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EDITED BY
Bin Zhou,
Hunan University, China

REVIEWED BY
Yuanzheng Li,
Huazhong University of Science and Technology, China
Jian Zhao,
Shanghai University of Electric Power, China
Yingjun Wu,
Hohai University, China

*CORRESPONDENCE
Xiaoyu Gong,
39191267@qq.com
Lei Xi,
xilei2014@163.com

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A new search direction of IPM for horizontal linear complementarity problems

Xiaoyu Gong[1,2]*, Lei Xi[2,3]* and Bo Yuan[3]

[1]School of Economics and Management, China Three Gorges University, Yichang, China, [2]Yichang Key Laboratory of Information Physics Fusion Defense and Control System (Three Gorges University), Yichang, China, [3]College of Electrical Engineering and New Energy, China Three Gorges University, Yichang, China

This study presents a new search direction for the horizontal linear complementarity problem. A vector-valued function is applied to the system of $xy = \mu e$, which defines the central path. Usually, the way to get the equivalent form of the central path is using the square root function. However, in our study, we substitute a new search function formed by a different identity map, which obtains the equivalent shape of the central path using the square root function. We get the new search directions from Newton's Method. Given this framework, we prove polynomial complexity for the Newton directions. We show that the algorithm's complexity is $O(\sqrt{n}\log\frac{n}{\epsilon})$, which is the same as the best-given algorithms for the horizontal linear complementarity problem.

KEYWORDS

linear complementarity, interior-point method, full-Newton step, complexity, HLCP

Introduction

Karmarkar (1984) found the first method of the interior point algorithm, so linear programming appeared as a dynamic field of research. Soon after, the interior point algorithm was able to resolve linear programming problems and other optimal problems such as semi-definite programming problems, high-order conic programming problems, and linear and nonlinear complementarity problems.

Then, Nestrov and Nemirovskii (1994) imported a new concept of self-concordant barrier functions to define the interior point method for solving the convex programming problem. In addition, Vieira (2007) proposed a different interior point algorithm using the kernel function.

It showed that linear complementarity problems have more significant adhibition in the economic field; the most significant model is the equilibrium model of the Arrow–Debreu market. (Kojima et al. (1992) proved that linear complementarity problems are equal to some models of equilibrium market, but that is not necessarily sufficient. Hence, Illés et al. (2010) analyzed the general linear complementarity problems' solvability.

Some special search directions play an important role in analyzing interior point algorithms.

A basic idea of primal-dual inter-point algorithms is to go through the central path to get the optimal solution. Later, Peng et al. (2002a) verified that the essence of Karmarkar's algorithm was just a special classical barrier function, which is a polynomial time algorithm. Later, Peng et al. (2002b) proposed a self-regular function and got the best iteration bound for a large-update algorithm for linear programming problems.

Moreover, Peng et al. (2002b) presented a new method for getting search directions called full-Newton methods; the new algorithm transformed the center equation $xs = \mu e$ using a function ϕ and then got the new search direction from Newton's method.

Because linear complementarity problems are closely related to linear programming problemsKarimi and Tuncel, 2020; Yamashita et al., 2021; Yang, 2022; Zhang et al., 2022a; 2022b), many interior-point algorithms (Mansouri et al., 2015) are designed from linear programming to linear complementarity problems, and all got polynomial time numerical results.

Furthermore, Wang and Bai (2009) and Wang and Bai (2012) proposed the second-order cone programming using a new full Nesterov–Todd step of the primal-dual method. Scheunemann et al. (2021) presented a barrier term for the infeasible primal-dual interior algorithm of small strain single crystal plasticity. Lu et al. (2020) proposed a two-step method for horizontal linear complementarity problems, and Asadi et al. (2019) presented a large-step infeasible algorithm for horizontal, linear complementarity problems.

The above-mentioned studies almost used the square root function, which obtained a form of the central path. The basic idea of the new function is named the difference of identity. In this study, we use the new square root function to define the search direction to solve horizontal linear complementarity problems and give the complementarity problems and give the complexity of the algorithm.

The interior algorithm of HLCP

Two square matrices $M, N \in R^{n\times n}$ are given, and $q \in R^n$ is a vector. The horizontal linear complementarity problems finds a pair of $x, y \in R^n$, such that

$$\begin{cases} Ny - Mx = q, \\ x^T y = 0, \\ x \geq 0,\ y \geq 0. \end{cases} \tag{1}$$

In this section, we study the horizontal linear complementarity problems (HLCP) based on the central path method to get the search directions.

