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State estimation for dynamic systems with higher-order autoregressive moving average non-Gaussian noise

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The classical Kalman filter is a very important state estimation approach, which has been widely used in many engineering applications. The Kalman filter is optimal for linear dynamic systems with independent Gaussian noises. However, the independence and Gaussian assumptions may not be satisfied in practice. On the one hand, modeling physical systems usually results in discrete-time state-space models with correlated process and measurement noises. On the other hand, the noise is non-Gaussian when the system is disturbed by heavy-tailed noise. In this case, the performance of the Kalman filter will deteriorate, or even diverge. This paper is devoted to addressing the state estimation problem of linear dynamic systems with high-order autoregressive moving average (ARMA) non-Gaussian noise. First, a triplet Markov model is introduced to model the system with high-order ARMA noise, since this model relaxes the independence assumption of the hidden Markov model. Then, a new filter is derived based on correntropy, instead of the commonly used minimum mean square error (MMSE), to deal with non-Gaussian noise. Unlike the MMSE, which uses only second-order statistics of error, correntropy can capture second-order and higher-order statistics. Finally, simulation results verify the effectiveness of the proposed algorithm.

KEYWORDS

kalman filter, higher-order autoregressive moving average, non-Gaussian, triplet markov model, correntropy

1 Introduction

State estimation is a very important problem in many engineering applications, such as energy internet, system control, tracking, and so on [Zandavi and Chung (2019); Zhang et al. (2022)]. These engineering applications are essentially a dynamic system, which is usually described as a state-space model. The hidden Markov model (HMM) is the one of the most commonly used state-space models (Zhang et al. (2018)). For linear case, the state estimation problem is generally solved by the well-known Kalman filter(KF) (Kalman (1960)), which is an optimal filter in the minimum mean square error (MMSE) sense. MMSE is one of the most commonly used cost function in the case of Gaussian noise, and MMSE approach is an estimator which minimizes the mean square error. In addition, a large number of nonlinear filters have been proposed to solve nonlinear estimation problems, such as extended Kalman filter, unscented Kalman filter, cubature Kalman filter, particle filter, to name but a few (Anderson and Moore (2012)).

Although the KF in general performs well, it has rigorous requirements, that is, the process and measurement noises of dynamic systems are independent and Gaussian. However, the independence and Gaussian assumptions do not always hold in practice (Zhang G. et al. (2021)). On one hand, in fact, the noise of most dynamic systems is correlated. Research has shown that modeling physical systems usually results in discrete-time state-space models with correlated process and measurement noises, and some practical applications are explained in (Saha and Gustafsson (2012)). In addition, the dynamic and measurement noise may even high-order (i.e., multi-step) correlated in some severe environments (Zhang D. et al. (2021)). On the other hand, the main reason for using the Gaussian assumption is that it is mathematically simple, but in fact, dynamic systems are usually disturbed by some heavy-tailed impulse noise (Roth et al. (2013)). When the independence and Gaussian assumptions are not satisfied, the KF may fail to output reliable estimation results.

To deal with correlated noise, the traditional method is to reconstruct an HMM by prewhitening processing, and then the classical KF can be used to estimate the state (Bar-Shalom et al. (2001)). Another solution is to characterize dynamic systems with correlated noise through more flexible state-space models, such as the pairwise Markov model (PMM) (Pieczynski and Desbouvries (2003)) and the triplet Markov model (TMM) (Ait-El-Fquih and Desbouvries (2006)). In the PMM, the state and measurement as a whole are regarded as a Markov process, which improves the modeling ability of complex dynamic systems. In the TMM, an auxiliary variable is introduced to completely describe the dynamic systems. This auxiliary variable can play a very significant role in some engineering applications. For example, it can characterize the uncertainty of parameters, non-stationarity and error sources. It has been proved that the TMM is more general than the HMM and the PMM, and it structural advantages make it more preferable in addressing some real-world applications, such as image segmentation (Derrode and Pieczynski (2004)), speech processing (Ait El Fquih and Desbouvries (2005)), target tracking (Zhang et al. (2017); Lehmann and Pieczynski (2020, 2021)), and so on.

For non-Gaussian noise, several approaches have been proposed, which are mainly divided into three categories (Izanloo et al. (2016)). The first is to replace the Gaussian distribution with a more extensive heavy-tailed distribution (Huang et al. (2017)). For example, the Student's t distribution is one of the most commonly used heavy-tailed distribution. The main disadvantage of heavy-tailed distributions is that they are usually analytically difficult, which brings about that related estimation approaches have no closed form solution. The second is the multiple model technique. In this approach, non-Gaussian noise is represented as a finite sum of Gaussian distribution (Shan et al. (2021)). The main difficulty of this approach is how to design the model set reasonably, and the disadvantage is that the amount of calculation will increase sharply with the increase of the number of models. The third is the Monte Carlo approach, in which a set of weighted random particles are employed to characterize the state (Liu et al. (2018)). Generally, sampling-based algorithms can be categorized into deterministic sampling method and random sampling method. Particularly, in the random sampling method, enough particles can approximate the real state with arbitrary precision, at the cost of expensive computation.

