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Teachers' mathematical work based on examples presented in the teaching of algebra in secondary education

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Both international examination results and scientific evidence related to the teaching and learning of school algebra point to the necessity for analyzing mathematical practice in the classroom. In this context, we have formulated the research objective of characterizing the mathematical work that teachers favor based on the examples that they implement in the teaching of topics within the domain of secondary school algebra. To this end, this study is situated within the interpretive paradigm utilizing an instrumental collective case study approach. The study design entails the selection of two representative cases of mathematics teachers in Chilean schools, with the objective of understanding and describing their mathematical work in the secondary school classroom based on the examples they present. The results show that the first teacher utilizes varied examples that activate distinct forms of mathematical thinking, promoting the activation of both instrumental and discursive genesis. In contrast, the second teacher follows a more traditional approach, with the use of examples to illustrate and motivate, and a direct presentation of knowledge, activating primarily semiotic genesis, in which a transition between the numerical and the algebraic is lacking. Ultimately, the results highlight the importance of deeper consideration of the choice of examples in the teaching of algebra and how these examples can influence students' learning. Furthermore, additional ideas are put forward for future work in this line of research associated with the use of examples in the classroom.

KEYWORDS

school algebra, secondary education, mathematical working space, teachers, examples

1 Introduction

Results on international examinations, such as the PISA 2018, reveal low scores among Chilean students (an average of 417 points in mathematics), situating them below the OECD average (OECD, 2019). These results have prompted continuous vigilance with regard to the quality of education, particularly in the area of teaching and learning of mathematics—with algebraic thinking being one of the central themes of Chile's national curriculum (Radford, 2013; Kieran et al., 2016)—and the particular difficulties associated with its teaching and learning (National Mathematics Advisory Panel, 2008; McCrory et al., 2012; Warren et al., 2016; Kieran, 2018). In this vein, diverse research projects have arisen directed at the study of

distinct notions of algebra, one of these being the notion of the function (Even and Tirosh, 1995; Hitt, 1998; Oehrtman et al., 2008; Dubinsky and Wilson, 2013; Bagley et al., 2015; Amaya De Armas et al., 2021; Henríquez-Rivas and Verdugo-Hernández, 2023; Martins et al., 2023).

Meanwhile, Billings (2008) and Mason (1996) highlight the fundamental role of teachers in comprehending patterns and developing an understanding of algebraic rules. Specifically, Mason (1996) indicates that it is teachers who must first understand generalization in order to transmit these concepts effectively to their students and, in doing so, incentivize the development of mathematical thinking. In this context, other studies have examined teacher training and the teaching of notable products (Graciano and Aké, 2019; Graciano-Barragan and Aké, 2020; Verdugo-Hernández and Coulange, 2021). For their part, Graciano-Barragan and Aké (2020) examine pre-service teachers' mathematical knowledge of notable products. Meanwhile, Chang and Tsai (2005) report the importance of investigating the teaching of notable products due to the difficulties that they present to students in the development of mathematical tasks.

Algebraic expressions have also been a focus of study in the teaching of algebra for various authors (e.g., Doran, 2018; Moschkovich et al., 2018), who have focused on their multi-semiotic character, coordinating as they do diverse representations in natural language, written symbols, and algebraic composite representations. Other works call attention to systems of linear equations, as in Segurade (2004), who focuses on the construction and application of a didactic sequence aimed at facilitating the learning and solving of systems of linear equations.

In terms of research addressing the study of the mathematical practices of the teacher in the classroom, this is a topic that has grown in value and validity in recent years (Badillo Jiménez et al., 2013; Climent et al., 2021; Wasserman and McGuffey, 2021; Ekmekci and Serrano, 2022; Nkundabakuraa et al., 2022; Wasserman, 2023). These studies call attention to diverse topics, school levels, and mathematical domains. For example, Zakaryan and Sosa (2021) analyze the knowledge put into play by a teacher in the classroom, identifying the role of symbols and mathematical conventions as well as the role of demonstrations and their main teaching methods (among other factors).

Other research has addressed the importance of class organization both inside and outside the classroom in order to achieve effective learning (e.g., Marder et al., 2023), while some studies have focused on the importance of feedback in the mathematics classroom (Stovner and Klette, 2022; Van der Kleij, 2023). Meanwhile, Rakes et al. (2022) examine the classroom practice of pre-service secondary mathematics teachers when implementing technology to aid in the conceptual understanding of mathematics. The results of these studies reveal that work must continue regarding the knowledge of future teachers in order for the use of technology to have a truly meaningful impact, as well as to increase the understanding of the complexity of classroom management during the teaching of specific classes.

Likewise, Salani and Jojo (2023) explore the choice of representations, examples, and classwork on algebraic equations by teachers in their classroom practice. Their results underscore that these selections do not promote practical activities that would foster the understanding and creation of conceptual connections. Meanwhile, Henríquez-Rivas et al. (2021) put forward that it is necessary to conduct research centered on the design, selection,

adaptation, and implementation of classwork and examples in the classroom in order to strengthen teachers' mathematical and didactic knowledge.

The above research points to the use of examples as fundamental for the learning of concepts, theorems, and techniques, among other topics (Zodik and Zaslavsky, 2008); it follows that their selection for the teaching of a given mathematical concept in the classroom is highly significant (Watson and Mason, 2005). Given the importance of the use of examples in teachers' classroom management and the need to study the mathematical work that takes place during teaching, the objective of the present study is *to characterize the mathematical work that professors favor based on the examples that they implement in the teaching of topics within the domain of secondary school algebra*. The results allow us to discuss contributions to the teacher's mathematical work in the classroom, centered on the use of examples associated with given mathematical contents, specifically the study of topics in algebra at the beginning of Chilean secondary education (corresponding to 14 years of age).

2 Theoretical framework

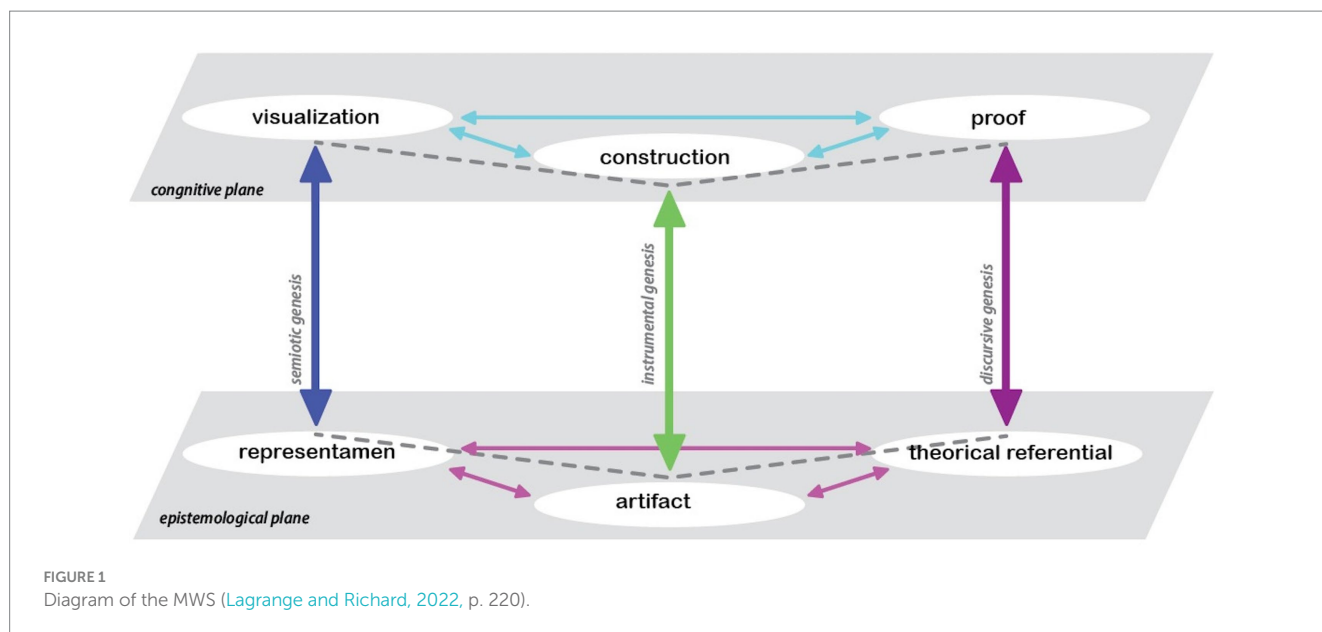
2.1 Mathematical working spaces

The theory of *Mathematical Working Spaces* (MWS) has been developed by various authors (Kuzniak, 2011; Kuzniak et al., 2016, 2022). In addition, it has been utilized in several recent studies (Verdugo-Hernández et al., 2022; Henríquez-Rivas et al., 2023; Henríquez-Rivas and Verdugo-Hernández, 2023) that demonstrate its potential as an analytical and methodological tool (Kuzniak and Nechache, 2021). The MWS model focuses on the process that individual (student, teacher, or other professional) undertakes in the face of a given mathematical problem, and it includes three types: personal MWS, reference MWS, and suitable MWS. For the purposes of this study, we focus on the suitable MWS of the teacher (Henríquez-Rivas et al., 2022) in relation to the implementation of teaching in the classroom.