We assume that (1) meets the need of the following two assumptions (Darvay, 2003).

Interior point condition

There are two vectors such that

$$Nx^0 - My^0 = q, \quad y^0 > 0, \quad x^0 > 0.$$

The monotonic property

There are two matrices (N, M) such that

$$Ny - Mx = 0 \rightarrow x^T y \geq 0 \ (x,\ y \in R^n).$$

From the above two assumptions, we can conclude that there is a solution for HLCP. We find an approximate solution by solving the following system:

$$\begin{cases} Ny - Mx = q, \\ xy = 0,\ x \geq 0,\ y \geq 0. \end{cases} \tag{2}$$

Using the path-following interior algorithm replaces the second equation of Eq. 2 with the parameterized equation $xy = \mu e$ $(\mu > 0)$; then, we get the following system:

$$\begin{cases} Ny - Mx = q, \\ xy = \mu e. \end{cases} \tag{3}$$

With $\mu > 0$, we can get the unique solution $(x(\mu),\ y(\mu))$ from system (3), and we call $(x(\mu),\ y(\mu))$ the μ-center of horizontal linear programming problem. With μ running through all positive numbers and when $\mu \rightarrow 0$, the central path exists and we get a solution for the horizontal linear programming problems (Kheirfam and Haghighi, 2019).

Search directions for HLCP

Considering the continuously differentiable ϕ: $R^+ \rightarrow R^+$ and the inverse function ϕ^{-1}, then (2.3) can be transformed into the following form:

$$\begin{matrix} Ny - Mx = q \\ \varphi\left(\frac{xs}{\mu}\right) = \varphi e \end{matrix}.$$

Applying Newton's method yields new search directions. Let

$$\begin{cases} N\Delta y - M\Delta x = 0 \\ \frac{y}{\mu}\varphi'\left(\frac{xy}{\mu}\right)\Delta x + \frac{x}{\mu}\varphi'\left(\frac{xy}{\mu}\right)\Delta s = \varphi(e) - \varphi\left(\frac{xy}{\mu}\right) \end{cases} \tag{4}$$

Let $dx = \frac{v\Delta x}{x}$, $dy = \frac{v\Delta y}{v}$; then,

$$\mu v(dx+dy)=y\Delta x+x\Delta y \quad (5)$$

$$dxdy=\frac{\Delta x\Delta y}{\mu} \quad (6)$$

From (5) and (6), (4) can be written in the form

$$\begin{cases}\bar{N}dy-\bar{M}dx=0\\ dx+dy=pv\end{cases} \quad (7)$$

At this time, $\bar{M}=MXV^{-1}, \bar{N}=MYV^{-1}$, $V=\mathrm{diag}(v)$.

We get different values for the pv from the φ function and obtain the search directions.

Now, for $pv=\frac{\varphi(e)-\varphi(v^2)}{v\varphi'(v^2)}$, we choose $\varphi(t)=t-\sqrt{t}$; then, from the new function, we get a new direction, and

$$pv=\frac{2(v-v^2)}{2v-e}\ v\in\left(\frac{1}{2},+\infty\right) \quad (8)$$

We define $qv=dx-dy$ from

$$(XV^{-1}dy)^T(YV^{-1}dx)=dx^Tdy$$

and monotonicity

Furthermore, let $dx=\frac{pv+qv}{2}$, $dy=\frac{pv-qv}{2}$; then,

$$dxdxy=\frac{p_v^2-q_v^2}{4} \quad (9)$$

Primal-dual interior-point algorithm for HLCP

1) Let $\epsilon>0$ be the accuracy parameter, $0<\theta<1$ the update parameter, $\theta=\frac{1}{27\sqrt{n}}$. Assume a strictly feasible point (x^0, y^0), s.t. $\delta(x^0,y^0,\mu^0)<\tau$.
2) If $\bar{x}y\le\epsilon$, then stop; otherwise, go to the next step.
3) According to (4), find (4) and $(\Delta x, \Delta y)$. We get $x=x+\Delta x$, $y=y+\Delta y$. Then, turn to step 2.

Convergence analyses

Lemma 4.1. Let (dx, dy) be a solution of (7). Then, we have $0\le d_x^Td_y\le 2\delta^2$.