In the past few years, the correntropy-based filtering technology has become an important orientation to solve the state estimation of dynamic systems with non-Gaussian noise (Kulikova (2017); Chen et al. (2017)). In information theory, correntropy is a significant mathematical tool to measure the similarity of two random variables. Unlike the commonly utilized MMSE cost function, which uses only second-order statistic of error, the correntropy captures second-order and higher-order information, and is more suitable for non-Gaussian noise, such as heavy-tailed impulsive noise. Several filtering algorithms based on correntropy have been designed in the framework of HMMs, and they are more robust to non-Gaussian noise than the KF and its variants.

In this paper, we are devoted to addressing the state estimation problem of linear dynamic systems with high-order autoregressive moving average (ARMA) non-Gaussian noise. A new Kalman-like filter is developed in the framework of the TMM based on correntropy. First, we resort to a linear TMM to describe dynamic systems with high-order ARMA noise, since the TMM is more general than the HMM. Second, based on the model, a new Kalman-like filter is derived by using correntropy cost function, instead of the commonly used MMSE cost function. Because correntropy can capture not only second-order but also higher-order statistics of error, the proposed algorithm is more robust to non-Gaussian noise than the traditional filter. Finally, simulation results show the effectiveness of the proposed algorithm.

The rest of the paper is organized as follows. **Section 2** is the modeling of linear dynamic systems with high-order ARMA noise. **Section 3** derives a new Kalman filter by using correntropy cost function in the framework of the TMM. In **Section 4**, we validate the proposed algorithm *via* simulations. Finally, conclusion is provided in **Section 5**.

2 Modeling of linear dynamic systems with high-order ARMA noise

2.1 Linear hidden markov model

Consider the following linear dynamic system

$$\begin{cases} x_{k+1} = F_k x_k + G_k w_k \\ z_k = H_k x_k + v_k \end{cases} \tag{1}$$

where k is the time index, $x_k \in \mathbb{R}^{n_x}$ is the state vector of dimension n_x, F_k the transition matrix, G_k is the process noise matrix, $w_k \sim \mathcal{N}(0, Q_k)$ is the process noise, $z_k \in \mathbb{R}^{n_z}$ is the measurement vector of dimension n_z, H_k is the measurement matrix, and $v_k \sim \mathcal{N}(0, R_k)$ is the measurement noise. $\mathcal{N}(m, P)$ denotes a Gaussian distribution with mean vector m and covariance matrix P.

In general, noise sequences $w = \{w_k\}_{k\in \mathrm{IN}}$ and $v = \{v_k\}_{k\in \mathrm{IN}}$ are assumed independent, jointly independent and independent of the initial state $x_0 \sim \mathcal{N}(\hat{x}_0, P_0)$. Then the state estimate can be obtained by the classical KF, which is an optimal filter in the MMSE sense. However, the independence and Gaussian assumptions that are typically assumed in the HMM do not always hold in practice, such as dynamic systems with high-order ARMA non-Gaussian noise. In this case, the KF may not output reliable estimation results.

2.2 Linear triplet markov model for dynamic systems with high-order ARMA noise

2.2.1 Linear triplet markov chain model

We resort to a linear TMM to describe a linear HMM with correlated noise. Let $x_k \in \mathbb{R}^{n_x}$ is the state vector, $z_k \in \mathbb{R}^{n_z}$ is the measurement vector, $r_k \in \mathbb{R}^{n_r}$ is an auxiliary variable, and $\zeta_k = \left[x_k^{\mathrm{T}}, r_k^{\mathrm{T}}, z_{k-1}^{\mathrm{T}}\right]^{\mathrm{T}}$. If $\zeta = \{\zeta_k\}_{k\in \mathrm{IN}}$ is a Markov process, the following system is called a linear TMM (Ait-El-Fquih and Desbouvries (2006)):

$$\underbrace{\begin{bmatrix} x_{k+1} \\ r_{k+1} \\ z_k \end{bmatrix}}_{\zeta_{k+1}} = \underbrace{\begin{bmatrix} \mathcal{F}_k^{xx} & \mathcal{F}_k^{xr} & \mathcal{F}_k^{xz} \\ \mathcal{F}_k^{rx} & \mathcal{F}_k^{rr} & \mathcal{F}_k^{rz} \\ \mathcal{F}_k^{zx} & \mathcal{F}_k^{zr} & \mathcal{F}_k^{zz} \end{bmatrix}}_{\mathcal{F}_k} \begin{bmatrix} x_k \\ r_k \\ z_{k-1} \end{bmatrix} + \underbrace{\begin{bmatrix} \xi_k^x \\ \xi_k^r \\ \xi_k^z \end{bmatrix}}_{\xi_k} \tag{2}$$

where $\xi = \{\xi_k\}_{k\in \mathrm{IN}}$ is zero mean white noise and independent of the initial state ζ_0.

