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Side Tributary Distribution of Quasi-Uniform Iterative Binary Tree Networks for River Networks

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Self-similarity and plane-filling are intrinsic structure properties of natural river networks. Statistical data indicates that most natural river networks are Tokunaga trees. Researchers have explored to use iterative binary tree networks (IBTNs) to simulate natural river networks. However, the characteristics of natural rivers such as Tokunaga self-similarity and plane-filling cannot be easily guaranteed by the configuration of the IBTN. In this paper, the generator series and a quasi-uniform iteration rule are specified for the generation of nonstochastic quasi-uniform iterative binary tree networks (QU-IBTNs). First, we demonstrate that QU-IBTNs definitely satisfy self-similarity. Second, we show that the constraint for a QU-IBTN to be a Tokunaga tree is that the exterior links must be replaced in the generator series with a neighboring generator that is larger than the interior links during the iterative process. Moreover, two natural river networks are examined to reveal the inherent consistency with QU-IBTN at low Horton-Strahler orders.

Keywords: horton-strahler (H-S) order, tokunaga tree, generator series, side tributary distribution, self-similarity, iterative binary tree network

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INTRODUCTION

River networks are typically considered to be dendritic, self-similar, and plane-filling in plane shape (Horton, 1945; Tokunaga, 1966; Mandelbrot, 1982; Peckham, 1995b). With this in mind, their scalings have been comprehensively studied. The first method for quantifying the scale of streams in a river network was given by Horton (1945), and then modified by Strahler (1952) to create what is now known as the Horton-Strahler (H-S) ordering method. This method gives every stream an H-S order according to its position in the confluencing structure of a river network. It has become one of the most basic concepts in river network topology. Furthermore, the Horton ratios, which are the bifurcation ratio R_B, the length ratio R_L, and the area ratio R_A, have been well established and verified (Horton, 1945; Strahler, 1952). These ratios reflect the self-similarity of a river network by approximately formed geometric sequences.

Tokunaga (1966), Tokunaga (1978) introduced a bivariate stream ordering method based on the H-S method. The Tokunaga ordering method quantifies different confluences by using the H-S order in pairs to describe the flow of a tributary into its main stem. The mean ratio of the number of streams with H-S order ω to the number of streams with H-S order $\omega+k$, which are the streams that

H-S order ω streams flow into, is defined as the side-branching ratio T_k. Furthermore, Tokunaga (1978, 1984) recognized that the side-branching ratio characterizes the self-similar structure of a binary tree river network. Therefore, a strict constraint for a binary tree network to be a Tokunaga tree is that T_k varies geometrically with k but independently with ω. By using this constraint of side-branching ratios, large natural river networks have been proved to be Tokunaga trees (Tokunaga, 1978; Peckham, 1995a; Mantilla et al., 2010; Zanardo et al., 2013; Gupta and Mesa, 2014).

Additionally, studies on similarity and fractals have made the synthetic generation of iterative networks an efficient tool for simulating real systems with self-similarity. Some of these synthetic iterative networks have been used for hydrological simulations of their analogous natural river networks (Claps et al., 1996; Menabde et al., 2001; Veitzer and Gupta, 2001; Wang and Wang, 2002; Hung and Wang, 2005). Furthermore, recursive replacement networks have been introduced to show that the Horton laws are well predicted (Peckham, 1995a; Veitzer and Gupta, 2000; Mcconnell and Gupta, 2008). Synthetic trees generated by the Tokunaga model have been used to statistically evaluate the Tokunaga self-similarity, and have been compared with natural river networks (Zanardo et al., 2013). Besides the iterative replacement networks, the diffusion-limited aggregation (DLA) model using random walk as synthetic mechanism can also be used for river network generation, and is similar to natural river networks (Masek and Turcotte, 1993). However, none of these synthetic networks have been proved to be self-similar or Tokunaga trees, either mathematically or theoretically. Therefore, it is unreasonable to use them to capture the properties of natural river networks.

Using synthetic iterative networks to simulate natural river networks has limitations that need to be overcome, mainly in two aspects. First, a commonly accepted and rigorous mathematical definition of generators for river networks is needed. The lack of a rigorous definition of generators (Wang and Wang, 2002; Troutman, 2005) results in an inability to guarantee the basicness and completeness of the generator population. A basic and complete binary generator series has been specially and explicitly defined and graphed by Zhang et al. (2009), and has been proved to be effective and fundamental by means of comparison with data from China and the United States (Zhang et al., 2009). Second, the constraints of Tokunaga trees should be strictly added during the generation of synthetic iterative networks.

The main objectives of this paper involve specifying a common mathematical framework so as to 1) reemphasize the standard generator series for iterative networks; 2) generate an iterative binary tree network using the generator series and iteration rules; and 3) find the appropriate constraint that guarantees a synthetic iterative network to be a Tokunaga tree.

This paper is organized as follows. *Materials and Method* briefly reviews the rules and definitions of the H-S method, Tokunaga ordering method, Tokunaga tree, Generators and rules for the iterative binary tree networks (IBTNs). In *Results*, the mathematical derivations of side tributary distribution are shown recursively and graphically for quasi-uniform iterative binary tree networks (QU-IBTNs). The sufficient and necessary conditions for a QU-IBTN to be a Tokunaga tree are discussed in *Discussion*. Two natural river networks are given as examples to verify the feasibility of QU-IBTNs.

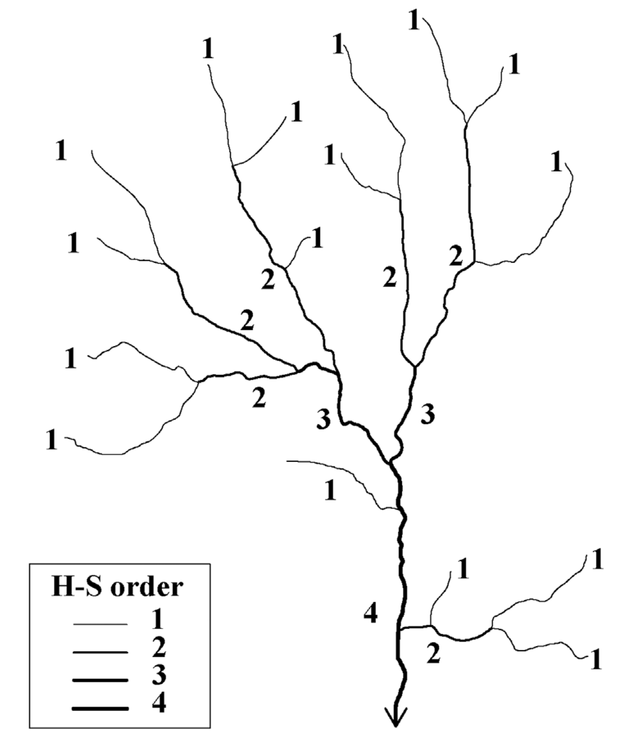

FIGURE 1 | H-S ordering method schematic diagram for the illustrated river network with H-S order 4. There are 16 streams with H-S order 1, 6 streams with H-S order 2, 2 streams with H-S order 3, and 1 stream with H-S order 4. The arrow at the bottom is the outlet of this river network.

MATERIALS AND METHOD

Tokunaga Tree

Horton (1945) and Strahler (1952) defined the classification method for the hierarchical structure of a river network by means of stream order as follows:

1) every source channel has an H-S order 1;
2) two streams with the same order, ω, confluence to a stream ordered $\omega + 1$; and
3) two streams with different orders, $u, v\,(u < v)$, confluence to a stream with ordered v.

The H-S order of the whole network, Ω, is defined as the highest order of all streams. The H-S ordering method is shown in **Figure 1**.

Tokunaga (1966) proposed an extended ordering method based on the H-S order. The Tokunaga ordering method reflects the topological relation of a side-branching tributary flowing into another stream with a higher order. His work

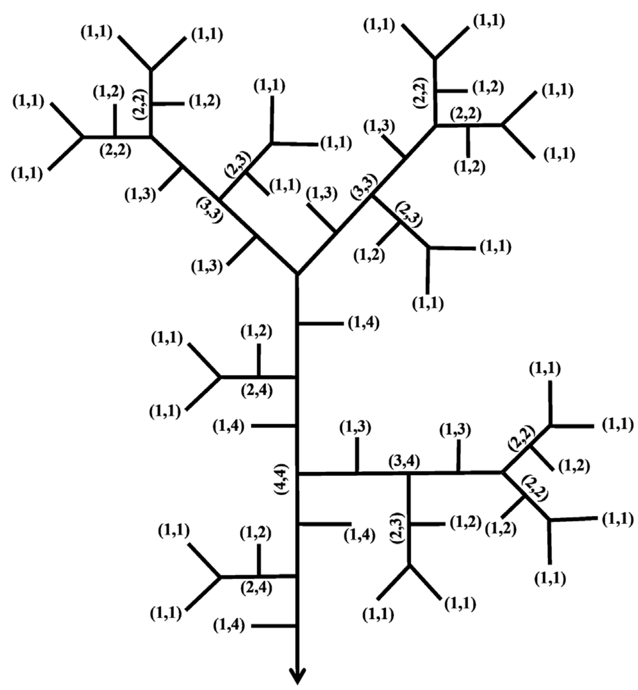

FIGURE 2 | Schematic diagram for the Tokunaga ordering method for a binary tree (from Zhang et al., 2009).

plays an important role in analyzing the topology of river networks because it represents the inherent self-similarity of a river network.

