An efficient spectral element method for two-dimensional magnetotelluric modeling

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We introduce a new efficient spectral element approach to solve the two-dimensional magnetotelluric forward problem based on Gauss−Lobatto−Legendre polynomials. It combines the high accuracy of the spectral technique and the perfect flexibility of the finite element approach, which can significantly improve the calculation accuracy. This method mainly includes two steps: 1) transforming the boundary value problem in the partial differential form into the variational problem in the integral form and 2) solving large symmetric sparse systems based on the combination of incomplete LU factorization and the double conjugate gradient stability algorithm through the spectral element with quadrilateral meshes. We imply the spectral element method on a resistivity half-space model to obtain a simple analytical solution and find that the magnetic field solutions simulated by the spectral element approach matched closely to the exact solutions. The experiment result shows that the spectral element solution has high accuracy with coarse meshes. We further compare the numerical results of the spectral element, finite difference, and finite element approaches on the COMMEMI 2D-1 and smooth models, respectively. The numerical results of the spectral element procedure are highly consistent with the other two techniques. All these comparison results suggest that the spectral element technique can not only give high accuracy for modeling results but also provide more detailed information. In particular, a few nodes are required in this method relative to the finite difference and finite element methods, which can decrease the relative errors. We then deduce that the spectral element method might be an alternative approach to simulate the magnetotelluric responses in two- or three-dimensional structures.

1 Introduction

As a special geo-electromagnetic method, magnetotelluric sounding can identify the resistivity or conductivity distributions in a geological medium based on harmonic and variable electromagnetic fields (Chave and Jones, 2012). Magnetotelluric sounding is based on naturally occurring electromagnetic fields, which can provide a comprehensive and continuous spectrum of the geo-electromagnetic field. This electrical resistivity, measured by comparing the electric field's horizontal component to the magnetic field on the surface, can detect a depth of several tens of kilometers associated with the acquisition frequency. With the rapid advancement in magnetotelluric modeling and inversion, it has become one of the

essential tools for recognizing deep geological structures (Unsworth, 2010; Avdeeva et al., 2012; Azeez et al., 2017; Nagarjuna et al., 2021) and geophysical investigations, such as geothermal exploration (Barcelona et al., 2013; Patro, 2017; Tarek et al., 2023), mineral deposit exploration (Benjamin et al., 2018; Jiang et al., 2022), and gas exploration (Zhang et al., 2014).

There are some numerical approaches for solving two-dimensional magnetotelluric forward problems, such as finite difference and finite element, and they are applied to two-dimensional magnetotelluric inversion (deGroot-Hedlin and Constable, 1990; Rodi and Mackie, 2001; Siripunvaraporn and Egbert, 2007; Lee et al., 2009; Kelbert et al., 2014; Guo et al., 2020; Liao et al., 2022). The finite difference numerical approach can solve partial differential equations for approximating first-order or second-order derivatives with a difference scheme (Pek and Verner, 1997; deGroot-Hedlin, 2006; Rao and Babu, 2006; Kumar et al., 2011). They also investigated the efficiency of computing two-dimensional magnetotelluric responses. This method also has high accuracy on the electric field and magnetic field components (Guo et al., 2018; Kalscheuer et al., 2018; Sarakorn and Vachiratienchai, 2018). Unfortunately, it is not easy to compute the resulting fields' accurate apparent resistivities and phases. As another important numerical approach, finite element can be applied to solve the two-dimensional magnetotelluric forward problem (Wannamaker et al., 1986; Key and Weiss, 2006; Franke et al., 2007; Lee et al., 2009; Sarakorn, 2017; Yao et al., 2021). It involves the hypothetical functional form of the model and the field in a small area of the specified geometry. The finite element method can introduce complex information from the real world to construct the initial model, including surface topography, and can also improve the flexibility of mesh discretization. However, it requires fine meshing to obtain high accuracy, which results in high computational costs. Some other numerical methods are also used to simulate two-dimensional magnetotelluric forward modeling, such as the boundary element (Xu and Zhou, 1997), the finite-volume (Du et al., 2016; Wang et al., 2019), the mesh-free (Wittke and Tezkan, 2014; Wittke and Tezkan, 2021), the domain decomposition (Bihlo et al., 2017), the numerical manifold (Liang et al., 2021), and the pseudo-spectral methods (Tong et al., 2020). These numerical methods provide an essential practical basis for two-dimensional magnetotelluric forward modeling.