The theory considers two planes: the epistemological plane, relative to the theory of a study domain (geometry, algebra, etc.), and a cognitive plane, which refers to the mental processes of the individual that are utilized when solving a math problem.

The epistemological plane is composed of three components: representamen (the set of tangible and concrete symbols); artifact (drawing tools, programs and software); and referential (theory being studied). Meanwhile, the cognitive plane is composed of the following three components: visualization (related to interpretation), construction (closely related to the artifact deployed and/or techniques used), and the proof (based on the referential utilized in a validation process). The epistemological and cognitive planes are articulated by three geneses running between them (Gómez-Chacón et al., 2016), which can be in constant interaction: semiotic, discursive, and instrumental genesis, which allow for the connection of the components of each plane to contribute to the analysis of mathematical work (Henríquez-Rivas et al., 2021). A diagram of the planes, components, and geneses is shown in Figure 1.

These three geneses can interact among themselves, generating pairs, which have been termed vertical planes (Kuzniak and Richard,



2014) and include the following: semiotic-instrumental [Sem-Ins], instrumental-discursive [Ins-Dis], and semiotic-discursive [Sem-Dis].

Kuzniak et al. (2016) employ the notion of *tool* to refer to that which must be utilized to take on a given problem. These tools emerge from the components of the epistemological plane, and through the different geneses, they are directed toward the cognitive plane. The categories of tools proposed by Kuzniak et al. (2016) include the following: (a) *semiotic tools*, understood as non-material tools used to operate on semiotic representations of mathematical objects; (b) *technological tools*, referring to artifacts such as drawing tools, routine techniques based on algorithms, or calculators that implement calculation algorithms; and (c) *theoretical tools*, corresponding to instruments based on logic and the properties of mathematical objects. Additionally, related research has introduced the notion of (d) *operational tools*, understood as tools utilized to solve a given problem, but which do not form part of the theoretical referential to which the problem belongs (Verdugo-Hernández et al., 2022).

Meanwhile, the analysis of routes of mathematical work, or activations of the components, geneses, and vertical planes, specifically define the work carried out by an individual; this is termed *circulation* in the working space (Montoya-Delgado et al., 2014). In these analyses, changes of domain can be identified based on the work in the origin domain and the possible shift to a different solution domain (Montoya-Delgado and Vivier, 2014).

2.2 Examples in mathematics

From the perspective of task design (Watson and Ohtani, 2015), the mathematical example occupies a unique place of study that highlights its use, selection, and adaptation by the teacher. Examples are understood as a means of communication and mediation between students and ideas, as well as being recognized for their importance for proposing educational improvements (Sullivan et al., 2015; Yerushalmy, 2015). Thus, the use of examples by teachers is not arbitrary, but rather requires planning and entails confronting various

difficulties, as methods for example selection do not exist (Zodik and Zaslavsky, 2008; Huntley, 2013). By the same token, according to Zazkis and Leikin (2007), researchers can learn about teachers' knowledge based on the examples that they produce.

The notion of the example, according to Watson and Mason (2005), is defined broadly as that which represents anything meant to help the student to generalize, covering diverse *uses*: as illustrations of concepts and principles; dynamic images for definitions and theorems; solved examples presented in textbooks or created by teachers as a means to demonstrate the use of a particular method; aids to learn to utilize, apply, and attain fluency with specific techniques in exercises; representations for inductive reasoning and later searching for patterns; or in concrete contextual situations that can be treated as cases for motivating interest in mathematics.

In the literature, diverse classifications of examples can be found (e.g., Rowland et al., 2003; Rowland, 2008). The present study is based on the system of categories of examples proposed by Figueiredo et al. (2007), who describe the following:

- *Definition*: the first examples that can be presented to students immediately following the definition of a concept, or a series of examples with common characteristics shown before the definition.
- *Representation*: referring to the first typical exercises or problem situations related to the application of the concept in question. The role of the teacher is less participatory, with the aim of promoting greater involvement of the student with the exercise or problem.
- *Characteristics*: a type of example that arises when the student examines the concept in greater depth and discovers its particularities; these are given as explanations to situations of doubt or confusion among students, and likewise they can be presented before such situations arise, in the case of more experienced teachers. This type of example is fundamental for the progression of learning in mathematics.
- *Internal applications*: examples that are presented in the phases of more in-depth examination of the concept; the applications can

include contents or concepts that have been taught previously or can relate to others that will be taught subsequently. They require more background in the concepts in question and arise at the end of the teaching of a concept that integrates with others to form a network of related concepts. Moreover, they are examples that should be written and consider new situations for students or the solving of problems that are strictly mathematical.

- *External applications*: a category of examples that apply to real life and to other academic areas. They entail a certain degree of difficulty and require conceptual depth and flexibility in order to be tackled.

The information detailed above points to the importance of research focused on the work of the teacher in terms of the examples used in the classroom (Sosa et al., 2016). In the case of the present study, the examples used by teachers in the classroom are analyzed with attention centered on the uses of examples, the example categories, and the mathematical work that they promote.

3 Methodology

To characterize the suitable MWS that teachers present based on examples for teaching topics in the domain of algebra, the present study has been formulated in the interpretive paradigm with a qualitative focus (Rehman and Alharthi, 2016), specifically using the model of an *instrumental collective case study* (Stake, 2007). This study design is justified based on the selection of two teachers or representative cases, allowing for the comprehension and description of their mathematical work in the classroom (Simons, 2011).

The two selected teachers have experience in teaching mathematics and both work in secondary education. At the time of class observation, the first teacher (P1) has 1 year of experience, and the second one (P2) has a master’s degree in Science Teaching, and 5 years of teaching experience. P1 works in a scientific-humanistic high school and P2 works in a Professional Technical high school (see Table 1). The unit of analysis consists in the mathematical work carried out by the teachers in 2022 during in-person algebra classes for ninth grade students (approximately 14 years of age) in two Chilean secondary schools.

3.1 Case selection, access, and context of each participant teacher

The criteria for selecting the two teachers (one teacher per course in each school) are instrumental in nature (Stake, 2007; Simons, 2011), since it is of central interest to this study to describe the mathematical work developed in the classroom based on the examples presented during teaching. Specifically, the criteria include the following: (1)

TABLE 1 Description of the case study participants.

Participant	School modality	Class topic
P1	Scientific-humanistic	Systems of linear equations
P2	Technical-professional	Algebraic expressions

Source: Authors.

diversity of school contexts, diverse schools located in the south-central zone of Chile have been selected, one whose modality is Scientific-Humanistic while that of the other is Technical-Professional; (2) *topics of study at the time of data collection*, since the recording of classes on algebra topics for ninth grade was coordinated with the teachers; (3) *researchers’ access*, which was a significant issue due to the difficulties of managing field work in schools during the post-pandemic period.

The procedure for approaching the participants initially involved securing the backing of the headmaster of each school to record classes and the informed consent of the participant teachers. It should be noted that the teachers participated in this study voluntarily, with the option of terminating their participation at any time. Likewise, in order to protect the identities of the participating informants and institutions, pseudonyms are used for each teacher (they will be referred to as P1 and P2), whose details are listed below in Table 1.

3.2 Data collection and analysis

Data collection was carried out using video-recordings of classes, their respective transcriptions, and non-participant observation. Two classes were recorded by each teacher, and out of these the first class was chosen for analysis, since in both cases it provides more information regarding the use of examples in teaching.

In relation to strategies of triangulation, this was based on triangulation by expert researchers (Arias, 2000), considering that the research team was composed of two university mathematics professors specialized in MWS who carried out the initial analysis process separately. Following this process, they then arrived at a consensus regarding the findings, allowing for consistency to be achieved in the results presented here.

In order to characterize the mathematical work of P1 and P2, an analysis methodology was adapted that has been utilized in other MWS studies (Henríquez-Rivas and Kuzniak, 2021; Kuzniak and Nechache, 2021). This methodology, outlined in Table 2, describes the principal actions of mathematical work carried out by the teacher in a given class.

For the data analysis, the following protocol was utilized, which identifies categories based on the examples implemented during teaching and the components, geneses, and vertical planes activated based on the circulation of the suitable MWS of each teacher. This protocol is detailed in Table 3.

TABLE 2 Stages in the analysis methodology [adapted from Henríquez-Rivas et al., 2021].

Stage of analysis	Description
1 Identification of examples and description of uses.	Examples proposed by the teachers in the classroom are identified, and their uses are described in relation to the mathematical work undertaken for teaching.
2 Analysis of circulations and categories of examples.	Based on the description of the examples used, these are analyzed and interpreted in terms of the geneses, components, and vertical planes activated in the circulation of the MWS and the categories of examples.