A stream with H-S order ω as a side-branching tributary flowing into a stream ordered ω' is assigned a Tokunaga order (ω, ω'), for which $\omega < \omega'$. A pair of streams as sources of a stream ordered $\omega + 1$ is assigned a Tokunaga order (ω, ω). **Figure 2** shows the Tokunaga ordering method applied to an example binary tree.

For a binary tree network, the Tokunaga stream number matrix $\boldsymbol{N}$ specifies the number of streams with Tokunaga order (ω, ω'), $N_{\omega,\omega'}$, as:

$$\boldsymbol{N} = \begin{pmatrix} N_{1,1} & N_{1,2} & N_{1,3} & \dots & N_{1,(\Omega-1)} & N_{1,\Omega} \\ & N_{2,2} & N_{2,3} & \dots & \dots & N_{2,\Omega} \\ & & & \dots & \dots & \dots \\ & & & & N_{(\Omega-1),(\Omega-1)} & N_{(\Omega-1),\Omega} \\ & & & & & N_{\Omega,\Omega} \end{pmatrix} \quad (1)$$

For example, the Tokunaga stream number matrix of the binary tree in **Figure 2** is:

$$\boldsymbol{N} = \begin{pmatrix} 22 & 11 & 6 & 4 \\ & 6 & 3 & 2 \\ & & 2 & 1 \\ & & & 1 \end{pmatrix} \quad (2)$$

Eq. 2 defines the ordering method in **Figure 2**, which shows 22 streams with Tokunaga order (1,1), 11 streams with Tokunaga order (1,2), and so on.

The Tokunaga side-branching ratio $T_{\omega,\omega+k}$ $(1 \le k \le \Omega - \omega)$, which is determined in terms of the ratio of the number of streams with the H-S order ω flowing into streams with the H-S order $\omega + k$ to the number of streams with the H-S order $\omega + k$, is defined as:

$$T_{\omega,\omega+k} = \frac{N_{\omega,\omega+k}}{N_{\omega+k}} = \frac{N_{\omega,\omega+k}}{\sum_{i=\omega+k}^{\Omega} N_{\omega+k,i}},\ 1 \le k \le \Omega - \omega \quad (3)$$

An upper triangular matrix $\boldsymbol{T}$, which is calculated in terms of the Tokunaga stream number matrix $\boldsymbol{N}$ by **Eq. 3**, is defined as the side-branching ratio matrix with a dimension of $\Omega - 1$. The side-branching ratio matrix of the binary tree in **Figure 2** is:

$$\boldsymbol{T} = \begin{pmatrix} 1 & 2 & 4 \\ & 1 & 2 \\ & & 1 \end{pmatrix} \quad (4)$$

The necessary condition for a network to be self-similar is that the side-branching ratio T_k is independent of ω (Peckham, 1995a), that is:

$$T_{\omega,\omega+k} = T_k \quad (5)$$

For a network to be a Tokunaga tree in a statistical sense, the side-branching ratio must satisfy the constraint (Peckham, 1995a):

$$T_k = ac^{k-1} \quad (6)$$

Here, a is the average number of streams of H-S order ω flowing into streams of order $\omega + 1$, and c is the average rate of increase of the side-branching ratios of side tributaries with different order.

In **Figure 2**, $T_1 = 1, T_2 = 2, T_3 = 4$, and consequently $T_k = 2^{k-1}$, which means that the binary tree in **Figure 2** is a Tokunaga tree with $a = 1$ and $c = 2$.

Iterative Binary Tree Networks

The basic elements of a synthetic iterative network are 1) the generators, which are the smallest units of an iterative network, and 2) the iteration rules, which specify the growth pattern of the network using the generators. Different combinations of generators and iteration rules result in different networks. Iterative network models must be based on a series of generators. Each generator should be unique, and the generator series should be complete (Zhang et al., 2009; Mantilla et al., 2010).

Generator Series

Self-similarity has been considered to be an inherent characteristic of river networks since Mandelbrot first described their fractal nature (Mandelbrot, 1982; Peckham, 1995b). Since then, various methods to create synthetic networks have been proposed based on generator iteration (Veitzer and Gupta, 2000; Wang and Wang, 2002; Hung and Wang, 2005; Zhang et al., 2009; Mantilla et al., 2010). However, generators are not sequential and complete until they are cataloged by their topological structure (Zhang et al., 2009;

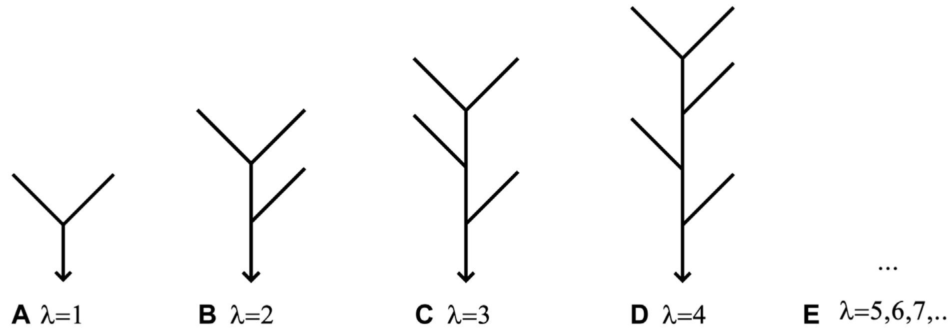

FIGURE 3 | Diagrammatic infinite generator series for the topological structure of iterative binary tree networks. Here, λ is the index of each generator [revised from Zhang et al. (2009)].

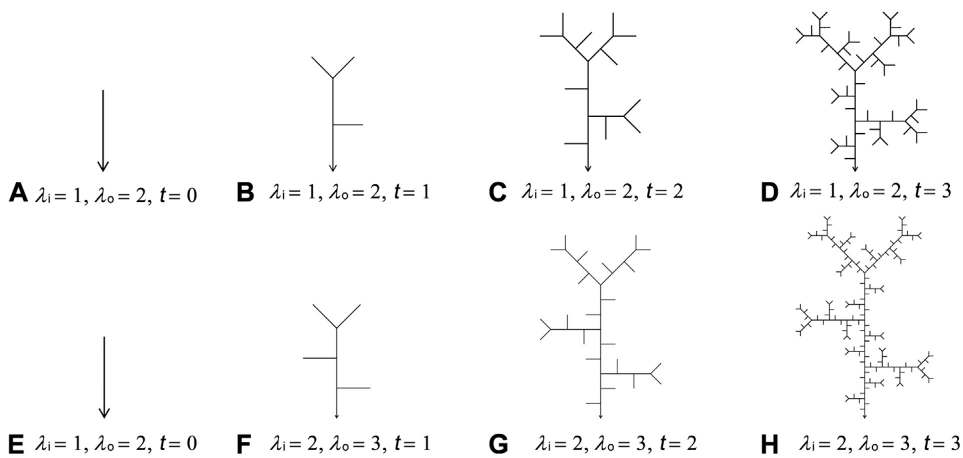

FIGURE 4 | QU-IBTNs **(A)** and **(E)**: initial cases (t = 0); **(B–D)**: generator $\lambda_i = 1$ and $\lambda_o = 2$ after 1, 2, and 3 iteration steps; **(F–H)**: generator $\lambda_i = 2$ and $\lambda_o = 3$ after 1, 2, and 3 iteration steps.

Mantilla et al., 2010). **Figure 3** shows the first four generators in a generator series. Each generator in the series can be denoted by an index λ, which is a positive integer in sequential order. The generator series in **Figure 3** is the basis of the iterative networks discussed in this paper.

Iteration Rule

The generation of a synthetic iterative network, specifically an ITBN, is based on iteration definitions and rules. First, we give some basic definitions for an iterative network. An exterior link is an unbroken section of stream that extends from a source to the first junction, and corresponds to a stream with H-S order 1, whereas an interior link connects two successive junctions or the last junction with the outlet (Shreve, 1966). The generators for the replacement of interior and exterior links are chosen from the series in **Figure 3** and are denoted by λ_i and λ_o, respectively. Both interior and exterior links are replaced by the interior generator λ_i and exterior generator λ_o, respectively, in each step of the iterative process. The length of each link is assumed to be 1 in this paper. We use one link as the initial case (t = 0), and then the exterior generator λ_o as the first iterative step (t = 1).

The iteration rule discussed in this paper is taking λ_i as the constant interior generator and λ_o as the constant exterior generator in each step of the iterative process. This iteration rule is quasi-uniform because of the consistent generators in every iterative step. Therefore, the iterative network generated by this rule is defined as a QU-IBTN. **Figure 4** shows two examples of how to generate QU-IBTNs using $\lambda_i = 1, \lambda_o = 2$ (**Figures 4A–D**) and $\lambda_i = 2$, $\lambda_o = 3$ (**Figures 4E–H**), respectively.

The invariance of generators in each iterative step for QU-IBTNs is strictly consistent with the definition of self-similarity. However, although the relevant iterative processes must not only have the sequentially and mathematically expressible generator series, but must also agree with the explicit given iteration rule, whether the QU-IBTN is a Tokunaga tree has never been examined graphically and recursively using a mathematical method.