Compared to other numerical approaches, the finite element method requires fine grids to obtain higher calculation accuracy. This will bring challenges, especially when computational resources are limited. Moreover, in practical geophysical applications, when the discrete meshes need be refined to a geo-electrical model, the convergence rate will decrease gradually, while the number of meshes and the computational cost can increase largely (Key and Weiss, 2006). The spectral method, as a novel approach, can provide the numerical approximation of partial differential equations (Zou and Cheng, 2018). In this numerical approach, the field in the computational domain can be approximated by polynomials or Fourier expansions. Since high-order orthogonal basis functions are applied in the spectral method, it has exponential convergence. In addition, the spectral interpolation points are densely distributed at the boundary, which can avoid the Runge phenomenon in the traditional high-order interpolation (Tong et al., 2020). The method that combines the finite element and spectral method is called the
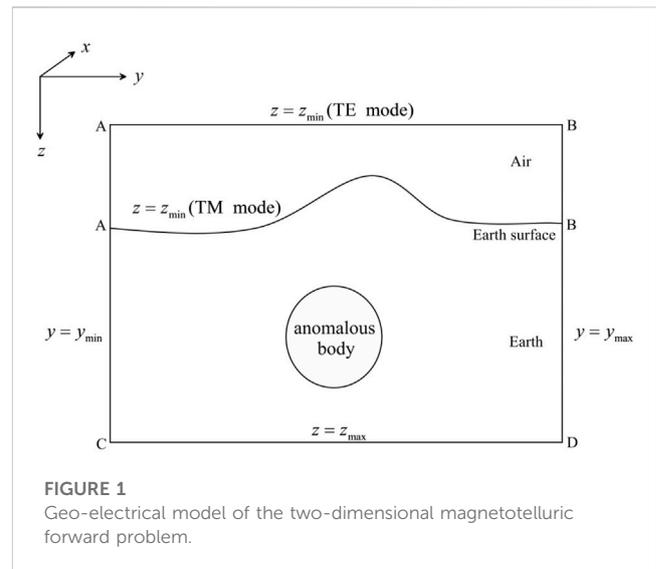
FIGURE 1
Geo-electrical model of the two-dimensional magnetotelluric forward problem.

spectral element method. In the past 20 years, geophysicists have dedicated these numerical methods to developing efficiency and accuracy. Some recent developments found that the spectral element approach can be seen as a high-order finite element method and its high-accuracy is derived from the properties of the spectral method (Patera, 1984). It can combine the high-accuracy of the spectral method and the flexibility of the finite element technique. Compared with the classical finite element method, the Runge phenomenon of isometric interpolation can be avoided using Gaussian orthogonal basis functions and Gaussian points (Xu et al., 2022). There are two types of spectral element methods, one based on Legendre polynomials and another based on Chebyshev polynomials. It is widely used in applications for wave propagation (Komatitsch and Tromp, 1999; Seriani and Oliveria, 2008; Luo et al., 2013; Trinh et al., 2019; Lyu et al., 2020), forward gravity modeling (Ghariti et al., 2018; Martin et al., 2017), and for geo-electromagnetic forward modeling problems (Zhou et al., 2016; Huang et al., 2019; Yin et al., 2019; Zhu et al., 2020; Huang et al., 2021; Weiss et al., 2023). However, it is rarely used in two-dimensional magnetotelluric forward modeling.