TABLE 3 Protocol for data analysis.

Criteria for analysis	Category	Description	
Examples	Categories of examples	Definition	Will be identified when examples are presented before or after defining a concept.
		Representation	First typical exercises or problem situations in the application of the concept.
		Characteristics	Examples that allow studying a concept at a higher level and discovering its particularities.
		Internal applications	These can be identified by more in-depth study of the concept in question, since they require a stronger background in it and integrate it with other concepts.
		External applications	Examples applied to the real world and other academic subjects; these entail a certain degree of difficulty that entails conceptual depth and flexibility.
		Recognizing the activation of the genres (semiotic, instrumental, and discursive) and the components of each plane (epistemological and cognitive) utilized. Likewise, within the different components of the epistemological plane, corresponding tools are identified: semiotic, technological, theoretical, and operational.	
Mathematical work	Genres and components	Examples applied to the real world and other academic subjects; these entail a certain degree of difficulty that entails conceptual depth and flexibility.	
	Vertical planes	Recognizing interactions between the genres through the vertical planes: [Sem-Ins], [Ins-Dis], [Sem-Dis], which are activated by whoever is doing the solving.	

Source: Authors.

4 Results

4.1 Case P1

In this case, three examples have been identified that P1 presented during a class on systems of linear equations, which are described and analyzed below.

4.1.1 Example 1: Activation of previous knowledge

4.1.1.1 Description of the use of the example

The teacher initiates the class by reminding students of contents taught in the previous class in order to connect them with the topic of the current class; among them, he mentions equations with infinite solutions, presenting the following example (1) “ $x + y = 5$,” which he uses as a manner of *illustration* of the concept:

P1: [...] I said that these equations had infinite solutions, right, cases like, for example, “ x ” plus “ y ” equals 5, for example “ x ” could have the value of 3 and “ y ” the value of 2, or “ x ” could equal 4 and “ y ” equal 1, and this equation was always going to be solvable, right, we were going to have infinite solutions, and these solutions, these values of “ x ” up there, I could represent them. Where? On the Cartesian plane, and there was the explanation of how the equation had infinite solutions, that the line was composed on infinite what? Infinite points, right, the problem is that when I joined this equation with another, we formed something that I called the system of linear equations [...].

P1 continues with the same equation in order to explain and illustrate what is occurring with the line. Likewise, he represents another line, “ $-x + 2y = 4$.” To do so, he graphs the equations of the lines in question (Figure 2) using geometric software.

The teacher invites his students’ reflection through a series of questions, for example: *What does a system of equations represent?* The following extract gives an account of this instance and of the answers of some of the students (S):

P1: [...] What do you think based on, what we got from the graphical method, what did it represent?

S: A system of linear equations.

P1: It represented two lines, what happened with those two lines?

S: They collide [...]

P1: They intersect, that’s the word, so, they intersect at a point on the Cartesian plane, and that point on the Cartesian plane, then, is the solution to that system, the solution that we have now looked for graphically [...].

P1 presents this initial example to illustrate the new concept to be studied, as well as to motivate the students based on topics that are already familiar (equation of a straight line).

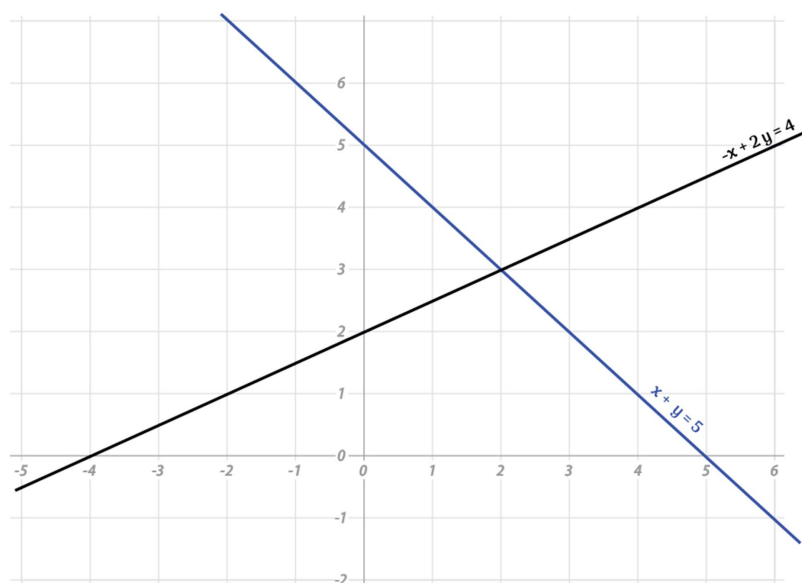


FIGURE 2
Graph of the lines provided by P1.

4.1.1.2 Analysis of circulation and category

In theoretical terms, the mathematical work of P1 begins with semiotic genesis with the linear equation, associated with the representamen, which is represented in the algebraic semiotic register. Then, the teacher carries out a conversion by representing the line, invoking the graphical register in the system of Cartesian coordinates and, in addition, he graphs another line that intersects at one point with the initial line so that the students, through their own perception, can deduce the notion of systems of linear equations. This activates the use of a technological tool that allows the construction of the lines mentioned. In this manner, the work of P1 is related to visualization of the iconic type, specifically associated with the perception of the mathematical object in the vertical plane [Sem-Ins].

In observing the development of the current suitable MWS of P1, it is apparent that he encourages spaces of reflection on the part of his students based on the representations provided, fostering coordination between visualization and discursive reasoning when he asks them to explain based on their knowledge, which activates the [Sem-Dis] vertical plane, but without considering the referential component to justify or validate their answers. Likewise, this example is classified in the *definition* category, as it involves an initial example provided before addressing the definition of the central topic of the class. When finishing this example, P1 poses a question in relation to the same example, “What situations is this useful for?” which could be considered to fall into the category *external application* if the teacher stopped working on the topic at that time. However, this is not the case, and the class progresses to the example presented below.

4.1.2 Example 2: Presenting a contextualized situation

4.1.2.1 Description of the use of the example

During the development of the class, P1 gradually goes into greater depth with the different examples he presents. At this point,

the teacher introduces example 2 (Figure 3), bolstered by the final question posed in the previous example, in which he asks, “*what would be a situation where the same thing does not happen?*” This question is used to propose a contextualized situation with the aim of motivating work with equations.

After presenting the situation in context, P1 indicates that this example can be addressed graphically or algebraically, but he favors the algebraic solution due to the question of immediate access to the resources needed to solve to the problem, including graph paper, a ruler, and a pencil, as illustrated in the following excerpt:

P1: [...] There in that phrase they are implicitly giving me my two equations together, linear with two variables, or two equations for straight lines, however we want to look at it, whether geometrically or with algebra, but today we are going to focus on solving it algebraically, since we aren't always going to have a ruler and graph paper available to solve the system of equations with the graphing method, so any idea how we could formulate an equation from this?

Next, P1 indicates that one of the first challenges of the problem is to approach it in algebraic terms. A student responds, correctly, that the representation is “ $x + y = 77$.” Then, P1 asks, “*How do we obtain the other equation?*” This generates the next dialogue with the students:

S: “ x ” plus two equals “ $2y$ ” plus two.

P1: that was “ x ” plus two equals...

S: “ $2y$ ” plus two.

P1: “ $2y$ ” plus two, and how did you get that?

Las edades de Marco (x) y Valeria (y) suman 77 años. Si en dos años más la edad de Marco será el doble de la de Valeria, ¿cuál es la edad de cada uno?

Pero en este caso, ¿ante que dificultad nos encontramos en primer lugar?

Translation:

The ages of Marco (x) and Valeria (y) have a sum of 77. If Marco's age will be twice Valeria's age in 2 more years, how old are they?

But in this case, what issue are we faced with right away?

FIGURE 3
Contextualized situation from example 2 (Translated by authors).

S: it says *if in two years Marco's age*, Marco is " x ," " x " plus two, so Valeria the " y " is " $2y$ " and you also need to add two to the " x 's."

P1: So, I'm going to correct you on one thing, that parenthesis, right, there's that parenthesis, we always forget it, right, so Marco's age is " x " with two more years, what do we have to do with his age to obtain the [...].

In this dialogue, the teacher makes a correction to the expression $x + 2 = 2y + 2$ indicated by the student, since the correct answer would be $x + 2 = 2(y + 2)$. In order to check whether the correction made on the board has been understood, P1 offers a similar verbal example in the following manner:

P1: Plus two, right, how old are you now Mateo?

S: I'm 15.

P1: In two years how old will you be?

S: 17.

P1: What did you do with your 15 years?

S: I added two.

P1: I added two, right, so here we do not know Marco's age, or Valeria's right, we have to add two to it, but it will be double, I mean, we'd need to multiply Valeria's age by two to figure out Marco's age [...].