2.2.2 Modeling high-order ARMA noise using TMM

In this section, we utilize a linear TMM to model dynamic systems with high-order ARMA noise (Zhang D. et al. (2021)). The TMM provides a general framework for these typical stochastic systems.

(1) High-Order ARMA Process Noise.

For high-order ARMA process noise, it can usually be written in the form of the following difference equation:

$$w_k = -\sum_{i=1}^{p^w} \alpha_i^w w_{k-i} + \sum_{i=0}^{q^w} \beta_i^w \xi_{k-i}^w \tag{3}$$

where ξ_k^w is white noise. Model (3) is a typical high-order ARMA model, in which p^w is the autoregressive order, q^w is the moving average order, and coefficient parameters α_i^w and β_i^w are determined by the spectral factor $H_k^w(z)$ of the power spectral density $\Phi_k^w(z)$ of the process noise w_k.

Suppose $\Phi_k^w(z)$ is a rational spectrum. According to the spectral decomposition theorem, there is a spectral factor satisfying

$$\Phi_k^w(z) = H_k^w(z) H_k^w(z)^* \tag{4}$$

where $(\cdot)^*$ represents complex conjugate transpose operation, and $H_k^w(z)$ can be written by

$$H_k^w(z) = C_k^w \left(zI - A_k^w\right)^{-1} B_k^w + D_k^w \tag{5}$$

Then, high-order ARMA process noise can be formulated by

$$\begin{cases} x_{k+1}^w = A_k^w x_k^w + B_k^w \xi_k^w \\ w_k = C_k^w x_k^w + D_k^w \xi_k^w \end{cases} \tag{6}$$

If the process noise in model 1) is high-order ARMA noise, it can be described by model (6). In this case, $\{x_k\}$ is no longer a Markov process, but $\left\{\left(x_k, x_k^w\right)\right\}$ is a Markov process. Let $\zeta_k = \left(x_k, r_k = x_k^w, z_{k-1}\right)$. Model 1) with (6) can be written in the form of linear TMM (2), i.e.,

$$\begin{bmatrix} x_{k+1} \\ x_{k+1}^w \\ z_k \end{bmatrix} = \begin{bmatrix} F_k & G_k C_k^w & 0 \\ 0 & A_k^w & 0 \\ H_k & 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_k^w \\ z_{k-1} \end{bmatrix} + \begin{bmatrix} G_k D_k^w \xi_k^w \\ B_k^w \xi_k^w \\ v_k \end{bmatrix} \tag{7}$$

Assuming white noise $\xi_k^w \sim \mathcal{N}(0, Q_k)$, the noise covariance matrix of model 7) is

$$\mathcal{Q}_k = \begin{bmatrix} G_k D_k^w Q_k \left(D_k^w\right)^{\mathrm{T}} G_k^{\mathrm{T}} & G_k D_k^w Q_k \left(B_k^w\right)^{\mathrm{T}} & 0 \\ B_k^w Q_k \left(D_k^w\right)^{\mathrm{T}} G_k^{\mathrm{T}} & B_k^w Q_k \left(B_k^w\right)^{\mathrm{T}} & 0 \\ 0 & 0 & R_k \end{bmatrix} \tag{8}$$

(2) High-order ARMA Measurement Noise.

For high-order ARMA measurement noise, it can also be written in the following form of difference equation

$$v_k = -\sum_{i=1}^{p^v} a_i^v v_{k-i} + \sum_{i=0}^{q^v} b_i^v \xi_{k-i}^v \tag{9}$$

where ξ_k^v is white noise, p^v and q^v are autoregressive order and moving average order, respectively. If the spectral density of high-order ARMA measurement noise is $\Phi_k^v(z)$ and the corresponding spectral factor is $H_k^v(z)$, similar to model (6), v_k can be modeled as follows

$$\begin{cases} x_{k+1}^v = A_k^v x_k^v + B_k^v \xi_k^v \\ v_k = C_k^v x_k^v + D_k^v \xi_k^v \end{cases} \tag{10}$$

If the measurement noise in model 1) is high-order ARMA noise, it can be described by model (10). Let $\zeta_k = (x_k, r_k = x_k^v, z_{k-1})$. Model 1) with (10) can be written in the form of linear TMM (2), i.e.,

$$\begin{bmatrix} x_{k+1} \\ x_{k+1}^v \\ z_k \end{bmatrix} = \begin{bmatrix} F_k & 0 & 0 \\ 0 & A_k^v & 0 \\ H_k & C_k^v & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_k^v \\ z_{k-1} \end{bmatrix} + \begin{bmatrix} G_k w_k \\ B_k^v \xi_k^v \\ D_k^v \xi_k^v \end{bmatrix} \tag{11}$$

Assuming white noise $\xi_k^v \sim \mathcal{N}(0, R_k)$, the noise covariance matrix of model 11) is

$$\mathcal{Q}_k = \begin{bmatrix} G_k Q_k G_k^{\mathrm{T}} & 0 & 0 \\ 0 & B_k^v R_k (B_k^v)^{\mathrm{T}} & B_k^v R_k (D_k^v)^{\mathrm{T}} \\ 0 & D_k^v R_k (B_k^v)^{\mathrm{T}} & D_k^v R_k (D_k^v)^{\mathrm{T}} \end{bmatrix} \tag{12}$$

(3) High-Order ARMA Process and Measurement Noises.