Proof. Because the pair [N, M] is in the monotone HLCP, we conclude that

$$\delta^2=||pv||^2=||dx+dy||^2=||dx||^2+||dy||^2+2dx^Tdy\ge 2dx^Tdy.$$

That is, $d_x^Td_y\le 2\delta^2$.

Lemma 4.2. Let $\delta=(x,y,\mu)<1$ and $e-2v<0$. Then, $(x_+,y_+)>0$.

Proof. Let

$$\forall\alpha\in[0,1],\ x_+(\alpha)=x+\alpha\Delta x,\ y_+(\alpha)=y+\alpha\Delta y.$$

Therefore,

$$x_+(\alpha)\,y_+(\alpha)=xy+\alpha(y\Delta x+x\Delta y)+\alpha^2\Delta x\Delta y \quad (10)$$

From (5) and (6),

$$\frac{1}{\mu}x_+y_+=v^2+\alpha v(dx+dy)+\alpha^2 dxdy \quad (11)$$

Due (7) to (9),

$$\frac{1}{\mu}x_+y_+=(1-\alpha)v^2+\alpha(v^2+vp_v)+\alpha^2\left(\frac{p_v^2}{4}-\frac{q_v^2}{4}\right).$$

Furthermore, from (8),

$$v^2+vp_v=v^2+\frac{2(v^2-v^3)}{2v-e}=\frac{(2v^2)}{2v-e} \quad (12)$$

From (11), we get

$$\frac{1}{\mu}x_+(\alpha)y_+(\alpha)=(1-\alpha)v^2+\alpha\left(e+\frac{(v-e)^2}{2v-e}+\alpha\frac{p_v^2}{4}-\alpha\frac{q_v^2}{4}\right) \quad (13)$$

From $e-2v<0$, we get $\frac{(v-e)^2}{2v-e}\ge-\frac{p_v^2}{4}$.

From (13), we obtain

$$\frac{1}{\mu}x_+(\alpha)y_+(\alpha)\ge(1-\alpha)v^2+\alpha\left(e-(1-\alpha)\frac{p_v^2}{4}-\alpha\frac{q_v^2}{4}\right),$$

$$x_+(\alpha)y_+(\alpha)>0\rightarrow||(1-\alpha)\frac{p_v^2}{4}+\alpha\frac{q_v^2}{4}||\le 1.$$

Using $||p_v||\ge||q_v||$, $\delta^2=\frac{||pv||2}{4}$.

Then,

$$||(1-\alpha)\frac{p_v}{4}||+||\alpha\frac{q_v^2}{4}||_\infty\le(1-\alpha)\frac{||p_v||_\infty^2}{4}+\alpha\frac{||q_v^2||_\infty}{4}.$$

Therefore, we get a conclusion that, for any $\alpha\in[0,1]$, the inequality $x_+(\alpha)y_+(\alpha)>0$ holds, which signifies that the signs of $x_+(\alpha)$ and $y_+(\alpha)$ do not change on the interval [0,1]. Hence, $x_+(0)>0$, $y_+(0)>0$ leads to $x_+(1)>0$, $y_+(1)>0$.

Lemma 4.3. Let $f: D\rightarrow(0,+\infty)$ be a decreasing function, where $D=[d,+\infty]$, $d>0$.

Furthermore, let $v\in R_+^N$ such that $\min(v)>d$. Then,

$$\left\|f(v)\cdot(v-e^2)\right\|\le f(\min(v))\cdot\left\|e-v^2\right\|\le f(d)\left\|e-v^2\right\|.$$

Proof.

$$\begin{aligned}||f(v)\cdot(e-v^2)||&=\sqrt{\sum_{i=1}^n(f(v_i))^2(1-v_i^2)^2}\\&\le f(\min(v))\cdot\sqrt{\sum_{i=1}^n(1-v_i^2)^2}\\&=f(\min(v))\cdot||(e-v^2)||\\&\le f(d)\cdot||(e-v^2)||.\end{aligned}$$

Lemma 4.4. Let $\delta=\delta(x,y,\mu)<\frac{1}{2}$, $2v-e>0$. Then,

$$v_+>\frac{1}{2}e,\ \delta=\delta(x_+,y_+,\mu)\le\frac{3-3\delta^2-3\sqrt{1-\delta^2}(1-2\delta^2)}{3-4\delta^2}.$$