Side Tributary Distribution of QU-IBTNs

The generation of a QU-IBTN begins with one link, the interior generator λ_i, and the exterior generator λ_o (i.e. the QU-IBTN is λ_o itself when t = 1). According to the iteration rule and Tokunaga

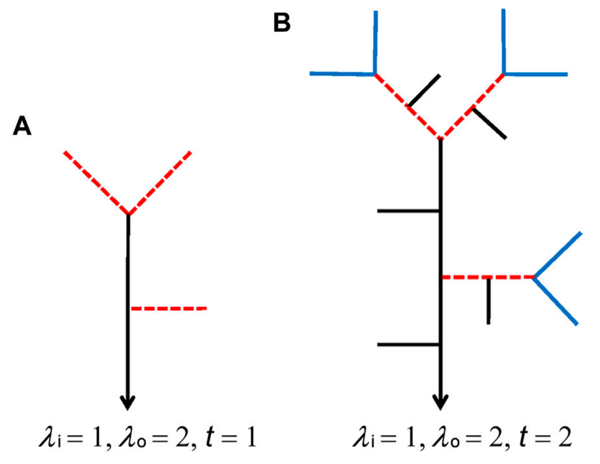

FIGURE 5 | Generation of sources in the iterative process. The sources at the second step are generated by the exterior links at the first step. The red dotted lines in both **(A)** and **(B)** are the exterior links of QU-IBTNs at the first step in **(A)**. The blue lines in **(B)** are the sources at the second step that are generated by the red dotted lines.

ordering method, there are five stream number generating laws that must be obeyed during the iterative process.

The components of the generating law equations are defined as follows:

$N_{t,\omega}$ is the number of streams with H-S order ω at the tth iterative step;
$N_{t,(\omega,\omega')}$ is the number of streams (ω,ω') at the tth iterative step;
M_t^{o} is the number of exterior links at the tth iterative step; and
M_t^{i} is the number of interior links at the tth iterative step.

Law 1: H-S Order Law

The H-S order of the QU-IBTN is $\Omega = t + 1$ after the tth iterative step, that is:

$$N_{t,\omega} = \sum_{\omega'=\omega}^{\Omega} N_{t,(\omega,\omega')} = \sum_{\omega'=\omega}^{t+1} N_{t,(\omega,\omega')} \quad (7)$$

Law 1 serves to replace the upper bound of the H-S order with the iterative step number in the calculations.

Law 2: Iteration Unchanging Law

The H-S orders of every stream and the entire network increase by 1 from the $(t-1)$th iterative step to the tth iterative step. Therefore, the number of streams (ω,ω') in the tth iterative step (i.e., $N_{t,(\omega,\omega')}$) equals the number of streams $(\omega-1,\omega'-1)$ in the $(t-1)^{\mathrm{th}}$ iterative step, that is:

$$N_{t,(\omega,\omega')} = N_{t-1,(\omega-1,\omega'-1)} \quad (8)$$

By recursion from the $(t-k)$th iterative step to the tth iterative step, the relationship between the numbers of streams is found to be:

$$N_{t,(\omega,\omega')} = N_{t-k,(\omega-k,\omega'-k)} \quad (9)$$

Law 2 ensures that the corresponding equality relationship between the number of streams in the different iterative steps is satisfied.

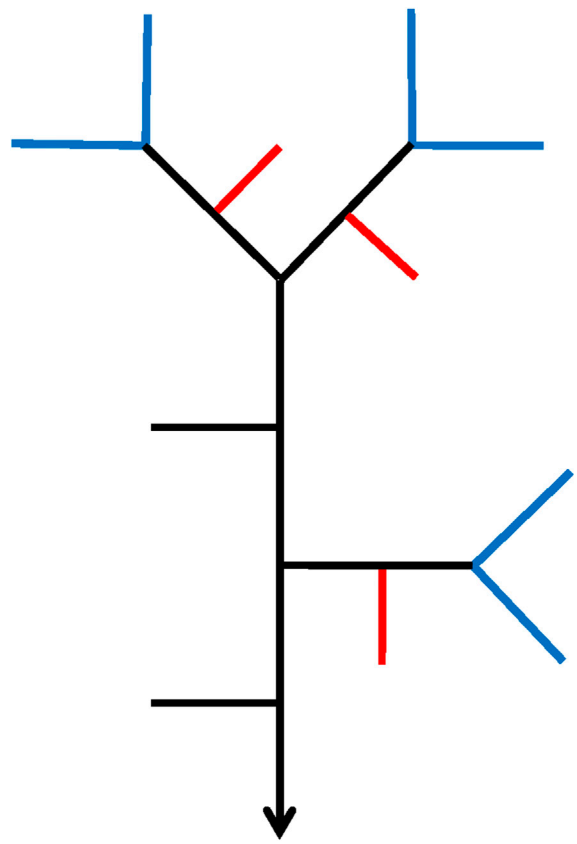

FIGURE 6 | The QU-IBTN generated by $\lambda_{\mathrm{i}} = 1$, $\lambda_{\mathrm{o}} = 2$, $t = 2$ at the second step. The red lines are the streams with Tokunaga order (1,2) and the blue lines are the sources, which are the streams with Tokunaga order (1,1).

Law 3: The Source Stream Law

Every pair of source streams with order $(1,1)$ at the tth iterative step comes from one exterior link at the $(t-1)$th iterative step, which is shown as an example of QU-IBTNs by the choices $\lambda_{\mathrm{i}} = 1$, and $\lambda_{\mathrm{o}} = 2$ in **Figure 5**.

Figure 5 shows the generation of sources in **Figure 5B** from exterior links in **Figure 5A**. Consequently, the relationship between the sources at the tth step and the exterior links in the $(t-1)$th step is found to be:

$$N_{t,(1,1)} = 2N_{t-1,1} = 2\sum_{\omega'=1}^{t} N_{t-1,(1,\omega')} \quad (10)$$

According to **Eq. 9** in Law 2 and **Eq. 10**, the number of streams with order (k,k) in the tth step can be determined by the number of exterior links at the $(t-k+1)$th step:

$$N_{t,(k,k)} = N_{t-k+1,(1,1)} \quad (11)$$

Law 4: The Neighbor-Ordered Side-branch Law

In every iterative step, the relationship between the number of streams with Tokunaga order (1,1) and order (1,2) that depend on the exterior generator λ_{o} is:

$$N_{t,(1,2)} = \frac{\lambda_{\mathrm{o}} - 1}{2} N_{t,(1,1)} \quad (12)$$

$\lambda_i = 1, \lambda_o = 2, t = 2$. The red lines are the three streams with Tokunaga order (1,2), and the black dotted lines are the six interior links.

$\lambda_i = 1, \lambda_o = 2, t = 3$. The blue dotted lines are the six streams with Tokunaga order (1,3) generated on the six black dotted lines.

FIGURE 7 | Streams with Tokunaga order (1,3) in QU-IBTNs are generated using $\lambda_i = 1$, $\lambda_o = 2$ from the second step in **(A)** to third iterative step in **(B)**.

Figure 6 shows a ratio of 3/6 for the number of streams with the order (1,2) and the number of streams with the order (1,1) in the QU-IBTN generated by the choices $\lambda_i = 1, \lambda_o = 2$ at the second iterative step.

According to **Eq. 9** in Law 2 and **Eq. 12**, the number of side branches that flow into streams that are 1 order greater (i.e., $(k, k+1)$) is:

$$N_{t,(k,k+1)} = N_{t-k+1,(1,2)} = \frac{\lambda_o - 1}{2} N_{t-k+1,(1,1)} \tag{13}$$

Law 5: The Greater-Ordered Side-branch Law

1) The streams with Tokunaga order (1,3) at the tth iterative step are generated by the $\frac{\lambda_o}{\lambda_o - 1} N_{t-1,(1,2)}$ interior links between the streams with Tokunaga order (1,2) at the $(t-1)$th iterative step. Additionally, each interior link produces λ_i streams with Tokunaga order (1,3). Therefore, the number of streams with Tokunaga order (1,3) is:

$$N_{t,(1,3)} = \left(\frac{\lambda_o}{\lambda_o - 1} N_{t-1,(1,2)}\right)\lambda_i = \frac{\lambda_i \lambda_o}{\lambda_o - 1} N_{t-1,(1,2)} \tag{14}$$

According to **Equation 9** and **14**, the number of side branches that flow into streams that are 2 orders greater [i.e., $(k, k+2)$] is:

$$N_{t,(k,k+2)} = N_{t-k+1,(1,3)} = \frac{\lambda_i \lambda_o}{\lambda_o - 1} N_{t-k,(1,2)} \tag{15}$$

Figure 7 shows the generation of streams with Tokunaga order (1,3) in the QU-IBTN with $\lambda_i = 1, \lambda_o = 2$, from the second to third iterative step.

2) The streams with Tokunaga order $(1, \omega' + 1), \omega' \geq 3$ at the tth iterative step are generated by the $(\frac{\lambda_i + 1}{\lambda_i} N_{t-1,(1,\omega')})$ interior links between streams with Tokunaga order $(1, \omega')$ at the $(t-1)$th iterative step. Each interior link produces λ_i streams with Tokunaga order $(1, \omega' + 1)$. Therefore, the number of streams with Tokunaga order $(1, \omega' + 1)$ is:

$$N_{t,(1,\omega'+1)} = \left(\frac{\lambda_i + 1}{\lambda_i} N_{t-1,(1,\omega')}\right)\lambda_i = (\lambda_i + 1) N_{t-1,(1,\omega')}, \omega' \geq 3 \tag{16}$$

Figure 8 shows the generation of streams with Tokunaga order (1,4) in the QU-IBTN with $\lambda_i = 1, \lambda_o = 2$ from the second to third iterative step.