To explain the example given, P1 mentions the "=" sign, which relates to the verb "will be" given in the situation, so the students associated the sign with this verb; this is expressed in the following extract:

P1: Right, somehow they are telling me, implicitly, that there is an equal sign, right, an equal sign when they tell me that Marco's age will be, that "will be" appears there, in other words, they are telling me that I need to formulate my equation with an equal sign.

Ultimately, P1 leaves two equations formulated and indicates that after studying systems of equations more, they will be solved.

4.1.2.2 Analysis of circulation and category

In terms of the MWS, initially P1 is seen to privilege semiotic genesis, specifically the activity of conversation between the representations in the initial register in natural language (the problem) and the algebraic register in the solution. To approach the example, the teacher privileges the coordination between the registers mentioned, which entails proposing a system of equations (representamen). Additionally, P1 highlights the interpretation of the "=" sign, establishing a relation between this unit of meaning of representation in algebraic language with the verb "will be" from the initial representation. In the sense of Duval (1995), this is linked to the semantic correspondence and univocality of these signifying elements. In general, the two representations that work in the example (in the natural and algebraic registers) are congruent, since the units of meaning could establish a semantic correspondence and univocality between the representations, in addition to the order of units they compose when establishing the conversion of the statement to the algebraic approach of the equations. This facilitates the understanding of the conversion of the situation presented.

Additionally, the work of P1 demonstrates activation through discursive genesis when verbally expressing an analogous situation to ensure that the students understand the conversion from the representation in natural language to algebraic language, as a test that allows for the validation of the solution given, but without resorting to the referential of the epistemological plane. Considering the aforementioned, the work of P1 entails the activation of the [Sem-Dis] vertical plane.

Thus, this example is classified in the *representation* category because it deals with a situation of typical application of the concept in question, in which students' participation is encouraged in the formulation of the system of linear equations.

4.1.3 Example 3: presentation of systems of linear equations

4.1.3.1 Description of the use of the example

After working through the previous example, P1 proceeds with example 3, which consists of three parts: first, he presents a definition

(Figure 4); then, he presents the steps associated with the substitution method (Figure 5); and lastly, he provides a graphical representation of the solution (Figure 6). This example is used to apply a method that allows students to attain *greater fluency* with the techniques that will be taught (in this class, the focus is on the teaching of the substitution method).

P1 presents the following definition on the board (Figure 4), which is accompanied by three systems of linear equations that he utilizes as an illustration.

In order to relate the definition with the systems of linear equations that he presents (Figure 4), the teacher asks the students to indicate the values that correspond to the variables *a*, *b*, *c*, *d*, *e*, and *f* in each case, as exhibited below.

P1: [...] it's another, distinct equation, we are always going to call it equation one and equation two, okay, and from there, we are going to solve the first equation one and the second equation two; in equation one, I would have to find the values of "a," "b" and "c," what are the values of "a," "b" and "c" in equation one? [...].

P1: Here, "a" goes with "x," we said that "a" had to be a rational number, what number accompanies the "x"?

S: One.

P1: One, what number goes with "y"?

S: One.

P1: What number goes with, excuse me, what number is to the right of the equal sign?

S: Two.

P1: And that value is?

S: "c."

P1: "c," so which values are going to be, now, "d," "e" and "f"?

S: One and one.

S: One, two, and two [...]

P1: We're going to call the letters unknowns, okay, the letters that you are telling me, unknowns, "x" and "y," it's best to always have both on the same side of the equation, and everything that's a number or numerical coefficient, the independent number that appears, always on the right side of the equation [...].

In the preceding extract, the teacher reinforces the notion of the unknown, which is also made explicit in the definition given in Figure 4. In the three examples presented, the teacher guides the

Un Sistema de ecuaciones lineales con dos incógnitas tiene la forma:

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases} \quad \text{Donde, } b, c, d, e, \text{ y } f \text{ son números racionales, } x \text{ e } y \text{ son las incógnitas}$$

La **solución del sistema** es la solución común en ambas ecuaciones y corresponde al punto de corte de las rectas asociadas a las ecuaciones.

Para **resolver un sistema de ecuaciones**, puedes utilizar diferentes métodos. A continuación, se representan los métodos gráfico, por igualación, por sustitución y por reducción.

$$\begin{cases} x + y = 2 \\ x + 2y = 2 \end{cases} \quad \begin{cases} x - 6y = -46 \\ 2x + y = -1 \end{cases} \quad \begin{cases} -\frac{1}{2}x + 2y = -8 \\ 2x + y = 5 \end{cases}$$

Translation:

A system of equations with two unknowns has the form:

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases} \quad \text{Where } a, b, c, d, e, \text{ and } f \text{ are rational number, and } x \text{ and } y \text{ are unknowns.}$$

The **solution of the system** is the solution common to both equations and corresponds to the intersection point of the lines of the two equations.

To **solve a system of equations**, you can use different methods. Here, the **graphical, equalization, substitution, and elimination** methods will be presented.

$$\begin{cases} x + y = 2 \\ x + 2y = 2 \end{cases} \quad \begin{cases} x - 6y = -46 \\ 2x + y = -1 \end{cases} \quad \begin{cases} -\frac{1}{2}x + 2y = -8 \\ 2x + y = 5 \end{cases}$$

FIGURE 4 Definition of system of linear equations, proposed as example 3 (Translated by authors).

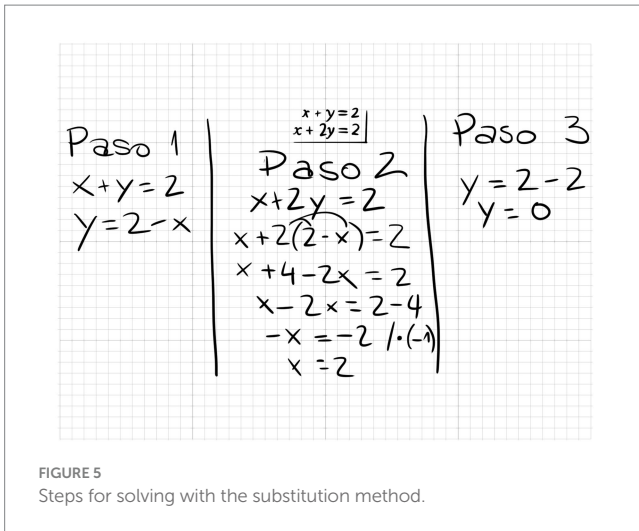


FIGURE 5 Steps for solving with the substitution method.

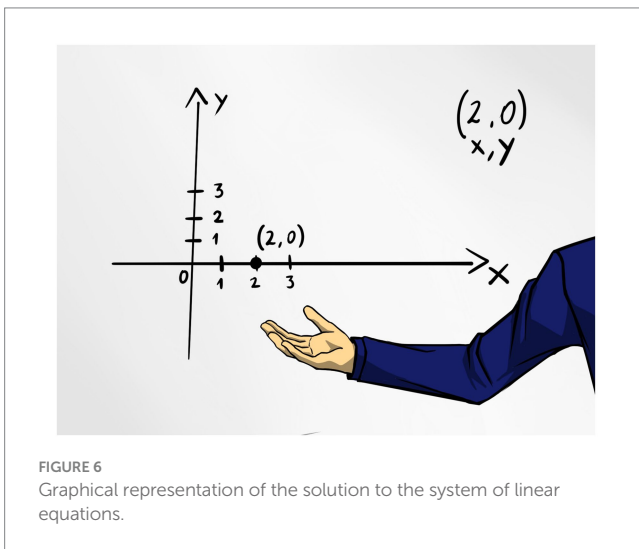


FIGURE 6 Graphical representation of the solution to the system of linear equations.

students to associate the values of a , b , c , d , e , and f , which are treated as rational numbers, with the definition shown in Figure 4. Likewise, he makes reference to a technique for solving the equations, but without representing it on the board.

In the second part, P1 indicates that the system of equations should be organized before being solved: “when it’s not organized, we are going to have to organize the unknowns, with both on one side and the numbers on the other side.” He continues with what he calls the *steps* of the substitution method. To do so, he considers the first system of linear equations, presented previously (see Figure 4), focusing on the steps that he is writing on the board (Figure 5). While writing on the board, he says the following:

P1: [...] okay, so, we are going to do the first method, the substitution method,..., these are the four steps that we need to follow to solve with the substitution method, the first step tells me I need to get rid of one of the unknowns “ x ” or “ y ,” handling equations, right, in which of the equations in this system,..., right, second step, replace that expression that we are going to obtain in the other equation,..., once I find the value of the other unknown

I’m going to replace it in each of the two equations, and I’m going to find the value of the unknown I did not have yet, then, what’s left is verifying [...].

It is worth noting that, in solving this example, P1 points out the imprecision on the part of the students in *canceling out a number* when the equation is solved. However, he lets this moment justify the procedure, pointing out the following:

P1: Okay, if we get rid of “ x ,” what’s going to happen to this equation?

S: Plus “ y ” equals two plus “ x .”

P1: Ok, “ y ” is going to equal 2 minus “ x ,” right, what did I do here? So from this step to this step...