Proof. From Lemma 4.2, we get

$$x_+ > 0,\ y_+ > 0.\ v_+ = \sqrt{\frac{x_+ y_+}{\mu}}.$$

Due to (4.4), as $\alpha = 1$, we get

$$v_+^2 = e - \frac{e - 2v - v^2}{2v^2} \cdot \frac{p_v^2}{4} - \frac{q_v^2}{4} \tag{14}$$

From $2v - e > 0$ and $v^2 + 2v - e > 0$, that is $v_+^2 \geq e - \frac{q_v^2}{4}$, then

$$\min(v_+) \geq \sqrt{1 - \frac{1}{4}\|q_v^2\|_\infty} \geq \sqrt{1 - \frac{\|q_v\|^2}{4}} = \sqrt{1-\delta^2} \tag{15}$$

By using the function $f(t) = \frac{t}{(2t-1)(1+t)} > 0$ for any $t > 0.5$, $f'(t) < 0$, *f* is monotone decreasing.

From Lemma 4.3,

$$\delta(x_+, y_+, \mu) \leq \frac{\sqrt{1-\delta^2}}{2(1-\delta^2) + \sqrt{1-\delta^2} - 1} \cdot \|e - v_+^2\| \tag{16}$$

Substituting $\sqrt{1-\delta^2}$ and making reductions, we get

$$f\left(\sqrt{1-\delta^2}\right) = \frac{1 - \delta^2 - \sqrt{1-\delta^2}(1 - 2\delta^2)}{\delta^2(3 - 4\delta^2)} \tag{17}$$

We have $1 < \frac{t^2+2t-1}{t^2} < 2$ for all $t > \frac{1}{2}$.

Moreover, $e - v_+^2 = \frac{e-2v-v^2}{2v^2} \cdot \frac{p_v^2}{4} + \frac{q_v^2}{4}$.

Thus,

$$\|e - v_+^2\| \leq 2\left\|-\frac{p_v^2}{4}\right\| + \left\|-\frac{q_v^2}{4}\right\| = 3\delta^2 \tag{18}$$

Using (16), (17), we obtain

$$\delta(x_+, y_+, \mu) \leq \frac{\left(3 - 3\delta^2 - 3\sqrt{1-\delta^2}\right)(1 - 2\delta^2)}{3 - 4\delta^2}.$$

Furthermore,

$$\delta(x_+, y_+, \mu) \leq \frac{3 - \left(1 - \sqrt{1-\delta^2}\right)}{3 - 4\delta^2} + \frac{3\delta^2\left(-1 + 2\sqrt{1-\delta^2}\right)}{3 - 4\delta^2}.$$

Let $\varphi_1(\delta) = \frac{3(1-\sqrt{1-\delta^2})}{3-4\delta^2}$, $\varphi_2(\delta) = \frac{3\delta^2(-1+\sqrt{1-\delta^2})}{3-4\delta^2}$.

For $\delta < \frac{1}{2}$ and then $4\delta^2 < 1$, $\frac{1}{3-4\delta^2} < \frac{1}{2}$, we obtain

$$\frac{1}{3}\varnothing_2(\delta) < \frac{4 - 2\sqrt{3}}{2}\delta^2 \tag{19}$$

A simple calculus yields

$$\frac{1}{3}\varnothing_2 < \frac{4 - 2\sqrt{3}}{2}\delta^2 \tag{20}$$

We have (19), (20).

We have $\frac{1}{3}(\varnothing_1(\delta) + \varnothing_2(\delta) < \frac{4-2\sqrt{3}+\sqrt{3}}{2}\delta^2 = \frac{3-\sqrt{3}}{2}\delta^2$.

Lemma 4.5. Let $\delta = \delta(x, y, \mu)$ and suppose that the vectors x+ and y + are obtained using a full-Newton step. Thus, $x_+ = x + \Delta x$, $y_+ = y + \Delta y$. We get $(x_+)^T y_+ \leq \mu(n + 3\delta^2)$.

If $\delta < \frac{1}{2}$, then we obtain $(x_+)^T y_+ \leq \mu(n + \frac{3}{4})$.

Lemma 4.6. Let

$$\delta = \delta(x, y, \mu) < \frac{1}{2},\ v > \frac{1}{2}, \mu_+ = (1 - \theta)\mu,\ v' = \sqrt{\frac{x_+\theta_+}{\mu_+}},\ \gamma = \sqrt{1-\theta},\ (0 < \theta < 1),$$

then $v' > \frac{1}{2}e$ and $\delta(x_+, y_+, \mu_+) < \frac{\sqrt{3}(\theta\sqrt{n} + 3\delta^2)}{-2\gamma^3 + \sqrt{3}\gamma^2 + 3\gamma}$.