If the process noise and measurement noise are high-order ARMA noises, they can be described by model 6) and model (10), respectively. Let $\zeta_k = (x_k, r_k = (x_k^w, x_k^v), z_{k-1})$. Model 1) with (6) and (10) can be written in the form of linear TMM (2), i.e.,

$$\begin{bmatrix} x_{k+1} \\ x_{k+1}^w \\ x_{k+1}^v \\ z_k \end{bmatrix} = \begin{bmatrix} F_k & G_k C_k^w & 0 & 0 \\ 0 & A_k^w & 0 & 0 \\ 0 & 0 & A_k^v & 0 \\ H_k & 0 & C_k^v & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_k^w \\ x_k^v \\ z_{k-1} \end{bmatrix} + \begin{bmatrix} G_k D_k^w \xi_k^w \\ B_k^w \xi_k^w \\ B_k^v \xi_k^v \\ D_k^v \xi_k^v \end{bmatrix} \tag{13}$$

Assuming white noises $\xi_k^w \sim \mathcal{N}(0, Q_k)$ and $\xi_k^v \sim \mathcal{N}(0, R_k)$, the noise covariance matrix of model 13) is

$$\mathcal{Q}_k = \begin{bmatrix} G_k D_k^w Q_k (D_k^w)^{\mathrm{T}} G_k^{\mathrm{T}} & G_k D_k^w Q_k (B_k^w)^{\mathrm{T}} & 0 & 0 \\ B_k^w Q_k (D_k^w)^{\mathrm{T}} G_k^{\mathrm{T}} & B_k^w Q_k (B_k^w)^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & B_k^v R_k (B_k^v)^{\mathrm{T}} & B_k^v R_k (D_k^v)^{\mathrm{T}} \\ 0 & 0 & D_k^v R_k (B_k^v)^{\mathrm{T}} & D_k^v R_k (D_k^v)^{\mathrm{T}} \end{bmatrix} \tag{14}$$

2.3 Restoration algorithm

Let $x_k^* = (x_k, r_k)$. Then model 2) can be written as

$$\begin{bmatrix} x_{k+1}^* \\ z_k \end{bmatrix} = \begin{bmatrix} \mathcal{F}_k^{x^*x^*} & \mathcal{F}_k^{x^*z} \\ \mathcal{F}_k^{zx^*} & \mathcal{F}_k^{zz} \end{bmatrix} \begin{bmatrix} x_k^* \\ z_{k-1} \end{bmatrix} + \begin{bmatrix} \xi_k^{x^*} \\ \xi_k^z \end{bmatrix} \tag{15}$$

where the initial state x_0^* and noise ξ_k are

$$x_0^* \sim \mathcal{N}(\hat{x}_0^*, P_0^*), \ \xi_k \sim \mathcal{N}\left(0, \underbrace{\begin{bmatrix} \mathcal{Q}_k^{x^*x^*} & \mathcal{Q}_k^{x^*z} \\ \mathcal{Q}_k^{zx^*} & \mathcal{Q}_k^{zz} \end{bmatrix}}_{\mathcal{Q}_k}\right) \tag{16}$$

For model 15) with (16), a Kalman-like filter, called triplet Kalman filter (TKF), has been derived to estimate the state x_k^*. For convenience, the recursive equations are summarized as follows (Ait-El-Fquih and Desbouvries (2006)).

Initialization:

$$\hat{x}_{0|0}^* = \hat{x}_0^*, \ P_{0|0} = P_0^* \tag{17}$$

$$\hat{\mathcal{F}}_k^{x^*x^*} = \mathcal{F}_k^{x^*x^*} - \mathcal{Q}_k^{x^*z} \left(\mathcal{Q}_k^{zz}\right)^{-1} \mathcal{F}_k^{zx^*} \tag{18}$$

$$\hat{\mathcal{F}}_k^{x^*z} = \mathcal{F}_k^{x^*z} - \mathcal{Q}_k^{x^*z} \left(\mathcal{Q}_k^{zz}\right)^{-1} \mathcal{F}_k^{zz} \tag{19}$$

$$\hat{\mathcal{Q}}_k^{x^*x^*} = \mathcal{Q}_k^{x^*x^*} - \mathcal{Q}_k^{x^*z} \left(\mathcal{Q}_k^{zz}\right)^{-1} \mathcal{Q}_k^{zx^*} \tag{20}$$