This paper proposes an efficient and accurate spectral element approach to compute the two-dimensional magnetotelluric responses of the boundary problem without measuring Earth's curvature. To benchmark the accuracy, we compare the numerical results of the spectral element forward algorithm with the analytical solutions and numerical results of the finite difference and finite element schemes. Although our approach can be applied to any two-dimensional geo-electromagnetic forward modeling, in this study, we demonstrate its implementation mainly in numerical experiments.

2 Boundary value problem

2.1 Electromagnetic equations

We define the z-axis at the depth and the x-axis along the geologic strike, as shown in Figure 1. Using a time-harmonic factor
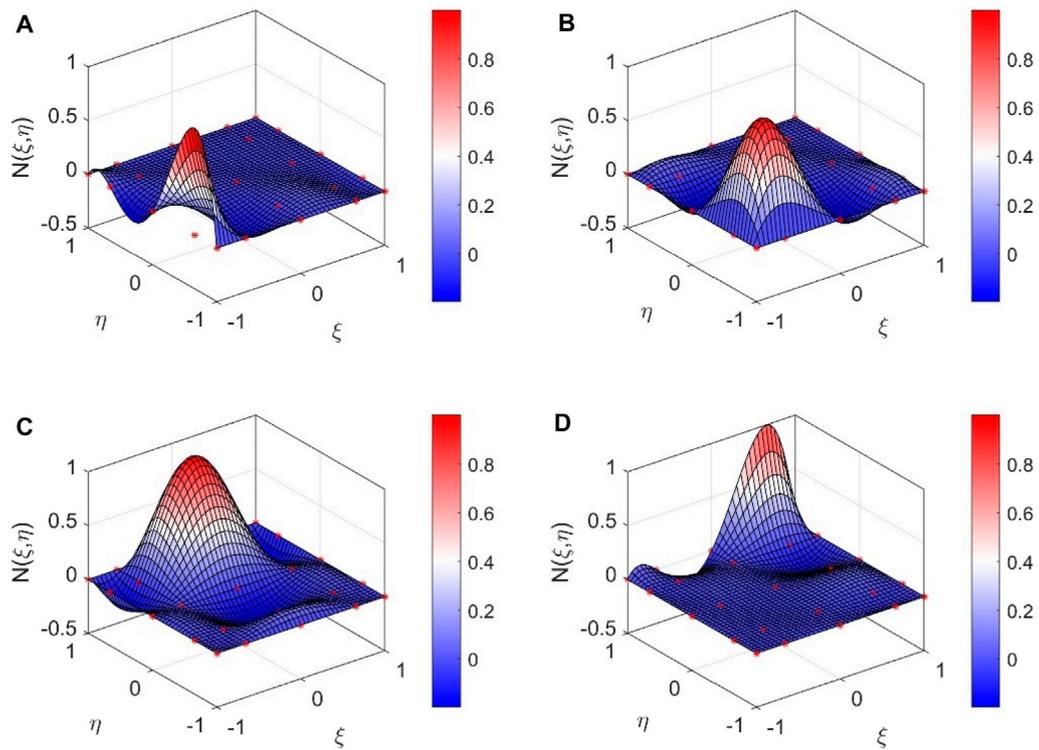
FIGURE 2
Two-dimensional spectral basis functions in part of order $p=4$. **(A)** $N_{12}(\xi,\eta) = \phi_1(\xi)\phi_2(\eta)$; **(B)** $N_{22}(\xi,\eta) = \phi_2(\xi)\phi_2(\eta)$; **(C)** $N_{34}(\xi,\eta) = \phi_3(\xi)\phi_4(\eta)$; and **(D)** $N_{45}(\xi,\eta) = \phi_4(\xi)\phi_5(\eta)$.
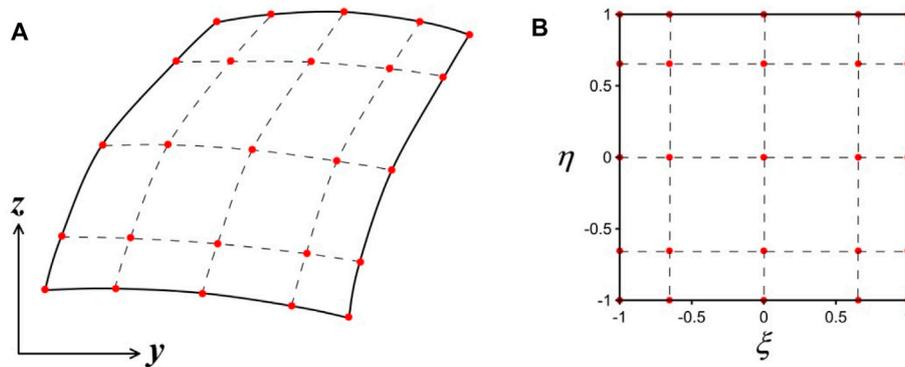
FIGURE 3
Mapping coordinate system of the spectral element of order $p = 4$. **(A)** Sub-element; **(B)** parent element.