According to **Equation 9** and **16**, the number of side branches that flow into streams that are ω' orders greater (i.e. $(k, k+\omega')$) is:

$$N_{t,(k,k+\omega')} = N_{t-k+1,(1,\omega'+1)} = (\lambda_i + 1) N_{t-k,(1,\omega')}, \ \omega' \geq 3 \tag{17}$$

Tokunaga Matrix N_t of IBTNs

During the tth iterative step, $(\lambda_o + 1)$ exterior links grow from the M^o_{t-1} exterior links and λ_i exterior links grow from the M^i_{t-1} interior links. Additionally, λ_o interior links grow from the M^o_{t-1} exterior links and $(\lambda_i + 1)$ interior links grow from the M^i_{t-1} interior links. **Figure 9** shows the numerical relationship of the exterior and interior links between the $(t-1)$th step and tth step.

Eq. 18 describes the relationship for increase of exterior and interior links between the $(t-1)$th and tth iterative step, illustrated in **Figure 9**, as:

$$\begin{cases} M^o_t = (\lambda_o + 1) M^o_{t-1} + \lambda_i M^i_{t-1} \\ M^i_t = \lambda_o M^o_{t-1} + (\lambda_i + 1) M^i_{t-1}, \qquad t \geq 2 \\ M^o_t = M^i_t + 1 \end{cases} \tag{18}$$

For the initial condition $t = 1$, we have $M^o_1 = \lambda_o + 1$, and $M^i_1 = \lambda_o$.

From **Eqs 10**, **Eqs 18**, it is clear that the number of exterior links at the $(t-1)$th step M^o_{t-1} and the number of streams with order (1,1) at the tth step $N_{t,(1,1)}$ are:

$$N_{t,(1,1)} = 2M^o_{t-1} = \left(2\lambda_o + 1 + \frac{\lambda_o - \lambda_i}{\lambda_o + \lambda_i}\right)(\lambda_o + \lambda_i + 1)^{t-1} + \frac{2\lambda_i}{\lambda_o + \lambda_i} \tag{19}$$

The Tokunaga matrix $\boldsymbol{N_t}$ of the QU-IBTN, which is generated by the interior generator λ_i and the exterior generator λ_o at the tth step, follows Law 1 through Law 5. Its diagonal elements, $\boldsymbol{N_t}(k, k)$, are calculated using **Eqs 11**, **18** in Law 3. The elements $\boldsymbol{N_t}(k, k+1)$ next to the diagonal elements come from **Eq. 13** in Law 4. The farther elements $\boldsymbol{N_t}(k, k+2)$ and $\boldsymbol{N_t}(k, k+\omega')$, in which $\omega' \geq 3$, are from **Eqs 15**, **17** in Law 5. The initial conditions are $N_{1,(1,1)} = 2$, $N_{1,(1,2)} = \lambda_o - 1$, and $N_{1,(2,2)} = 1$. The dimension of the matrix $\boldsymbol{N_t}$ is $\Omega \times \Omega$ (i.e., $(t+1) \times (t+1)$).

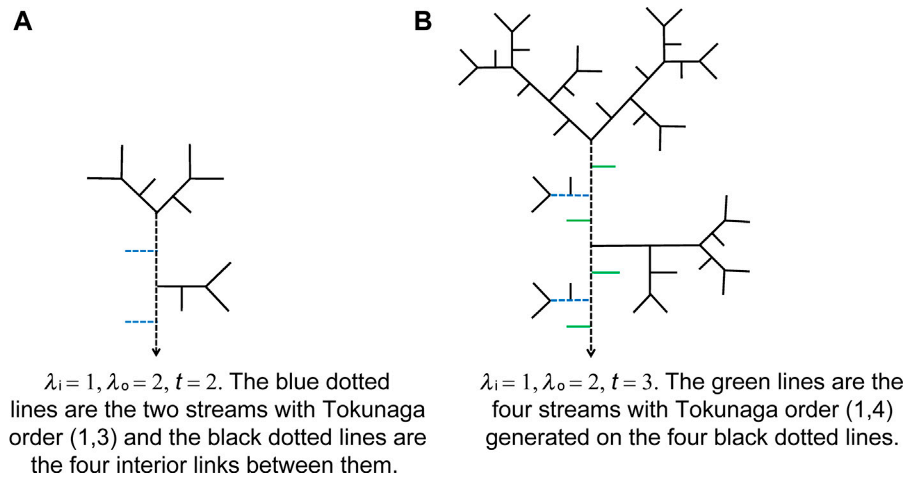

FIGURE 8 | Streams with Tokunaga order (1,4) in QU-IBTNs are generated using $\lambda_i = 1$, $\lambda_o = 2$ from the second step in **(A)** to third iterative step in **(B)**.

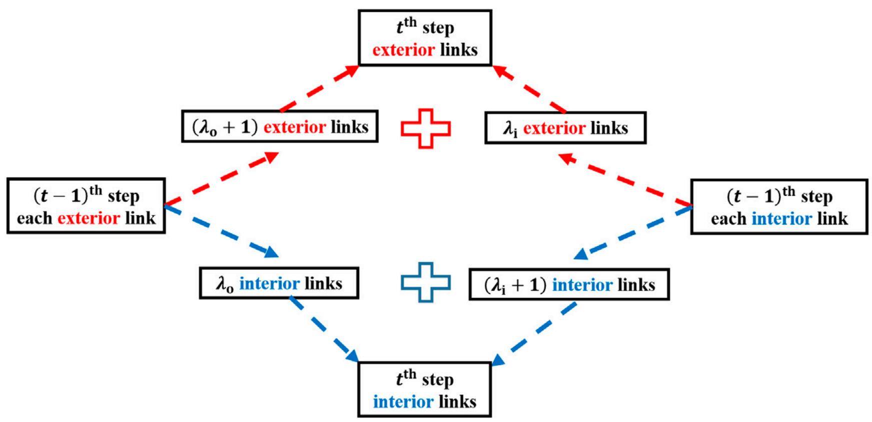

FIGURE 9 | The generation of each interior and exterior link from the $(t-1)$th to the tth iterative step.

Using the first row of $\boldsymbol{N_t}$ as an example, we find that:

$$
\begin{aligned}
&\left(N_{t,(1,1)} \quad N_{t,(1,2)} \quad N_{t,(1,3)} \quad N_{t,(1,4)} \quad N_{t,(1,5)} \cdots \right) \\
&= \left(N_{t,(1,1)} \quad \frac{\lambda_o - 1}{2} N_{t,(1,1)} \quad \frac{\lambda_i \lambda_o}{\lambda_o - 1} N_{t-1,(1,2)} \right. \\
&\quad \left. \times (\lambda_i + 1) N_{t-1,(1,3)} \quad (\lambda_i + 1) N_{t-1,(1,4)} \quad \cdots \right) \\
&= \left(N_{t,(1,1)} \quad \frac{\lambda_o - 1}{2} N_{t,(1,1)} \quad \frac{\lambda_i \lambda_o}{2} N_{t-1,(1,1)} \right. \\
&\quad \left. \times (\lambda_i + 1) \frac{\lambda_i \lambda_o}{\lambda_o - 1} N_{t-2,(1,2)} \quad (\lambda_i + 1)^2 N_{t-2,(1,3)} \cdots \right) \\
&= \left(N_{t,(1,1)} \quad \frac{\lambda_o - 1}{2} N_{t,(1,1)} \quad \frac{\lambda_i \lambda_o}{2} N_{t-1,(1,1)} \right. \\
&\quad \left. \times (\lambda_i + 1) \frac{\lambda_i \lambda_o^2}{2} N_{t-2,(1,1)} \, (\lambda_i + 1)^2 \frac{\lambda_i \lambda_o}{2} N_{t-3,(1,1)} \cdots \right)
\end{aligned}
\tag{20}
$$

In the following equations, $\hat{a} = \frac{\lambda_o - 1}{2}$, $\hat{b} = \frac{\lambda_i \lambda_o}{\lambda_o - 1}$, and $\hat{c} = \lambda_i + 1$. The matrix $\boldsymbol{N_t}$ is in the form of the general terms $m_{i,j}$ $(1 \le i, j \le t + 1 = \Omega)$ as follows:

In the first column of $\boldsymbol{N_t}$:

$$m_{1,1} = N_{t,(1,1)} \tag{21}$$

In the second column of $\boldsymbol{N_t}$:

$$m_{1,2} = \hat{a} m_{1,1}, \; m_{2,2} = N_{t-1,(1,1)} \tag{22}$$

In the third column of $\boldsymbol{N_t}$:

$$m_{1,3} = \hat{b} \hat{a} m_{2,2}, \; m_{2,3} = \hat{a} m_{2,2}, \; m_{3,3} = N_{t-2,(1,1)} \tag{23}$$

In the j^{th} column $(4 \le j \le t)$, the parameter $\hat{c}$ appears in the terms shown below:

$$
\begin{aligned}
m_{j,j} &= N_{t-j+1,(1,1)}, \; m_{j-1,j} = \hat{a} m_{j-1,j-1}, \\
m_{i,j} &= \hat{c}^{j-i-2} \hat{b} \hat{a} m_{j-1,j-1}, \; 1 \le i \le j - 2
\end{aligned}
\tag{24}
$$