Prediction:

$$\hat{x}_{k|k-1}^* = \hat{\mathcal{F}}_{k-1}^{x^*x^*} \hat{x}_{k-1|k-1}^* + \mathcal{Q}_{k-1}^{x^*z} \left(\mathcal{Q}_{k-1}^{zz}\right)^{-1} z_{k-1} + \hat{\mathcal{F}}_{k-1}^{x^*z} z_{k-2} \tag{21}$$

$$P_{k|k-1}^* = \hat{\mathcal{F}}_{k-1}^{x^*x^*} P_{k-1|k-1}^* \left(\hat{\mathcal{F}}_{k-1}^{x^*x^*}\right)^{\mathrm{T}} + \hat{\mathcal{Q}}_{k-1}^{x^*x^*} \tag{22}$$

Update:

$$e_k = z_k - \mathcal{F}_{k-1}^{zx^*} \hat{x}_{k|k-1}^* - \mathcal{F}_{k-1}^{zz} z_{k-1} \tag{23}$$

$$R_{e,k} = \mathcal{F}_{k-1}^{zx^*} P_{k|k-1}^* \left(\mathcal{F}_{k-1}^{zx^*}\right)^{\mathrm{T}} + \mathcal{Q}_{k-1}^{zz} \tag{24}$$

$$K_k = P_{k|k-1}^* \left(\mathcal{F}_{k-1}^{zx^*}\right)^{\mathrm{T}} R_{e,k}^{-1} \tag{25}$$

$$\hat{x}_{k|k}^* = \hat{x}_{k|k-1}^* + K_k e_k \tag{26}$$

$$P_{k|k}^* = \left(I - K_k \mathcal{F}_k^{zx^*}\right) P_{k|k-1}^* \tag{27}$$

The TKF is also an optimal filter in the MMSE sense. It in general performs well in Gaussian noise. However, it performance will deteriorate or even diverge when applied to non-Gaussian systems, since the TKF is derived under MMSE criterion, which only uses second-order statistics of error. To solve this problem, in the next section, a new filter is developed by using correntropy cost function, which utilizes not only second-order but also higher-order statistics information.

3 Correntropy-based triplet kalman filter

3.1 Correntropy

Correntropy is a very useful metric tool to measure the similarity of two random variables in information theory

(Chen et al. (2017)). For variables X and Y, the correntropy is defined by

$$C(X,Y) = \mathrm{E}[\kappa(X,Y)] = \int \kappa(x,y)\, df_{XY}(x,y) \tag{28}$$

where $\mathrm{E}[\cdot]$ is an expectation operator, $\kappa(\cdot,\cdot)$ is a kernel function, and $f_{XY}(x,y)$ is the joint probability density function of X and Y. Generally, $f_{XY}(x,y)$ is unknown, and only a limited amount of data is provided. Thus, the correntropy can be computed by

$$\hat{C}(X,Y) = \frac{1}{N}\sum_{i=1}^{N} \kappa(x,y). \tag{29}$$

There are many options for kernel function. In this paper, we choose the Gaussian kernel function

$$\kappa(x,y) = \mathrm{G}_\sigma(x_i - y_i), \tag{30}$$

where σ is the kernel size, and $\mathrm{G}_\sigma(x_i - y_i) = \exp\left(-\frac{\|x_i - y_i\|^2}{2\sigma^2}\right)$. The Gaussian kernel function is positive definite and bounded. When $X = Y$, it takes the maximum value.

For the Gaussian kernel function, its Taylor series expansion can be written as

$$C(X,Y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \sigma^{2n} n!} \mathrm{E}\left[(X-Y)^{2n}\right]. \tag{31}$$

It can be seen that the correntropy is in essence the weighted sum of all even-order moments of error. Compared with the MMSE, which uses only the second-order statistics of error, correntropy captures the second-order and higher-order statistics.

3.2 Main results

In this section, a new filter, called correntropy-based TKF (CTKF), is derived by using correntropy under TMM. For clarity, we first provide the main results, and then give the mathematical derivation.

The initialization and prediction steps of the CTKF are the same as those of the TKF, and its update step is summarized as follows:

$$e_k = z_k - \mathcal{F}^{zx^*}_{k-1}\hat{x}^*_{k|k-1} - \mathcal{F}^{zz}_{k-1} z_{k-1} \tag{32}$$

$$\lambda_k = \mathrm{G}_\sigma\left(\|e_k\|_{(Q_k^{zz})^{-1}}\right) \tag{33}$$

$$R_{e,k} = \mathcal{F}^{zx^*}_{k-1} P^*_{k|k-1} \left(\mathcal{F}^{zx^*}_{k-1}\right)^{\mathrm{T}} + Q^{zz}_{k-1} \tag{34}$$

$$K^\lambda_k = \lambda_k P^*_{k|k-1} \left(\mathcal{F}^{zx^*}_{k-1}\right)^{\mathrm{T}} R^{-1}_{e,k} \tag{35}$$

$$\hat{x}^*_{k|k} = \hat{x}^*_{k|k-1} + K^\lambda_k e_k \tag{36}$$

$$P^*_{k|k} = \left(I - K^\lambda_k \mathcal{F}^{zx^*}_k\right) P^*_{k|k-1} \tag{37}$$

