As a result, this allows us to define a quantum game scheme in many different ways. However, various techniques to describe a game in the quantum domain can imply different quantum game results. Therefore, it would be convenient to specify that some quantum schemes work under some further restrictions. We have been working under the assumption that a quantum scheme is invariant with respect to isomorphic transformations of an input game. We have shown that this requirement may be essential tool in defining a quantum scheme. The protocol that replicates classical correlated equilibria is an example that does not satisfy our criterion. The refined definition for a quantum game scheme may also be useful to generalize protocols. Our work has shown that dependence of local unitary operators in the MW scheme on the number of strategies in a classical game is not linear. In fact, the generalized approach to  $n \times m$  bimatrix game can be identified with a game of dimension  $n! \times m!$ .

## AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and approved it for publication.

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