In the $(t+1)^{th}$ column, the exterior generator λ_o affects the elements as follows:

$$
\begin{aligned}
m_{t+1,t+1} &= 1, \; m_{t,t+1} = \hat{a} m_{t,t} = \lambda_o - 1, \\
m_{i,t+1} &= \hat{c}^{t-1-i} \hat{b} \hat{a} m_{t,t} = \lambda_o \lambda_i \hat{c}^{t-1-i}, \; 1 \le i \le t - 1
\end{aligned}
\tag{25}
$$

The recursion of elements in **Eqs 21–25** form the matrix $\boldsymbol{N_t}$as:

$$\boldsymbol{N_t}=\begin{pmatrix} m_{1,1} & \hat{a}m_{1,1} & \hat{b}\hat{a}m_{2,2} & \hat{c}\,\hat{b}\hat{a}m_{3,3} & \cdots & \cdots & \hat{c}^{t-3}\hat{b}\hat{a}m_{t-1,t-1} & \hat{c}^{t-2}\hat{b}\hat{a}m_{t,t} \\ & m_{2,2} & \hat{a}m_{2,2} & \hat{b}\hat{a}m_{3,3} & \cdots & \cdots & \hat{c}^{t-4}\hat{b}\hat{a}m_{t-1,t-1} & \hat{c}^{t-3}\hat{b}\hat{a}m_{t,t} \\ & & m_{3,3} & \hat{a}m_{3,3} & \cdots & \cdots & \cdots & \cdots \\ & & & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & m_{t-2,t-2} & \hat{a}m_{t-2,t-2} & \hat{b}\hat{a}m_{t-1,t-1} & \hat{c}\hat{b}\hat{a}m_{t,t} \\ & & & & & m_{t-1,t-1} & \hat{a}m_{t-1,t-1} & \hat{b}\hat{a}m_{t,t} \\ & & & & & & m_{t,t} & \hat{a}m_{t,t} \\ & & & & & & & 1 \end{pmatrix} \tag{26}$$

The sum of the $(k+1)^{\text{th}}$ row in the matrix $\boldsymbol{N_t}$ is the number of streams with H-S order 1at the $(t-k)^{\text{th}}$ iterative step, which is also essentially the number of the exterior links. The number of exterior links at the $(t-k)^{\text{th}}$ iterative step is:

$$M_{t-k}^{\mathrm{o}}=\sum\nolimits_{j=k+1}^{t+1}\boldsymbol{N_t}(k+1,j) \tag{27}$$

Furthermore, the number of streams (1,1) at the $(t-k+1)^{\text{th}}$ iterative step (i.e., $N_{t-k+1,(1,1)}$) is twice the number of the exterior links at the $(t-k)^{\text{th}}$ iterative step (i.e., M_{t-k}^{o}) for any $1\le k\le t-1$ according to **Eq. 10** in Law 3. Therefore:

$$\frac{N_{t-k+1,(1,1)}}{M_{t-k}^{\mathrm{o}}}=\frac{m_{k,k}}{M_{t-k}^{\mathrm{o}}}=\frac{m_{k,k}}{\sum_{j=k+1}^{t+1}\boldsymbol{N_t}(k+1,j)}=2,\ 1\le k\le t-1 \tag{28}$$

Side-Branching Ratio Matrix $\boldsymbol{T_t}$ of IBTNs

The side-branching ratio matrix $\boldsymbol{T_t}$ is composed of the side-branching ratio $T_{i,k+1}$ as its element $\boldsymbol{T_t}(i,k)$. According to the definition of $T_{i,k+1}$ in **Eq. 3**, the expression for $T_{i,k+1}$ is:

$$\boldsymbol{T_t}(i,k)=T_{i,k+1}=\frac{\boldsymbol{N_t}(i,k+1)}{\sum_{j=k+1}^{t+1}\boldsymbol{N_t}(k+1,j)}=\frac{\boldsymbol{N_t}(i,k+1)}{m_{k,k}/2} \tag{29}$$

We can get the expression for $\boldsymbol{N_t}(i,k+1)$ in $\boldsymbol{N_t}$ as:

$$\boldsymbol{N_t}(i,k+1)=\begin{cases}\hat{a}m_{k,k}, & i=k\\ \hat{c}^{k-i-1}\hat{b}\hat{a}m_{k,k}, & i\le k-1\end{cases} \tag{30}$$

We modify the form of $\boldsymbol{N_t}(i,k+1)$ in **Eq. 29** using **Eq. 30**; therefore, the final form for $\boldsymbol{T_t}(i,k)$ is:

$$\boldsymbol{T_t}(i,k)=\begin{cases}2\hat{a}, & i=k\\ 2\hat{c}^{k-i-1}\hat{b}\hat{a}, & i\le k-1\end{cases} \tag{31}$$

By combining **Eq. 3** and the condition $\boldsymbol{N_t}(t+1,t+1)=1$ in the t^{th} column of the matrix $\boldsymbol{T_t}$, we get:

$$\boldsymbol{T_t}(i,t)=T_{i,t+1}=\frac{\boldsymbol{N_t}(i,t+1)}{\boldsymbol{N_t}(t+1,t+1)}=\boldsymbol{N_t}(i,t+1) \tag{32}$$

Using **Eqs 31**, **32**, we get the matrix $\boldsymbol{T_t}$ as follows:

$$\boldsymbol{T_t}=\begin{pmatrix} \lambda_{\mathrm{o}}-1 & \lambda_{\mathrm{i}}\lambda_{\mathrm{o}} & \lambda_{\mathrm{i}}\lambda_{\mathrm{o}}(\lambda_{\mathrm{i}}+1) & \cdots & \lambda_{\mathrm{i}}\lambda_{\mathrm{o}}(\lambda_{\mathrm{i}}+1)^{t-3} & \lambda_{\mathrm{o}}\lambda_{\mathrm{i}}(\lambda_{\mathrm{i}}+1)^{t-2} \\ & \lambda_{\mathrm{o}}-1 & \lambda_{\mathrm{i}}\lambda_{\mathrm{o}} & \cdots & \lambda_{\mathrm{i}}\lambda_{\mathrm{o}}(\lambda_{\mathrm{i}}+1)^{t-4} & \lambda_{\mathrm{o}}\lambda_{\mathrm{i}}(\lambda_{\mathrm{i}}+1)^{t-3} \\ & & \cdots & \cdots & \cdots & \cdots \\ & & & \lambda_{\mathrm{o}}-1 & \lambda_{\mathrm{i}}\lambda_{\mathrm{o}} & \lambda_{\mathrm{o}}\lambda_{\mathrm{i}}(\lambda_{\mathrm{i}}+1) \\ & & & \cdot & \lambda_{\mathrm{o}}-1 & \lambda_{\mathrm{o}}\lambda_{\mathrm{i}} \\ & & & \cdot & & \lambda_{\mathrm{o}}-1 \\ & & & \cdot & & \end{pmatrix} \tag{33}$$

The efficient and necessary condition to be a self-similar network is that $\boldsymbol{T_t}$ must to be a Toeplitz matrix according to **Eq. 5**. The QU-IBTNs with the interior generator λ_{i} and exterior generator λ_{o} are definitely self-similar because the elements on the diagonal are equal in **Eq. 33**, which is a Toeplitz matrix. This is also shown in terms of the results for side-branching ratios:

$$T_1=\lambda_{\mathrm{o}}-1,\ T_k=\lambda_{\mathrm{i}}\lambda_{\mathrm{o}}(\lambda_{\mathrm{i}}+1)^{k-2},\ 2\le k\le t \tag{34}$$

For strict Tokunaga self-similarity, the elements in $\boldsymbol{T_t}$ must satisfy **Eq. 6**. Therefore, the necessary condition for a QU-IBTN to be a Tokunaga tree is:

$$\lambda_{\mathrm{o}}=\lambda_{\mathrm{i}}+1 \tag{35}$$

No matter which values of λ_{o} and λ_{i} are selected to generate a QU-IBTN, this QU-IBTN must be self-similar. However, a QU-IBTN is a Tokunaga tree only when $\lambda_{\mathrm{o}}=\lambda_{\mathrm{i}}+1$. This means that when we use QU-IBTNs to simulate natural river networks, which are Tokunaga trees, we need to generate a QU-IBTN with the special condition that $\lambda_{\mathrm{o}}=\lambda_{\mathrm{i}}+1$.

To demonstrate the constraints on self-similarity versus Tokunaga self-similarity, we provide the following two examples.

Example 1: $\lambda_o=\lambda_i$

When the generators $\lambda_{\mathrm{o}}=\lambda_{\mathrm{i}}$, the side-branching ratios given by **Eq. 34** are:

$$T_1=\lambda_{\mathrm{o}}-1, T_k=\lambda_{\mathrm{o}}^2(\lambda_{\mathrm{o}}+1)^{k-2},\ 2\le k\le t \tag{36}$$

The side-branching ratio T_k $(1\le k)$ contains different constants, but does not define a geometric series. Therefore, the QU-IBTN with $\lambda_{\mathrm{o}}=\lambda_{\mathrm{i}}$ is self-similar but not a Tokunaga tree, as pointed out by Peckham (1995a).