S: You cancel out the “ x ” by subtracting.

P1: Right, just like you are telling me, I cancel out the “ x ” by subtracting, but do you guys remember that I told you that it wasn’t exactly like that?

S: How was is not like that?

P1: Canceling out the “ x ” by subtracting, do things get canceled out by subtracting, canceled out by adding?

S: No, teacher.

P1: Right, I explained it to you guys at some point, it must have been in one or two classes.

It should be noted that the final step of verifying the solution obtained is only mentioned verbally by P1—it is not written on the board. Meanwhile, in the third part of the example, P1 graphs the point (2,0) to illustrate the solution on the board (Figure 6).

In the next part of the class, the teacher proceeds with the second system of linear equations presented (see Figure 4), which is solved using the same method. In this case, he gives space for students to solve the example on their own and encourages interaction among them in the process. After this, the teacher solves the example on the board and graphs it to visualize the results. P1 tells the students that there are two algebraic methods to solve systems of equations (addition/subtraction and reduction) left to cover for the next class.

4.1.3.2 Analysis of circulation and category

In terms of theoretical analyses, in the first part of the example (see Figure 4), the referential component of the epistemological plane is present, provided by the presentation of the definition of a system of linear equations. Likewise, the activation of semiotic genesis in the work of P1, and especially the activity of identification of the signifying units (Duval, 1995) in the equations represented in the algebraic register when distinguishing the variables and unknowns, are also present.

In the second part, P1 privileges semiotic genesis and approaches in the algebraic register when solving the equations, using the

substitution method. Likewise, this method is used as a semiotic tool, resulting in the activation of the [Sem-Ins] vertical plane.

It should be noted that P1 implicitly alludes to certain properties of real numbers used to solve equations (additive inverse, multiplicative inverse). However, this entails the implicit activation of the referential component, since it is only related to the fact of *clearing* an equation. Additionally, P1 adheres to the definition obtained from a school-level textbook (Fresno Ramírez et al., 2022), which is not adapted for the teacher—this is evidenced when he describes “canceling out the x by subtracting” instead of referring to the additive inverse of the unknown x . This highlights aspects of the current suitable MWS of P1 regarding the teaching of systems of equations and his intention (implicit) to provide explanations of the operations used in this case.

The third part of the work of P1 privileges semiotic genesis, given the visualization of the solution obtained being represented in the graphical register using the system of Cartesian coordinates. The process of visualization observed is related to the iconic type, as it entails simply seeing the solution that corresponds to a point on the plane (representamen), which is represented freehand, constructed without the use of any artifact.

Thus, the example analyzed is associated with the *definition* category since it involves the first example presented immediately after defining a system of linear equations, in order to solve using the substitution method. The second system that the students must solve is associated with the *representation* category, since it involves an exercise applying the method that P1 has already taught, in which his participation is less active with the aim of providing more space for the participation and mathematical work of the students.

4.2 Case P2

In this case, four examples have been identified that P2 presents during a class on algebraic expressions; they are described and analyzed below.

4.2.1 Example 1: Activation of previous knowledge

4.2.1.1 Description of the use of the example

The first example presented by P2 is used to *illustrate* an algebraic technique studied in the previous class (see Figure 7), which entails

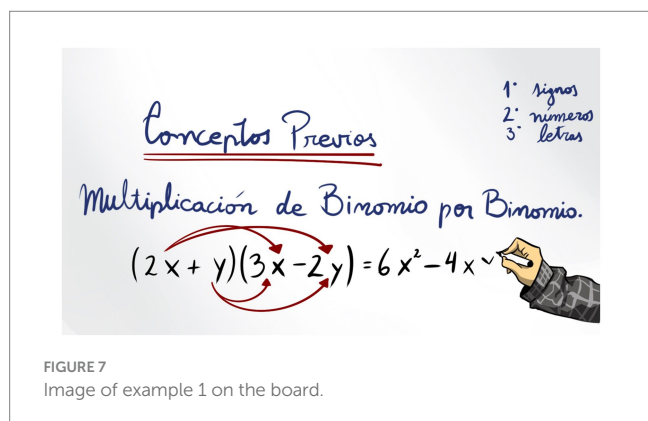


FIGURE 7 Image of example 1 on the board.

determining the product of the terms of two binomials, treated as variables. After, the teacher explains how to reduce the like terms, describing three steps (which she refers to as “rules”) for solving the problem that the students must follow and writing them on the board:

P2: Ah, you guys are telling me the order that I need to follow to do this multiplication, do you remember what we said about when we were multiplying in this case between algebraic terms, how we were going to have rules? Right? First, we are going to consider the signs, then we are going to consider the numbers, perfect, and then the multiplication, that is, in this case [...] the letters.

To solve the problem, the professor asks the students, “Do you remember this? We do a distribution,” this in order to calculate the multiplication between the terms and follow the steps. After this, the professor states the following:

P2: This first term I’m going to distribute across the binomial right, I have to multiply it by the binomial, then this “ y ,” I’m going to multiply it by the “ $3x$ ” and finally this “ y ” also, I’m going to multiply it by the “ $-2y$,” right, we were working on that, we said that we were going to do this here, first considering signs, first signs, the “ $2x$ ” with the “ $3x$,” both of them are positive or negative?

To avoid the having the students make errors when multiplying one term with another that is negative, the teacher indicates the negative sign on the board with her marker in the initial algebraic expression (see Figure 7). In this manner, the work becomes following the steps and analyzing the signs: “Positive and positive, what does that give me?,” in order to obtain and algebraic result.

4.2.1.2 Analysis of circulation and category

Regarding the circulation of the mathematical work that the teacher carries out in the classroom, there is an emphasis on semiotic genesis established by the algebraic expressions she presents and her algebraic procedures (in the sense of Duval, 1995), which involve the representamen and the process of visualization of this mathematical object. In this sense, despite the difficulty presented by the multiplication of two binomials—given that monomials of a numerical-algebraic type must be multiplied, and the management of the negative sign—the spotlight is on the distributive property of the multiplication over addition, operating from left to right. Thus, what occupies P2’s attention the most is demonstrating an algebraic development based on three steps, like a mechanical algorithm that does not justify, neither in terms of the design of a task applied to a given context, nor in a mathematical sense, placing scarce emphasis on the distributive property and reducing its use to that of an operational tool.

Based on the above, the intention of activating the referential on which the algebraic operations are based is evident, as the teacher alludes informally to the distributive property of the multiplication over the addition of the numbers (without indicating which numerical system it corresponds to). However, this is not made explicit but rather sidelined, given the emphasis on following what P2 terms *the rule*, which likewise indicates a rigid suitable MWS, one attached to sequential steps related to the distributivity of the product of the binomial. Hence, in terms of the vertical planes activated, the [Sem-Ins] plane emerges as favored,

along with a tendency to resort to the referential of the epistemological plane, without this being addressed in a clear or explicit manner.

We associate this example with the *definition* category, since P2 presents it immediately following the study of multiplication between binomials. Here, its role is to demonstrate the form in which this type of problem is solved under a rigid and passive organization of learning.

4.2.2 Example 2: Calculation of areas of figures

4.2.2.1 Description of the use of the example

The second example is used by P2 to *illustrate* a type of geometric exercise, studied in previous classes, that involves using two representations (figures and algebraic language). The initial representation is given by P2 in the figural register and the development is expressed in the algebraic register. In this case, the teacher draws two prototypical figures freehand (Hershkowitz, 1989) in order for their areas to be determined (Figure 8). She comments, “Look, there is a formula, in this case to be able to determine the area of this, like your classmate is saying, I’m going to multiply the base by the height.”

By way of “didactic contract” (Brousseau, 1988), the students must assume that the figures represented are a rectangle and a square, respectively, which is neither explained nor questioned. Likewise, certain magnitudes are assumed regarding letters *a* and *b* in order to express the length of the sides of the figures, but when observing the figures represented, the lengths are different for each, with the same letter *a* corresponding to different lengths, which is also not questioned (see Figure 9). At this point, P2 reminds the students how to express the multiplication of distinct monomials, stating the following:

P2: “*a*” times “*b*,” “*ab*,” very good, remember that when they were different letters we kept them, the same goes here; the area of this square, which would be the base? Which would be the height?”

The class works on this example as an extension of example 1, which entails the use of letters to represent numbers or magnitudes multiplied as algebraic expressions, and is linked to the next topic, notable products. Furthermore, P2 promotes interaction with the students more explicitly than in the work carried out in the previous example.

4.2.2.2 Analysis of circulation and category

From the perspective of P2’s mathematical work, an emphasis on semiotic genesis can be observed, as the central part of the work is based on the conversions (Duval, 1995) of the objects in question (rectangle and square), represented initially in the figural register, to work in the algebraic register and the corresponding procedures in this register of solution, in order to determine the area of the figures. Likewise, the freehand drawing done by P2 on the board stands out as it was done without the use of any type of artifact (material or technological) that could help show students the visualization of the figures. In this sense, the visualization that the professor uses for support corresponds to the perception of objects and their forms, without justification of the drawings or attention to the figural units of the representations and their decomposition; for this reason, the drawings are associated with iconic visualization (in the sense of Duval, 2016).