If $\theta = \frac{1}{27\sqrt{n}}$, $n \geq 4$, we have $\delta(x_+, y_{+,}\mu_+) \leq \frac{1}{2}$.

Proof. $v' = \frac{1}{\gamma}\mu_+$ from Lemma 4.4 $\mu_+ > \frac{1}{2}e$, $\mu' > \frac{1}{2}e$.

Consider $h(t) = \frac{t}{(2t-\gamma)(\gamma+t)}$, $(t > \frac{\gamma}{2})$; we get

$$\frac{v' - v'^2}{2v' - e} = \frac{1}{\gamma}h(v_+)(\gamma^2 e - v_+^2).$$

For h’(t) < 0, for h’(t) < 0, we get that h is a decreasing function.

Using (4.9), we have

$$\|\gamma^2 e - v_+^2\| = \|(1-\theta)e - v_+^2\| \leq \|-\theta e\| + \|e - v_+^2\| < \theta\sqrt{n} + 3\delta^2$$

$$\delta(x_+, y_+, \mu_+) \leq \frac{\sqrt{3}(\theta\sqrt{n} + 3\delta^2)}{-2\gamma^3 + \sqrt{3}\gamma^2 + 3\gamma}.$$

Using $g(\gamma) = \frac{1}{-2\gamma^3 + \sqrt{3}\gamma^2 + 3\gamma}$, $\gamma \in (0, 1)$, we have

$$g'(\gamma) = \frac{6\gamma^2 - 2\sqrt{3}\gamma - 3}{\left(-2\gamma^3 + \sqrt{3}\gamma^2 + 3\gamma\right)^2} < 0\ (0 < \gamma < 1).$$

This implies that g is decreasing.

We get $\delta(x_+, y_+, \mu_+) \leq \sqrt{3}(\frac{1}{27} + \frac{3}{4})\sqrt{\frac{53}{54}} < \frac{1}{2}$.

Lemma 4.7. We assume that the (x^0, y^0) is strictly feasible $\mu^0 = \frac{(x^0)^T y^0}{n}$ and $\delta(x^0, y^0, \mu) < \frac{1}{2}$, and assume that the two vectors x^k and y^k are obtained by the algorithm; then, after k iterations k and $(x^k)^T y^k \leq \epsilon$.

Proof. From lemma 4.5,

$$\left(x^k\right)^T y^k < \mu^k\left(n + \frac{3}{4}\right) = (1-\theta)^k \mu^0\left(n + \frac{3}{4}\right) \leq \epsilon.$$

Taking logarithms on two sides, then we get

$$k\log(1-\theta) + \log\left(\mu^0\left(n + \frac{3}{4}\right)\right) \leq \log\epsilon.$$

From $\theta \leq -\log(1-\theta)$, we obtain

$$k\theta \geq \left(\mu^0\left(n + \frac{3}{4}\right)\right) - \log\,\epsilon = \log\,\frac{\mu^0\left(n + \frac{3}{4}\right)}{\epsilon}.$$

Because the self-dual embedding allows us to propose without any loss of generality that $x^0 = y^0 = e$, we have $\mu^0 = 1$.

Theorem 4.1. Suppose that x0 = y0 = e. If we consider the default values for θ and τ, we get that the algorithm just requires no more than $O(\sqrt{n}\log\frac{n}{\epsilon})$ interior-point iterations. The conclusion satisfies $x^T y \leq \epsilon$

Conclusion and future works

This study proposed a primal-dual path-following algorithm for the horizontal linear complementarity problem based on a new search direction, which differs from those available. We analyzed this algorithm and illustrated that the proposed algorithm has $O(\sqrt{n}\log\frac{n}{\epsilon})$ iteration complexity bound. Some interesting topics remain for future research. Firstly, we can extend the algorithm to linear complementarity problems over symmetric cones. Secondly, we can develop the infeasible interior point algorithm based on the method given in this study.

Data availability statement

The raw data supporting the conclusion of this article will be made available by the authors without undue reservation.

Author contributions

XG: algorithm analysis; LX: astringency; BY: feasibility study.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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