Example 2: $\lambda_o=\lambda_i+1$

When the exterior link generator and the interior link generator satisfy the condition $\lambda_{\mathrm{i}}=\lambda_{\mathrm{o}}-1$, the side-branching ratios given by **Eq. 34** are:

$$T_k=(\lambda_{\mathrm{o}}-1)\lambda_{\mathrm{o}}^{k-1},\ k\ge 1 \tag{37}$$

According to **Eq. 37**, this QU-IBTN is a Tokunaga tree with $a=\lambda_{\mathrm{o}}-1$ and $c=\lambda_{\mathrm{o}}$. Based on statistics, the necessary condition for a QU-IBTN to be a Tokunaga tree in **Eq. 37** is consistent with the result presented by Veitzer and Gupta (2000).

RESULTS

Natural River Networks

In the following, we use the Yellow River in China (H-S order 11) and the Amazon River in South America (H-S order 12) as examples to verify whether or not they follow the rules of QU-IBTNs and the constraint of a Tokunaga tree. **Figure 10** shows the river networks extracted from digital elevation model (DEM) data with a 30 m resolution (Li et al., 2018; Li et al., 2020).

Supplementary Table S1 (shown in Supplementary Material) shows the Tokunaga matrices of the stream numbers $N_{\omega,\omega'}$ for both the Yellow River and the Amazon River.

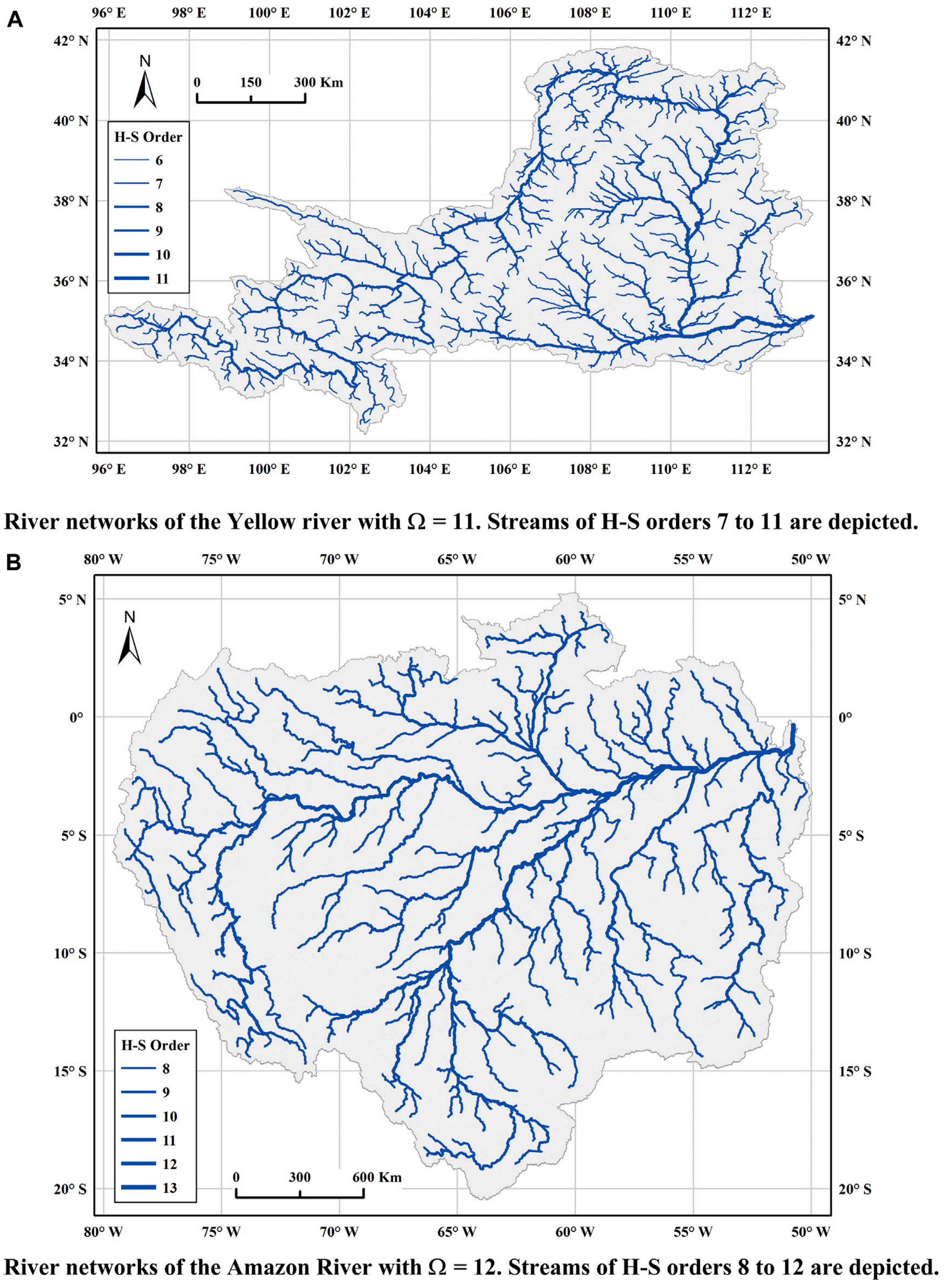

FIGURE 10 | River networks of the Yellow River and the Amazon River with streams of no less than an H-S order 7 are shown in the figure.

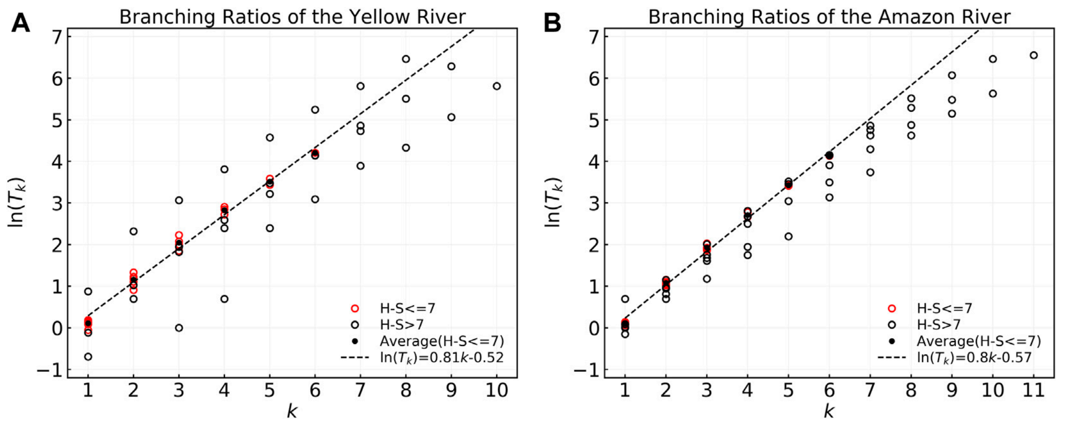

FIGURE 11 | Branching ratios, T_k, of the Yellow River and theAmazon River for every k. The black circles are the branching ratios shown in the shaded sections of the matrices of **Supplementary Table S2**. The red circles are the remaining data in the matrices. The black points are the average values of the black circles. The dotted black line is the fit of the black points using the least squares regression method.

The similarities Between Natural River Networks and Tokunaga Trees

Supplementary Table S2 (shown in Supplementary Material) shows the side-branching ratio matrices for the side-branching ratios $T_{i,j}$ based on **Eq. 3** for the two rivers.

Supplementary Table S2 also shows that the branching ratio values have large fluctuations for main stems with large H-S orders, as shown in **Figure 11**.

The statistical side-branching ratio matrices of the two rivers in **Supplementary Table S2** and **Figure 11** show that the statistical results of T_k vary greatly compared with the theoretic self-similarity derivation, which is uniform for a fixed k. The black circles in **Figure 11** denote the T_k data in the shaded sections of **Supplementary Table S2** that are less affected by local geomorphology and terrain, and therefore are more concentrated in their distribution. The red circles in **Figure 11** are the T_k data excluding the shaded sections of **Supplementary Table S2**; these are seen to scatter far away from each other. We now use streams with H-S orders 1 to 7 (i.e., the shaded sections of **Supplementary Table S2** and the black circles in **Figure 11**) to analyze the Tokunaga self-similarity of the two natural river networks. The values of the black points in **Figure 11**, which are the statistical average values, $\overline{T_k}$, of T_k for each k $(1 \le k \le 6)$, are (1.11, 3.19, 7.76, 16.91, 33.53, 67.09) for the Yellow River and (1.11, 2.93, 6.90, 14.97, 30.86, 62.41) for the Amazon River.

The side-branching ratio series in **Figure 11** were verified to satisfy as $1.34 \times 2.25^{k-1}$ and $1.26 \times 2.23^{k-1}$ using the least squares method, as shown in **Figure 11** in terms of the dotted black lines with coefficients of determination $R^2 > 0.99$.

Therefore, the Tokunaga parameters can be evaluated using $a_{\text{Yellow}} \cong 1.34$, $c_{\text{Yellow}} \cong 2.25$ and $a_{\text{Amazon}} \cong 1.26$, $c_{\text{Amazon}} \cong 2.23$.