From a discursive perspective, the oral language used by P2 in the explanations given in class is not very rigorous and is far from the

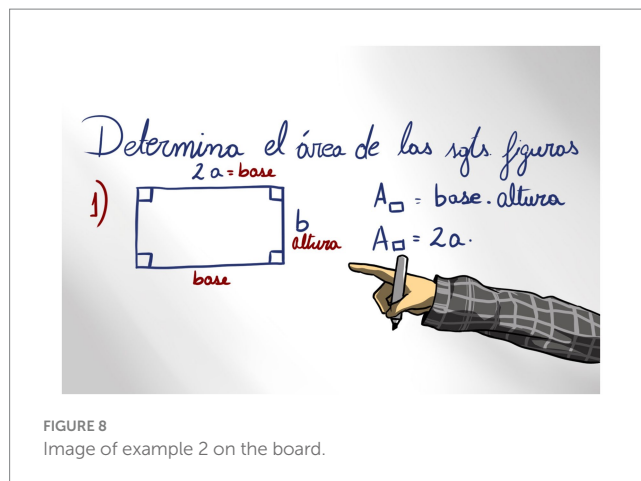


FIGURE 8 Image of example 2 on the board.

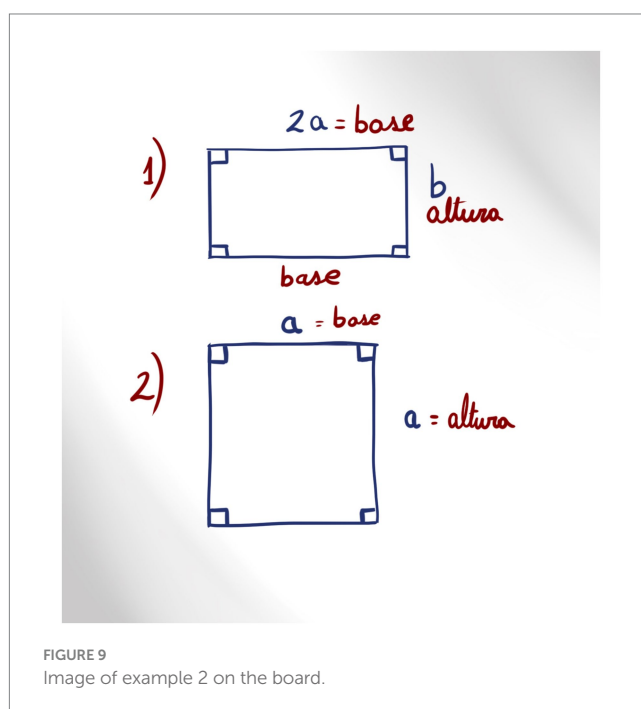


FIGURE 9 Image of example 2 on the board.

referential from the epistemological point of view. In a similar manner, this is observed in the justification of her freehand drawings. Therefore, the privileged plane in this case is [Sem-Ins], with the absence of any artifact in the construction of the quadrilaterals. In this case, the example falls within the category of *representation*, as it entails exercises typical of the application of the topic of study, which are presented as an extension of the first example seen. Additionally, P2’s intention to promote students’ work in a more active manner can be observed, even though it is ultimately she who solves the problem.

4.2.3 Example 3: Guided application of an algebraic technique

4.2.3.1 Description of the use of the example

Initially, the teacher dictates—for the students to write in their notebooks—the development of the square of a binomial. In doing so, she says the following:

P2: The square of a binomial or squared binomial is equal to the square of the first term, more or less, twice the product, plus, the second term squared [...] and please note this down, I'm going to put it up here on the board.

The teacher indicates that what they have written down corresponds to a definition, and she records it on the board in an algebraic manner, using a and b as variables and also using the numbers 1 and 2 as signs to orient the development of the terms (Figure 10).

Afterward, she presents the example meant to illustrate an algebraic technique, the development of the square of a binomial. First, she presents the square of a binomial that contains the numerical-algebraic expressions $2x$ and y , utilizes the equal sign, and on the left side of the expression frames the development of the notable product with spaces in place of the variables (Figure 11), indicating the following:

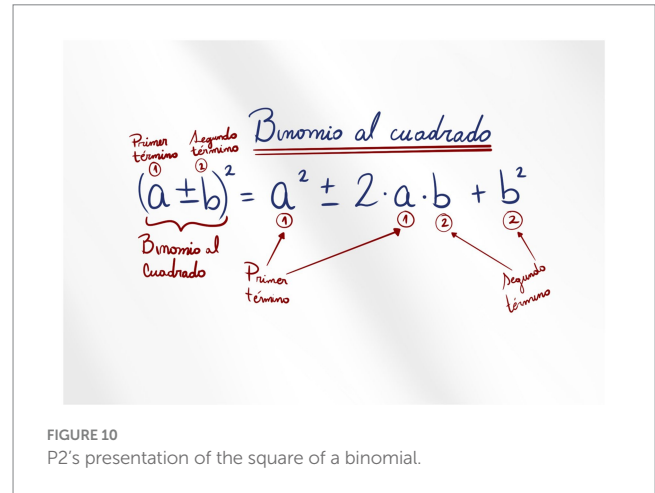


FIGURE 10 P2's presentation of the square of a binomial.

P2: We're going to apply the definition that we just made, but in a particular exercise.

This guided example is used to attain fluency in the solving of exercises that involve the application of the algorithm presented, which the teacher calls a *formula*, supported by signs (numbers 1 and 2) to orient the completion of the algebraic procedure. These signs have an algorithmic value within the example presented, and the teacher indicates them with a differently colored marker (see Figure 12) so the students can follow the steps of the procedure presented. Next, to validate this procedure, she presents the same example based on the distributive property of multiplication over addition (without indicating in which numerical system), performing the operations from left to right and reducing the like terms (Figure 13).

In this part of the lesson, P2 states the following:

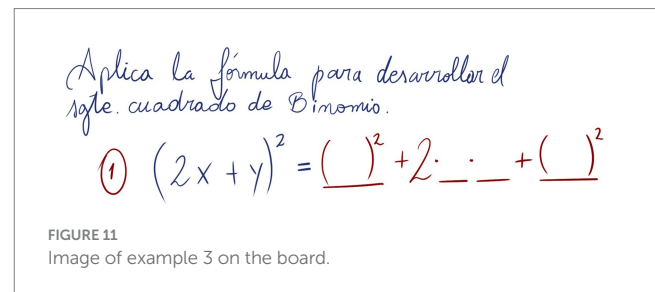


FIGURE 11 Image of example 3 on the board.

P2: In this case, we are going to apply the definition of the square of a binomial, okay, let us apply it directly, and then we are going to do it in a distributive way, which I think is the more common way to do it for you guys.

Lastly, she compares the equality of the solutions with both procedures, which are provided using the formula and a property of the numerical systems, to validate their results.

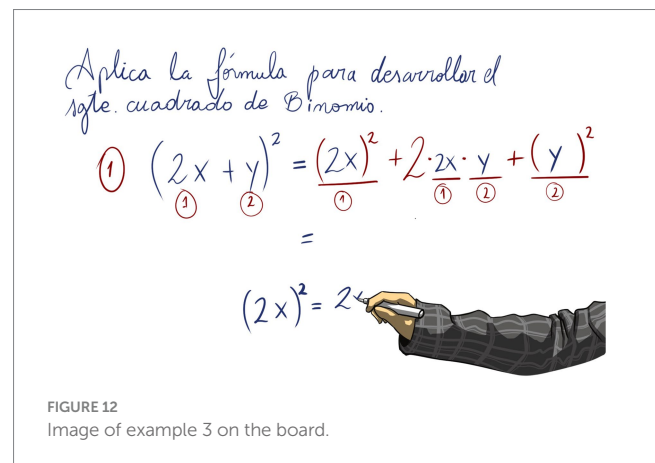


FIGURE 12 Image of example 3 on the board.

4.2.3.2 Analysis of circulation and category

The teacher begins with what she calls the *definition of the square of a binomial*, giving a formal air to the object of study, which activates the referential component. Next, in this example, used to illustrate and validate the application of an algebraic technique in exercises on the square of a binomial, the teacher privileges semiotic genesis, placing attention on certain procedures in the register of algebraic representation. In addition, the teacher uses a solution algorithm (formula), which is not justified and which has value as a semiotic tool. The signs employed in this instance of work (numbers 1 and 2) have an iconic value (Peirce, 1978), as they are used to direct the expressions that must be completed in the spaces provided. In this manner, up to this point, the plane activated is [Sem-Ins].