Proof. For the linear TMM (2), we have

$$\begin{bmatrix} \hat{x}^*_{k|k-1} \\ z_k \end{bmatrix} = \begin{bmatrix} I \\ \mathcal{F}^{zx^*}_k \end{bmatrix} x^*_k + \begin{bmatrix} 0 \\ \mathcal{F}^{zz}_k \end{bmatrix} z_{k-1} + \eta_k \tag{38}$$

where I and 0 are identity and zeros matrices, and

$$\eta_k = \begin{bmatrix} -\left(x^*_k - \hat{x}^*_{k|k-1}\right) \\ w^z_k \end{bmatrix} \text{ with } \mathrm{E}\left[\eta_k \eta_k^{\mathrm{T}}\right] = \begin{bmatrix} P^*_{k|k-1} & 0 \\ 0 & Q^{zz}_k \end{bmatrix} \tag{39}$$

To address non-Gaussian noise, we use correntropy instead of MMSE to derive update equations. The cost function based on correntropy is established by

$$\begin{aligned} J\left(x^*_k\right) = {} & \mathrm{G}_\sigma\left(\left\|z_k - \mathcal{F}^{zx^*}_k x^*_k - \mathcal{F}^{zz}_k z_{k-1}\right\|_{(Q_k^{zz})^{-1}}\right) \\ & + \mathrm{G}_\sigma\left(\left\|x^*_k - \hat{x}^*_{k|k-1}\right\|_{(P^*_{k|k-1})^{-1}}\right) \end{aligned} \tag{40}$$

Then the optimal estimation of x^*_k is $\hat{x}^*_k = \arg\max_{x^*_k} J\left(x^*_k\right)$, which can be obtained by

$$\begin{aligned} \frac{\partial J\left(x^*_k\right)}{\partial x^*_k} = {} & \frac{1}{\sigma^2}\mathrm{G}_\sigma\left(\left\|z_k - \mathcal{F}^{zx^*}_k x^*_k - \mathcal{F}^{zz}_k z_{k-1}\right\|_{(Q_k^{zz})^{-1}}\right) \\ & \left(\mathcal{F}^{zx^*}_k\right)^{\mathrm{T}} \left(Q^{zz}_k\right)^{-1} \left(z_k - \mathcal{F}^{zx^*}_k x^*_k - \mathcal{F}^{zz}_k z_{k-1}\right) \\ & - \frac{1}{\sigma^2}\mathrm{G}_\sigma\left(\left\|x_k - \hat{x}^*_{k|k-1}\right\|_{(P^*_{k|k-1})^{-1}}\right)\left(P^*_{k|k-1}\right)^{-1} \\ & \left(x_k - \hat{x}^*_{k|k-1}\right) \\ = {} & 0. \end{aligned} \tag{41}$$

Equation 41 can be written by

$$\begin{aligned} \Psi_k x^*_k = {} & \left(P^*_{k|k-1}\right)^{-1} \hat{x}^*_{k|k-1} + \lambda_k \left(\mathcal{F}^{zx^*}_k\right)^{\mathrm{T}} \left(Q^{zz}_k\right)^{-1} \\ & \left(z_k - \mathcal{F}^{zz}_k z_{k-1}\right) \end{aligned} \tag{42}$$

where

$$\Psi_k = \left(P^*_{k|k-1}\right)^{-1} + \lambda_k \left(\mathcal{F}^{zx^*}_k\right)^{\mathrm{T}} \left(Q^{zz}_k\right)^{-1} \mathcal{F}^{zx^*}_k, \tag{43}$$

$$\lambda_k = \frac{\mathrm{G}_\sigma\left(\left\|z_k - \mathcal{F}^{zx^*}_k x^*_k - \mathcal{F}^{zz}_k z_{k-1}\right\|_{(Q_k^{zz})^{-1}}\right)}{\mathrm{G}_\sigma\left(\left\|x^*_k - \hat{x}^*_{k|k-1}\right\|_{(P^*_{k|k-1})^{-1}}\right)}. \tag{44}$$

Adding and subtracting a term $\lambda_k \left(\mathcal{F}^{zx^*}_k\right)^{\mathrm{T}} \left(Q^{zz}_k\right)^{-1} \mathcal{F}^{zx^*}_k \hat{x}^*_{k|k-1}$ on the right-hand side of (42), we have

$$\begin{aligned} \Psi_k x^*_k = {} & \Psi_k \hat{x}^*_{k|k-1} + \lambda_k \left(\mathcal{F}^{zx^*}_k\right)^{\mathrm{T}} \left(Q^{zz}_k\right)^{-1} \\ & \left(z_k - \mathcal{F}^{zx^*}_k \hat{x}^*_{k|k-1} - \mathcal{F}^{zz}_k z_{k-1}\right). \end{aligned} \tag{45}$$