The Similarities Between Natural River Networks and QU-IBTNs

We construct $\boldsymbol{N\_R_t}$ matrices with the same dimension of $\boldsymbol{N_t}$ in **Supplementary Table S1** in terms of:

$$\boldsymbol{N\_R_t}(k,k+1) = \frac{\boldsymbol{N_t}\ (k,k+1)}{\boldsymbol{N_t}\ (k,k)},\ 1 \le k \le \Omega - 1 \quad (38)$$

$$\boldsymbol{N\_R_t}(k,k+2) = \frac{\boldsymbol{N_t}\ (k,k+2)}{\boldsymbol{N_t}\ (k+1,k+2)},\ 1 \le k \le \Omega - 2 \quad (39)$$

and

$$\boldsymbol{N\_R_t}(i,k) = \frac{\boldsymbol{N_t}\ (i,k)}{\boldsymbol{N_t}\ (i+1,k)},\ 1 \le i \le k-3,\ 4 \le k \le \Omega \quad (40)$$

The $\boldsymbol{N\_R_t}$ matrices for the Yellow River and the Amazon River using **Eqs 38–40** are listed in **Supplementary Table S3** (shown in Supplementary Material).

According to the standard form of the Tokunaga matrix $\boldsymbol{N_t}$ in **Eq. 26**, $\hat{a}$ values are expressed in **Eq. 38** by the ratios of $\boldsymbol{N_t}\ (k,k+1)$ to $\boldsymbol{N_t}\ (k,k)$ $(1 \le k \le \Omega - 1)$; $\hat{b}$ values are expressed in **Eq. 39** by the ratios of $\boldsymbol{N_t}\ (k,k+2)$ to $\boldsymbol{N_t}\ (k+1,k+2)$ $(1 \le k \le \Omega - 2)$; and $\hat{c}$ values are expressed in **Eq. 40** by the ratios of $\boldsymbol{N_t}(i,k)$ to $\boldsymbol{N_t}(i+1,k)$ $(1 \le i \le k-3,\ \ 4 \le k \le \Omega)$.

We calculate λ_{o} and λ_{i} for each iterative step from every row of the matrix $\boldsymbol{N\_R_t}$ in **Supplementary Table S3**, as shown in the Methods section. The matrices of the λ_{o} and λ_{i} values are defined as $\boldsymbol{N\_\lambda_t}$ in **Supplementary Table S4** (shown in Supplementary Material).

In the $\boldsymbol{N\_\lambda_t}$ matrices in **Supplementary Table S4**, the first element in each row is the value of the exterior generator λ_{o} for each iterative step, and the following elements in the row are the interior generators λ_{i} for each corresponding iterative step. **Figure 12** shows the values of the generators λ_{o} and λ_{i} for each iterative step in **Supplementary Table S4** for both the Yellow River and the Amazon River.

1) The exterior generators λ_{o} (i.e., the gray solid points) vary slightly for the higher iterative steps (greater than 6) in **Supplementary Table S4** and **Figure 12**. At the higher iterative steps, the streams are closer to the source streams, which allows them to evolve freely according to the same formative mechanism because they have enough space to

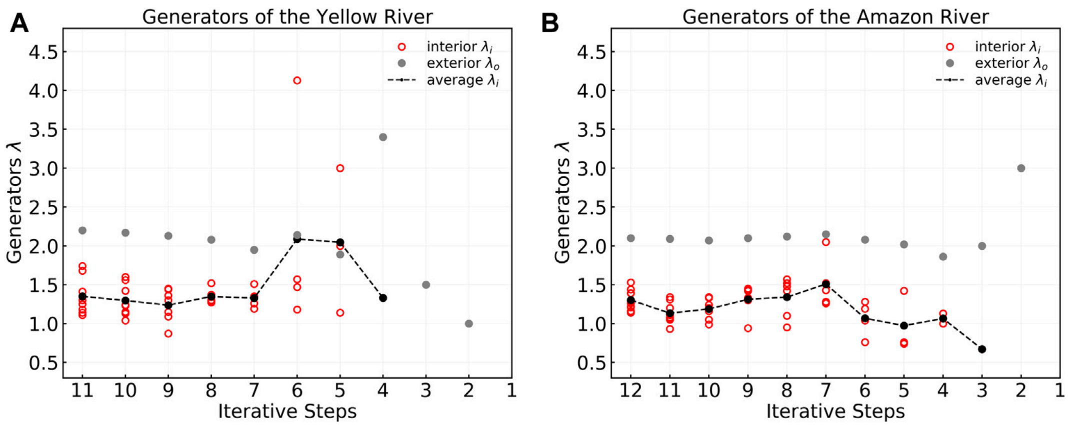

FIGURE 12 | Generators of the Yellow River and the Amazon River for each iterative step. The gray solid points are the exterior generators λ_o. The black circles are the interior generators λ_i. The dotted black line corresponds to the average value of the interior generators λ_i for each row, excluding those in the last column of **Supplementary Table S4**. Analysis and Conclusions for **Supplementary Table S4** and **Figure 12**.

grow. Freedom and space allows streams between different iterative steps in a river network, and even for different river networks, to be uniform and similar, which can be seen in terms of the uniformity of the generators λ_o.

2) The interior generators λ_i (i.e., the black circles) are concentrated and approaching uniform for the higher iterative steps in **Supplementary Table S4** and **Figure 12**. The last column in the Yellow River N_λ_t matrix in **Supplementary Table S4** shows the interior generators for the main stem of the Yellow River basin. The changes in this column are results from the geomorphology and terrain, which constrain the generators on the main stems. The Amazon River N_λ_t matrix also shows the same changes in this column. For the lower iterative steps, the generators shown in **Supplementary Table S4** are generators for streams with high H-S orders, which are also influenced by the geomorphology and terrain. To satisfy the conditions of uniformity of generators and similarity of the river networks, we need to exclude the streams that are heavily constrained by the geomorphology and terrain.
3) The interior generators λ_i (i.e., the black circles) vary slightly at iterative steps 7 to 11 for the Yellow River, and steps 8 to 12 for the Amazon River, as shown in **Supplementary Table S4** and **Figure 12**. The average values for each iterative step (i.e., the dotted black line) are almost stable between iterative steps 7 and 11 for the Yellow River and steps 8 and 12 for the Amazon River. The exterior generators λ_o are consistent at these iterative steps. The stability and consistency of the generators λ_o and λ_i at high iterative steps confirms that the two natural river networks follow the rules of QU-IBTNs in a statistical sense. We use the average values of λ_o and λ_i in the shaded section in **Supplementary Table S4** from iterative steps 7–11 and 8–12 separately to evaluate the generator of the Yellow River and the Amazon River. The calculations for the generators are provided in the Methods section. The statistical averages for the exterior generator and the interior generator are evaluated using $\lambda_{o-\text{Yellow}} \cong 2.11$, $\lambda_{i-\text{Yellow}} \cong 1.34$ for the Yellow River and $\lambda_{o-\text{Amazon}} \cong 2.10$, $\lambda_{i-\text{Amazon}} \cong 1.26$ for the Amazon River.

TABLE 1 | The generators (λ_i, λ_o) and Tokunaga parameters (a,c) for the Yellow River and the Amazon River.

Rivers	(λ_i, λ_o)	(a, c)	$\lambda_o - \lambda_i$	c − a
Yellow River	(1.34, 2.11)	(1.33, 2.25)	0.77	0.91
Amazon River	(1.26, 2.10)	(1.26, 2.23)	0.84	0.97

Table 1 lists the generators and Tokunaga parameters for the Yellow River and the Amazon River.

Analysis and conclusions for **Table 1**:

1) In our analysis, we have removed the main stems with high H-S orders at the low iterative steps, as these are controlled by the local geomorphology and terrain. The QU-IBTN rules and Tokunaga self-similarity are well demonstrated using the Yellow River and the Amazon River in terms of the uniformity and equality of the generators and branching ratios for streams with low H-S orders at high iterative steps.
2) According to the sufficient and necessary condition for a QU-IBTN to be a Tokunaga tree in **Eq. 35**, we should have $a = \lambda_i$, $c = \lambda_o$ and $\lambda_o - \lambda_i = c - a = 1$. However, from **Table 1**, there is a difference between c and λ_o because of the different data and methods used for calculation.

DISCUSSION

The QU-IBTNs proposed above illustrate how to generate iterative binary tree networks simulating natural river networks. The complete mathematic iterative steps with graphic deduction are demonstrated in iterative orders. The iterative process is sustained by the self-similarity theory. The synthetic QU-IBTNs are close to the natural river networks in similarity characteristics parameters by two examples.

When generated by exterior generator $\lambda_o = 2$ and interior generator $\lambda_i = 1$, the QU-IBTN is equivalent to the Shreve model (Shreve, 1966), for which the bifurcation ratio is $R_B = 4$ and the Tokunaga parameters are $a = 1$, $c = 2$ (Peckham, 1995a). This is the only case of a plane-filling IBTN (Zhang et al., 2009). Therefore, only the Shreve model can generate a plane-filling Tokunaga tree, and only the IBTN plane-filling Tokunaga tree fits the Shreve model.

Although in theory the Shreve model should be the most ideal topology network that approaches natural river networks because it is a Tokunaga tree and plane-filling, there is an obvious contradiction that topological parameters such as $R_B = 4$, $a = 1$, and $c = 2$ of the Shreve model are far from those of natural river networks [Average values of R_B, a and c are around 4.4, 1.1 and 2.5 (Peckham, 1995a; Dodds and Rothman, 1999; Dodds, 2000; Pelletier and Turcotte, 2000; Mantilla et al., 2010; Zanardo et al., 2013; Wang et al., 2016)]. Furthermore, the entire natural river network is determined not only by topological characteristics (such as stream number, R_B and Tokunaga parameters), but also geometrical characteristics (such as stream length and confluence angle) that depend on the energy conservation and the relation between sediment and water discharge. Consequently, the Shreve model is an ideal model to simulate the topology of river networks, but it is an unsatisfactory model for simulating the geometrical properties of natural river networks. Furthermore, a layout for an ideal plane-filling Tokunaga tree is still needed.