In a second part of the example presented by P2, with the intention of validating the previous algorithmic procedure, the work

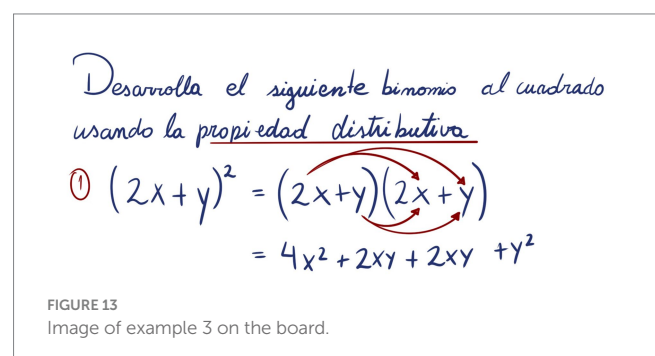


FIGURE 13 Image of example 3 on the board.

undertaken activates the referential component of the epistemological plane, as it relies upon a property of numerical systems (distributivity of multiplication over addition), carries out certain algebraic operations, and reduces like terms. Thus, in this part of the work, the [Sem-Dis] plane is activated. It should be noted that the aforementioned property is not justified, it is only used as an operational tool, and the link that could be established in this example with concrete numbers to support the validation presented is not addressed by P2.

In terms of the category of this classroom example, it is classified as *definition* since it is presented immediately after the teacher demonstrates what she calls the definition of the square of a binomial. First it is dictated in natural language, and then it is written algebraically. In this manner, example 3 follows the development of this *model* defined by the teacher (see Figure 10).

4.2.4 Example 4: Guided development with the use of representations

4.2.4.1 Description of the use of the example

This is the last example provided by P2, which is initiated with the presentation of the square of a binomial represented in the algebraic semiotic register; she indicates that students will need to show its development in pictorial form. This example is used to motivate tasks related to the development of the square of a binomial and validate its development. The teacher states the following:

P2: [...] in this case we find our regularity that always must be fulfilled in the square of a binomial, and now we are going to represent it and develop a squared binomial in a pictorial way, okay, that is going to be using, going to be using, guys, some geometric figures, to be able to represent it [...].

The teacher initiates this example with the expression $(a+b)^2$, presented in example 3, then writes it as the product of two binomials, $(a+b)(a+b)$, and establishes the relation between these expressions and the area of a square and its sides (see Figure 14), stating the following:

P2: [...] when I develop in this case the square of a binomial, I am in this case calculating the area of a square, if you look here, I am here multiplying one side, this is going to be a side of the square, and this here is going to be the other side of the square, I'm multiplying side by side.

Continuing, the teacher links the above with the representation of a square, drawn freehand, the length of whose sides is $(a+b)$, and she demonstrates the decomposition of the initial square into four figures, two squares and two rectangles, based on the lengths of their sides (see Figure 15).

In this manner, she calculates the total area of the initial square and shows, based on the decomposition of this figure, the calculation of the areas of each figure obtained (Figure 15), which she relates to the corresponding algebraic expressions.

Lastly, she shows that the sum of the areas of the figures obtained from the decomposition of the initial square is equal to the algebraic result of the development of the square of a binomial.

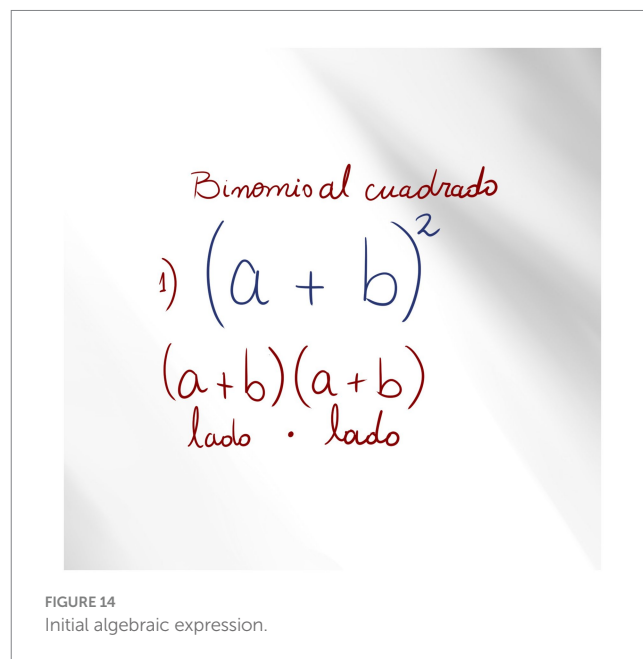


FIGURE 14
Initial algebraic expression.

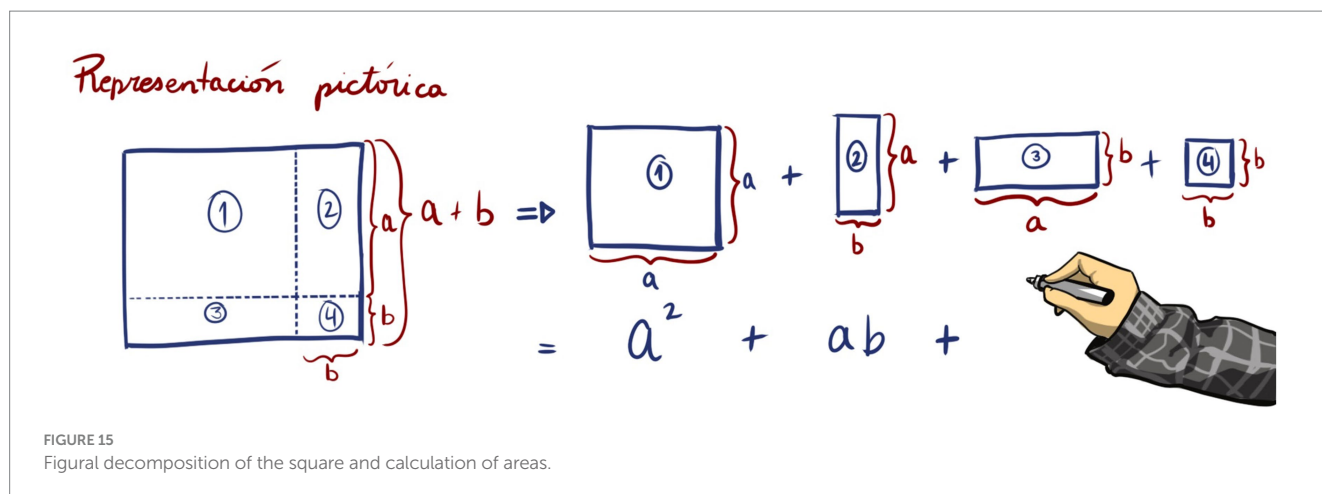
4.2.4.2 Analysis of circulation and category

From a theoretical perspective, this example is distinguished by the fact that the work of P2 places attention on semiotic genesis and the visualization of figures represented in the representamen. Delving into the semiotic perspective (Duval, 1995), on the one hand, the cognitive activity of conversion can be observed, as the initial representation is provided in the algebraic register and a transformation is produced to another representation in a distinct register (figural). It is important to note that these two representations are congruent since a semantic correspondence and univocality can be established between them, and moreover in the order of the units that compose each representation. This facilitates the explanation and understanding of the procedure involved in this example.

With regard to the operational activity carried out with the representation in the figural register, this is related to the process of non-iconic visualization (Duval, 2016), because while P2 relies on what can be seen in the figure, at the same time, she carries out the decomposition of the figure into its figural units within the same dimension (2D to 2D), which they reconfigure in a different way, using two squares and two rectangles that she calls (1), (2), (3), and (4). This activity is related to the inventor-artisan type of visualization, especially heterogeneous heuristic decomposition, since the decomposition is done with different figures.

From the above, the teacher privileges the conversion between the registers of algebraic and figural semiotic representation, then validates the algebraic procedure presented in example 3 through an approach with representations in the figural register, which entails recalling the calculation of areas of squares and rectangles (referential) from the domain of geometry. This is related with the appearance of the [Sem-Dis] vertical plane in the mathematical work of the discursive-graphical proof (Richard, 2004) carried out by the teacher.

Likewise, the intention for the *change of domain* stands out in the example presented by P2, from the domain of origin, the algebraic, to the geometric domain in the solution. It is noteworthy that P2 returns to the domain of origin, which allows her to validate the algebraic procedures undertaken in the previous examples.



In terms of the category of this example, it is classified within *internal applications* since it is presented in order to delve deeper into the class topic, and it involves the integration of geometric concepts that were taught in previous courses (parallelograms and the calculation of areas). This example is written and developed by the teacher on the board, with the participation of the students to contribute to certain procedures, and it also involves a mathematical task that is new for the students.

5 Discussion and conclusions

The cases of P1 and P2 presented in this study are developed and implemented by teachers in the classroom. Thus, attending to the research objective initially described, it is noteworthy that the mathematical work of P1 is characterized by encouraging interaction through posing questions and providing spaces for students to work in an autonomous manner. This generates a classroom dynamic that allows students to solve the example presented before the teacher offers an answer, or for them to work toward the solution jointly. It is important to mention that P1 relies on an interactive whiteboard during the full class, which allows him to resort to previous examples or examples from previous classes with ease, as the board saves the work done in previous classes, differing markedly from a typical whiteboard.