Thus, the estimation of x^*_k can be computed by

$$\hat{x}^*_{k|k} = \hat{x}^*_{k|k-1} + K^\lambda_k \left(z_k - \mathcal{F}^{zx^*}_k \hat{x}^*_{k|k-1} - \mathcal{F}^{zz}_k z_{k-1}\right) \tag{46}$$

where

$$K^\lambda_k = \Psi_k^{-1} \lambda_k \left(\mathcal{F}^{zx^*}_k\right)^{\mathrm{T}} \left(Q^{zz}_k\right)^{-1} \tag{47}$$

Note that the parameter λ_k is function of x_k^*. For simplicity, let $x_k^* \approx \hat{x}_{k|k-1}^*$ in (44), and λ_k can be obtained by

$$\lambda_k = G_\sigma\left(\left\| z_k - \mathcal{F}_k^{zx^*} \hat{x}_{k|k-1}^* - \mathcal{F}_k^{zz} z_{k-1} \right\|_{\left(Q_k^{zz}\right)^{-1}} \right) \tag{48}$$

In addition, parameter σ plays an important role in correntropy-based filters. Inspired by (Kulikova (2017)), this paper adopts an adaptive method to choose σ, i.e.,

$$\sigma = \left\| z_k - \mathcal{F}_k^{zx^*} \hat{x}_{k|k-1}^* - \mathcal{F}_k^{zz} z_{k-1} \right\|_{\left(Q_k^{zz}\right)^{-1}} \tag{49}$$

In this section, a CTKF is developed to address the estimation problem of dynamic systems with high-order ARMA non-Gaussian noise. Instead of the commonly used MMSE criterion, which uses only second-order statistics of error, correntropy is employed to derive the filter, since it can captures second-order and higher-order statistics of error. It can be seen that the structure of CTKF is similar to that of TKF, except that an extra scale parameter λ_k is involved. The scale parameter is computed according to correntropy criterion to control the gain matrix K_k^λ, which results in that the CTKF in general performs well for non-Gaussian noise. In addition, the CTKF has a simple form, which facilitates its practical application.

4 Numerical simulations

In this section, two scenarios, i.e., dynamic system with high-order ARMA Gaussian and non-Gaussian noise, are taken into account to verify the effectiveness of the TMM and CTKF. In model (1), the state is $x_k = \left[p_{x,k}, v_{x,k}, p_{y,k}, v_{y,k}\right]^T$, and relevant

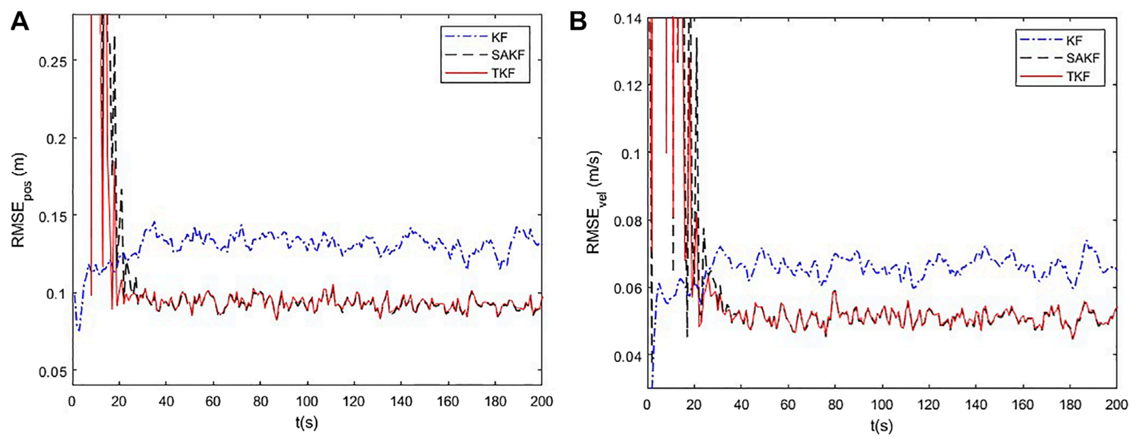

FIGURE 1
RMSE results for different filters. **(A)** Position RMSE. **(B)** Velocity RMSE.

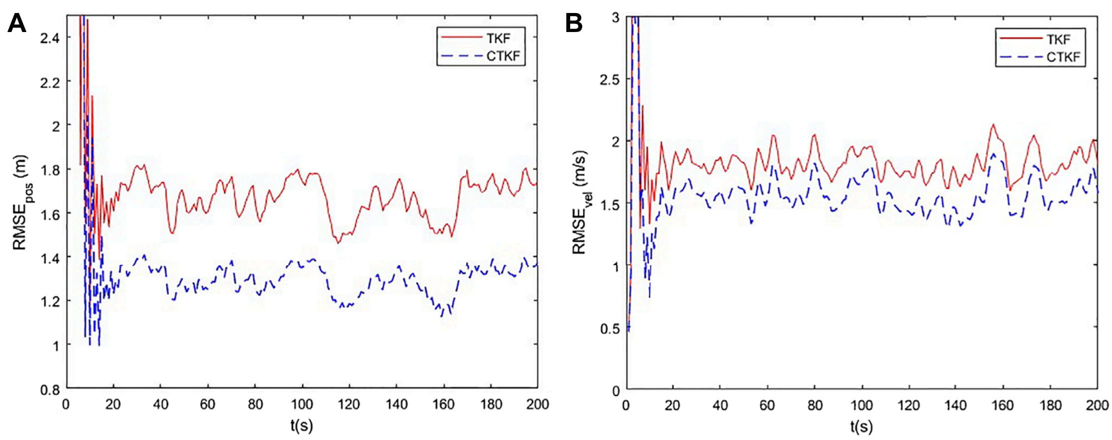

FIGURE 2
RMSE results for different filters. **(A)** Position RMSE. **(B)** Velocity RMSE.