We have supposed that the hypothesis of the uniformity of link length in the IBTN may lead to differences in the bifurcation ratio as compared with natural river networks under the condition of plane-filling. The link length and confluence angles of IBTNs should be redefined according to the values of the stream length ratios of natural river networks when considering plane-filling.

Some aspects of the linkages between the structure of river networks and the processes that shape them remain somewhat unclear and difficult to understand (Abrahams, 1984; Montgomery and Dietrich, 1992; Perron et al., 2012; Seybold et al., 2017). A considerable body of research is available on the topologic structure associated with stream numbers and self-similarity of river networks (Peckham, 1995a; Dodds, 2000; Pelletier and Turcotte, 2000; Veitzer and Gupta, 2000; Mcconnell and Gupta, 2008; Mantilla et al., 2010; Zanardo et al., 2013). Recently, confluence angles (also referred to as junction angles) of a number of natural river networks, which represent a geometric component of river network structure, have been statistically evaluated and shown to depend on the climatic setting (Devauchelle et al., 2012; Seybold et al., 2017). These treatments of confluence angles may provide important clues in further studies to explore the following two questions: 1) where does a stream emerge from an unchannelized region? and 2) how far downstream does this stream extend until it merges into another stream?

CONCLUSION

In this paper we provide a complete mathematical and graphical deduction of QU-IBTNs with specified generator series and iteration rule. Our conclusions are as follows:

1) For the QU-IBTN generated by the generator series and iteration rule in this paper, five intrinsic stream number laws—which determine the distribution of source streams and side-branches following into streams of greater orders—are graphically and recursively analyzed and satisfied.
2) As defined in this paper, the QU-IBTN are demonstrated to be self-similar.
3) The sufficient and necessary constraint for a QU-IBTN to be a Tokunaga tree is that the exterior links must be replaced with a neighboring generator in the generator series that is larger than the interior links during the iterative process. This defines a generation method for a simulated network that is identical to a natural river network in topology.
4) Two natural river networks, i.e. the Yellow River, China and the Amazon River, South America, are shown to be Tokunaga trees and QU-IBTNs within specified H-S order scales.

The self-similarity of river networks is a classical topic, and there are many researchers working on this topic using varied methods, including mathematicians who are good at fractal theory (Kovchegov and Zaliapin, 2020). Other researchers might use random and mathematical foundations in their own research. Our manuscript here is an intuitive, accessible and understandable initial tool to measure the self-similarity of river networks. The advance of generator series, the integer iterative rules showing directly by graphs, and the corresponding mathematical derivations are new and different from others' methods. The QU-IBTNs are built just in a few finite iterative steps as shown in the figures. QU-IBTNs in random scales will be generated by coding in the future. However, to some extent we are doing the same thing as other researchers-just find a way to demonstrate the self-similarity and Tokunaga property of river networks. This is what we think we can share with other researchers, and help make progress in river network simulations in the future.

METHODS

The Method for Calculating λ_o and λ_i From the Matrix N_R_t

For the k^{th} row in matrix $\boldsymbol{N\_R_t}$, $\Omega - k + 1$ is the corresponding iterative step. The matrix $\boldsymbol{N\_\lambda_t}$ expresses the generators λ_o and λ_i. The exterior generator at the $(\Omega - k + 1)^{th}$ iterative step is:

$$\begin{aligned}\lambda_o(\Omega - k + 1) &= \boldsymbol{N\_\lambda_t}(k, k+1) \\ &= 2 \times \boldsymbol{N\_R_t}(k, k+1) + 1,\ 1 \le k \le \Omega - 1 \qquad \text{(M1)}\end{aligned}$$

The interior generators in the k^{th} row (i.e., the $(\Omega - k + 1)^{th}$ iterative step) are:

$$\lambda_{\text{i\_1}}(\Omega-k+1) = \boldsymbol{N\_\lambda_t}(k,k+2) = \frac{\lambda_{\text{o}}(k)-1}{\lambda_{\text{o}}(k)} \times \boldsymbol{N\_R_t}(k,k+2),\ 1\le k\le \Omega-2 \quad \text{(M2)}$$

and

$$\boldsymbol{N\_\lambda_t}(k,i) = \frac{\lambda_{\text{i\_1}}(\Omega-k+1)}{\lambda_{\text{i\_1}}(\Omega-k)} \times \boldsymbol{N\_R_t}(k,i),\ 1\le k\le i-3,\ 4\le i\le \Omega \quad \text{(M3)}$$

In the matrix $\boldsymbol{N\_R_t}$, the elements $\boldsymbol{N\_\lambda_t}(k,k+1)$ $(1\le k\le \Omega-1)$ in Equation (M1) correspond to the exterior generator at the $(\Omega-k+1)$th iterative step. The elements $\boldsymbol{N\_\lambda_t}(k,k+2)$ and $\boldsymbol{N\_\lambda_t}(k,i)$ $(k+3\le i\le \Omega)$ are the interior generators at the $(\Omega-k+1)$th iterative step.

For the Yellow River and the Amazon River, the statistically averaged values for $\lambda_{\text{o-Yellow}}$ and $\lambda_{\text{o-Amazon}}$ are:

$$\lambda_{\text{o-Yellow}} = \frac{\sum_{k=1}^{5}\lambda_{\text{o}}(12-k)}{5} = \frac{\sum_{k=1}^{5}\boldsymbol{N}_{\lambda t}(k,k+1)}{5} = 2.11 \quad \text{(M4)}$$

and

$$\lambda_{\text{o-Amazon}} = \frac{\sum_{k=1}^{5}\lambda_{\text{o}}(13-k)}{5} = \frac{\sum_{k=1}^{5}\boldsymbol{N}_{\lambda t}(k,k+1)}{5} = 1.34 \quad \text{(M5)}$$

For the Yellow River and the Amazon River, the statistically averaged values for $\lambda_{\text{i-Yellow}}$ and $\lambda_{\text{i-Amazon}}$ are:

$$\lambda_{\text{i-Yellow}} = \frac{\sum_{k=1}^{5}\sum_{i=k+2}^{10}\boldsymbol{N}_{\lambda t}(k,i)}{5} = 2.10 \quad \text{(M6)}$$

and

$$\lambda_{\text{i-Amazon}} = \frac{\sum_{k=1}^{5}\sum_{i=k+2}^{11}\boldsymbol{N}_{\lambda t}(k,i)}{5} = 1.26 \quad \text{(M7)}$$

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/**Supplementary Material**, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

LZ, the corresponding author, did all the mathematics derivations. KYW organized the manuscript. TJL did the derivations and provided the data. XL, BYG, and GXC collected the data and plot figures. JHW and YFH processed the data and revised the manuscript.

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SUPPLEMENTARY MATERIAL

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fenvs.2021.792289/full#supplementary-material

Supplementary Table S1 | The Tokunaga matrices $\boldsymbol{N_t}$ for the Yellow River and the Amazon River.

Supplementary Table S2 | The side-branching ratio matrices $\boldsymbol{T_t}$ for the Yellow River and the Amazon River.

Supplementary Table S3 | The $\boldsymbol{N\_R_t}$ matrices for the Yellow River and the Amazon River.

Supplementary Table S4 | The $\boldsymbol{N\_\lambda_t}$ matrices of generators λ_{o} and λ_{i} for the Yellow River and the Amazon River.

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GLOSSARY

ω H-S order [-]

N_ω the number of streams with H-S order ω [-]

$R_\mathbf{B}$ bifurcation ratio (i.e. the ratio of the number of streams [-])

(ω, ω') a stream as a side-branching tributary with H-S order ω that flows into a stream ordered ω', assigned by a Tokunaga order (ω, ω') [-]

$N_{\omega,\omega'}$ the number of streams with Tokunaga order (ω, ω') [-]

$\boldsymbol{N}$ Tokunaga stream number matrix composed by $N_{\omega,\omega'}$ [-]

$T_{\omega,\omega+k}$ the ratio of the number of streams with the H-S order ω that flow into streams with the order $(\omega+k)$ to the number of streams with the H-S order $(\omega+k)$ [-]

$\boldsymbol{T}$ Tokunaga side-branching ratio matrix composed by $T_{\omega,\omega+k}$ [-]

T_k the side-branching ratio [-]

a the average number of streams with the H-S order ω that flow into streams ordered $\omega+1$[-]

c the average increasing rate of the side-branching ratio of side tributaries with a different order [-]

λ, $\lambda_\mathbf{o}, \lambda_\mathbf{i}$ generator, exterior generator, interior generator [-]

t iterative steps [-]

$M_t^\mathbf{o}, M_t^\mathbf{i}$ the number of exterior links and interior links at the tth iteration step [-]

$N_{t,(\omega,\omega')}, N_{t,\omega}$ the number of streams with Tokunaga order (ω, ω') and H-S order ω at the tth iterative step [-]

$\boldsymbol{N_t}$, $\boldsymbol{T_t}$ Tokunaga matrix and side-branching ratio matrix of the QU-IBTN, which is generated by the interior generator λ_i and the exterior generator λ_o at the tth step [-]

N_R_t, N_λ_t a transformation of $\boldsymbol{N_t}$; a transformation of $\boldsymbol{N\_R_t}$ and represents the $\boldsymbol{\lambda_o}$ and λ_i values [-]