In the case of P2, the teacher herself is the one who develops the examples during the class, offering brief opportunities for interaction in which the students can state their answers during the development of the examples presented, or possibly come to alternative solutions other than what the teacher exhibits. The majority of the lesson is based on the presentation of distinct examples by the teacher, which, by not fostering the spaces necessary for dialogue and interaction, causes the class to become routine and behaviorist in manner.

Here, we present a synthesis of the analysis associated with the mathematical work of P1 and P2, based on the examples that they implemented in the teaching of secondary-level algebra topics (based on the Chilean national curriculum). Thus, the geneses, components, and virtual planes activated are described, as well as the classification of the examples involved in each case. Table 4 summarizes the analysis results based on the mathematical work of each teacher.

Table 4 demonstrates that P1, in example 1, activates the three geneses: semiotic, instrumental, and discursive. In particular, discursive genesis is associated with discursive reasoning, since it asks that students explain the procedure when solving a problem based on their knowledge. In this example, it is notable that, in contrast to the others, all of the geneses are activated, which is likewise reflected in the classroom interaction observed.

In example 2, P1 activates the [Sem-Dis] plane, where the activation of semiotic genesis is most prevalent, through the intention to associate natural language with that of the equation, which is numerical-algebraic. Likewise, in this example P1 activates the proof process through validation, but without resorting to the referential of the epistemological plane. From a didactic perspective, the importance of distinguishing and specifying the nature of an explanation and a proof is presented (Balacheff, 2000). Finally, in example 3, P1's activation of semiotic genesis and instrumental genesis stands out, facilitating the presence of the [Sem-Ins] plane. This is due to the fact that the system proposed is associated with a graphical representation, which is executed freehand.

In comparison to P1, P2 demonstrates a more traditional form of teaching in which she directly transmits a large store of knowledge that she has available. In general, the class is based on the presentation of four examples that P2 develops, and the students must copy in their notebooks (or take photos of) exactly what the professor writes on the board. In terms of MWS, examples 1 and 3 are representative of her suitable MWS when indicating *the rule* in the development of the square of a binomial and fomenting the completion of the square of a binomial following a visual pattern, respectively.

In terms of P2's example 1, it is related to a common type of task associated with the solution of *developing*, *factoring*, and *reducing* present in many Chilean classrooms, as attested to by Croset (2009) and Pilet (2012). In addition, there is a notable absence of relevant mathematical arguments, which has been addressed by other studies (Abou-Raad and Mercier, 2009; Croset, 2009; Assude et al., 2012). Likewise, in example 2, the teacher could have taken better advantage of the conversions between registers of semiotic representations and the transition between the numbers and the algebra, incorporating numerical examples that would allow for the questioning and validation of the algebraic result on the part of the students. It could have been an instance capable of promoting discursive reasoning to

TABLE 4 Summary of current suitable MWS of each teacher.

Teacher (P)	Example number	Typification of theoretical aspects									
		Semiotic genesis (SG)		Instrumental genesis (IG)		Discursive genesis (DG)		Analysis of examples			
		Representamen	Visualization	Artifact	Construction	Referential	Proof	Use of example	Classification		
P1	1	X	X (iconic)	X (technological tool)	X		X (explanation)	Illustrate	Definition		
	2	X	X			X (proof)	Contextualize	Representation			
	3	X	X (iconic)	X (semiotic tool)	X (freehand)	X	X (explanation)	Attain fluency	Definition and representation		
P2	1	X	X	X (operational tool)		X (implicit)		Illustrate	Definition		
	2	X	X (iconic)		X (freehand)		X (explanation)	Illustrate	Representation		
	3	X	X (iconic)	X (semiotic and operational tool)		X (definition)	X (validation)	Illustrate	Definition		
	4	X	X (non-iconic)		X (freehand)	X	X (validation)	Motivate	Internal application		

validate numerical and algebraic procedures, or this could also have been achieved through the use of technology or particular cases that would permit students to validate the results. In terms of MWS, it could be possible to foster instances of links with the work of semiotic genesis with the others (instrumental and discursive), which would imply offering an improved and more robust task given in class for the example in question. A revision of *a priori* planning and analysis could be beneficial in this regard.

Based on the analyses, it appears that P2 does not devote sufficient attention to the idea of *continuity* between the numerical and the algebraic (Kieran, 1992). This idea refers to the transition from the study of common number concepts to elementary algebra, which occurs between primary and secondary school. The examples that P2 utilizes serve to introduce algebraic expressions, but they lack a transition phase to help students pass from working with numbers to using algebraic symbols and expressions effectively (Pilet and Grugeon-Allys, 2021). In this sense, as stated by Ruiz-Munzón et al. (2012), this transition process is not immediate or spontaneous. For this reason, a contribution to the suitable MWS of P2 could consider the design and implementation of tasks that consider numerical-algebraic work and the distinct stages in the process of “algebrization,” which involves the understanding of algebraic expressions, equations, and formulas.

Thus, it emerges as relevant to discuss P2’s teaching and consider what such a discussion could raise in more challenging terms and with the use of diverse examples that point toward learning based on abilities proposed by the Chilean national curriculum [such as representing and solving problems, among others (Ministerio de Educación de Chile, 2015)]. In the line of Early Algebra (Kieran et al., 2016), the teaching of algebra based on the capacity to represent relationships between quantities should be underscored. A contribution in this sense could include the solving of elementary arithmetic exercises that require converting problems presented in natural language into the algebraic register to arrive at their solution. From the perspective of examples, this could encompass diverse uses and different problem types based on their classification. In particular, it would be beneficial to go into greater depth using examples that allow for the handling of the minus sign, which can present eventual obstacles or difficulties for students (Croset, 2009).

In the presentation of P2’s class, there was no gradual progression observed in terms of the complexity of the topic at hand on the part of the teacher, which could have been more effective for the learning of her students. However, a highlight from her lesson was the intention to focus on domain changes (from algebra to geometry), which she demonstrates in the development of example 4.

In view of the examples used by both teachers (see Table 4), it is evident P1 employs examples with diverse uses, beginning with an example to illustrate, then to contextualize, and finally increase fluency, which suggests clear organization of teaching in this aspect. In the case of P2, of the four examples utilized, three serve to illustrate and only one to motivate, which could have been taken better advantage of to facilitate domain changes, connect with other concepts in an explicit manner, and foster processes of non-iconic visualization on the part of the students. In terms of example classification, an inclination toward the definition and representation types can be observed in both cases analyzed. In this sense, both teachers could have included better alternatives with the aim of helping students to generalize (Watson and Mason, 2005).

An additional important factor to highlight is the modality of the schools of the two teachers in this study. In the case of P1, the school is Scientific-Humanist, while in the case of P2, the school is Technical-Professional. For P2, the potential importance of a professional review and attention to future proposals is underscored, for which this work could serve as an input or reference. In this sense, an improved scenario would be for teaching to give greater consideration to situated learning and the integration of other disciplines in contexts related to technical education, as observed by [Henríquez-Rivas et al. \(2023\)](#). This implies work based on task design and an interdisciplinary focus ([Watson and Thompson, 2015](#); [Stillman et al., 2023](#)).

In light of the results obtained, this study offers an opportunity to explore mathematical work in terms of examples and how they are implemented by teachers in the classroom. In this line of thinking, we can discuss the categories of examples and how the MWS model allows for them to be characterized and explored in greater depth, especially in terms of mathematical work and its relation to the specific examples used within a given category. Additionally, we can analyze how the MWS could vary based on the teaching style of every educator and analyze together (researchers, teachers-in-training, in-service teachers) which examples contribute to improved learning.

Ultimately, the choice of examples for learning is fundamental for the development of the class, since it can both limit and increase the understanding of a given topic of study for students. For this reason, it is imperative to reinforce and expand the use of examples in the organization and planning of the suitable MWS of teachers and their work in the classroom. This will allow for the numerical-algebraic transition to be harnessed in the best manner possible for learning. From the perspective of the MWS, future research could address the design of signature tasks ([Kuzniak et al., 2022](#)) that underpin this type of choice for teachers. Furthermore, additional studies could be carried out on the planning of examples for the classroom and the analysis of their effects in the classroom, which implies focusing on both the current and potential suitable MWS ([Henríquez-Rivas et al., 2022](#)). In this manner, remaining attentive to examples can be viewed as an opportunity to improve teaching and strengthen mathematical work in the epistemological and cognitive planes, which can be addressed from other mathematical domains as well as from teacher education in simulated contexts.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

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Ethics statement

The studies involving humans were approved by Comité de Ética Científico Universidad Católica del Maule. The studies were conducted in accordance with the local legislation and institutional requirements. The participants provided their written informed consent to participate in this study.

Author contributions

CH-R: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. PV-H: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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