matrices are

$$F_k = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, G_k = \begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix},$$

$$H_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{50}$$

where $T = 1$ is the sampling period. The spectral factors of process and measurement noises are

$$H^w(z) = \frac{z^4 - 0.4z^3 + 0.9z^2 - 0.1z - 0.3}{z^4 - z^3 + 0.5z^2 + 0.2z - 0.4} \tag{51}$$

$$H^v(z) = \frac{z^6 + 0.6z^5 + 0.4z^4 + 0.3z^3 - 0.08z^2 + 0.05z + 0.01}{z^6 + 0.8z^5 + 0.6z^4 + 0.2z^3 - 0.09z^2 - 0.08z + 0.01} \tag{52}$$

Case 1: $\xi_k^w \sim \mathcal{N}(0, Q_k)$ and $\xi_k^v \sim \mathcal{N}(0, R_k)$ are Gaussian noises, where $Q_k = \operatorname{diag}(0.01^2, 0.01^2)$ and $R_k = \operatorname{diag}(0.1^2, 0.1^2)$. For comparison, the standard Kalman filter (KF), the traditional state augmented Kalman filter (SAKF) (Bar-Shalom et al. (2001)), and the triplet Kalman filter (TKF) are tested. Besides, the root mean square error (RMSE) is used to evaluate estimation performance, which is computed by

$$\text{RMSE} = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(\left(x_k^i - \hat{x}_k^i\right)^2 + \left(y_k^i - \hat{y}_k^i\right)^2\right)} \tag{53}$$

where x_k^i and y_k^i are the true values at time k in the ith Monte Carlo trail, and $\hat{x}_k^i$ and $\hat{y}_k^i$ are the corresponding estimation values. The number of Monte Carlo trails is $M = 200$.

Position and velocity RMSE results are provided in **Figure 1**. It can be seen that the TKF and SAKF have similar estimation performance, and are better than the standard KF. High-order ARMA process and measurement noises do not meet the independence assumption, resulting in poor estimation performance of the KF. The TKF and SAKF are essentially equivalent, and they are optimal in the MMSE sense. The former models dynamic system with high-order ARMA noise through TMM, and the latter deals with high-order ARMA noise through prewhitening technique. Simulation results show that TMM can accurately model dynamic systems with high-order ARMA noise.

Case 2: $\xi_k^w \sim \mathcal{N}(0, Q_k)$ and $\xi_k^v \sim \mathcal{N}(0, R_k)$ are Gaussian noise disturbed by shot noise with probability of 0.2, where $\mathrm{Q_k}$ and $\mathrm{R_k}$ are the same as those in case 1, and the shot noise is generated by $0.1 \times \text{randi}([5, 10])$. Symbol randi([a, b]) denotes that an integer is returned from the uniform distribution of $[a, b]$.

For comparison, the TKF and the proposed CTKF are tested. Position and velocity RMSE results are provided in **Figure 2**. It can be seen that the CTKF performs better than the TKF. Non-Gaussian noise results in the poor estimation performance of the TKF, since it adopts the MMSE criterion, which uses only second-order statistic of error. The CTKF shows stronger robustness to non-Gaussian noise, because the adopted correntropy cost function can capture second-order and higher-order statistics of error. Simulation results show that the CTKF is an effective state estimation method for dynamic systems with high-order ARMA non-Gaussian noise.

5 Conclusion

In this paper, a new filter is designed to solve the state estimation problem of dynamic systems with high-order ARMA non-Gaussian noise. In this filter, high-order ARMA process and measurement noises are modeled in the TMM framework, and then the recursive algorithm is derived by using corretropy cost function. On the one hand, the TMM is more general than the HMM, and it can directly model dynamic systems with high-order ARMA noise. On the other hand, correntropy can capture second-order and higher-order statistics of error, and is more suitable for non-Gaussian noise than the MMSE cost function, which uses only second-order statistics of error. In addition, the CTKF has a simple form, which facilitates its practical application.

Data availability statement

The original contributions presented in the study are included in the article/**Supplementary Material**, further inquiries can be directed to the corresponding author.

Author contributions

GZ provided the idea of the work and organized the manuscript, LZ performed the experiment, FL provided revisions to this paper, XL designed experimental scenarios, and NF and SD conducted data analysis.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Supplementary material

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fenrg.2022.990267/full#supplementary-material

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