$e^{-i\omega t}$, Maxwell's equations can then be written as (Wannamaker et al., 1987; Yao et al., 2021)

$$\nabla \times \boldsymbol{E} = i\omega\mu\boldsymbol{H} \qquad (1)$$

$$\nabla \times \boldsymbol{H} = \sigma\boldsymbol{E} \qquad (2)$$

where $\boldsymbol{E}$ means the electric field, $\boldsymbol{H}$ represents the magnetic field, ω denotes the angular frequency, $\mu = 4\pi \times 10^{-7}\,\text{H/m}$ is the magnetic permeability, and σ is the conductivity.

For a two-dimensional structure, due to $\partial\boldsymbol{E}/\partial x \equiv 0$ and $\partial\boldsymbol{H}/\partial x \equiv 0$, we can expand the curl operators in Eqs 1, 2 as follows:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = i\omega\mu H_x \qquad (3)$$

$$\frac{\partial E_x}{\partial z} = i\omega\mu H_y \qquad (4)$$

$$\frac{\partial E_x}{\partial y} = -i\omega\mu H_z \qquad (5)$$

FIGURE 4
Non-zero sparse elements of the discretization coefficient matrix obtained a 3 × 3 grid with a fourth-order polynomial spectral element approach.
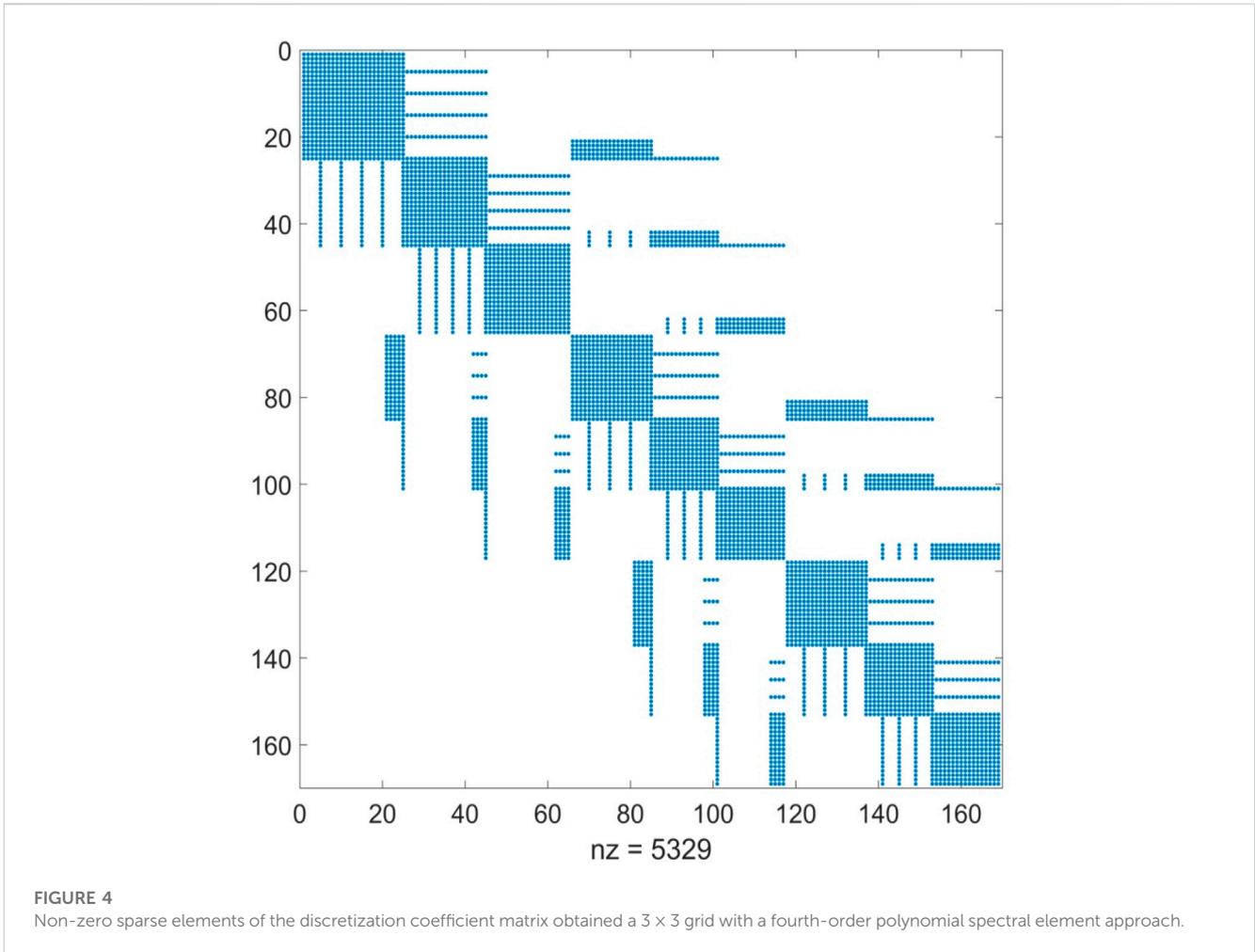
and

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \sigma E_x \tag{6}$$

$$\frac{\partial H_x}{\partial z} = \sigma E_y \tag{7}$$

$$\frac{\partial H_x}{\partial y} = -\sigma E_z \tag{8}$$

The electromagnetic equations are more complex than homogeneous media for two-dimensional modeling, where resistivity changes occur in the y-axis and z-axis. According to Eqs 4–6, E_x the TE mode can yield a second-order Helmholtz equation:

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + i\omega\mu\sigma E_x = 0 \tag{9}$$

Meanwhile, for the TM mode, H_x yields another second-order Helmholtz equation:

$$\frac{\partial}{\partial y}\left(\frac{1}{\sigma}\frac{\partial H_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\sigma}\frac{\partial H_x}{\partial z}\right) + i\omega\mu H_x = 0 \tag{10}$$

Then the electric field E_x or the magnetic field H_x in the Helmholtz-type equation can be rewritten as

$$\nabla \cdot (\tau \nabla u) + \lambda u = 0 \tag{11}$$

where u, τ, and λ represent different meanings relying on the different polarized modes. In the TE mode,

$$u = E_x, \tau = \frac{1}{i\omega\mu}, \lambda = \sigma \tag{12}$$

and in the TM mode,

$$u = H_x, \tau = \frac{1}{\sigma}, \lambda = i\omega\mu \tag{13}$$

2.2 Boundary conditions

We restrict the computational region for Eq. 11 to a two-dimensional bounded domain $\Omega = [y_{min}, y_{max}] \times [z_{min}, z_{max}]$, as shown in Figure 1. Therefore, the boundary conditions in the computational domain can be expressed as

$$u|_{z=z_{min}} = 1 \text{ (in AB)} \tag{14a}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=y_{min}} = 0 \text{ (in AC)}, \left.\frac{\partial u}{\partial y}\right|_{y=y_{max}} = 0 \text{ (in BD)} \tag{14b}$$
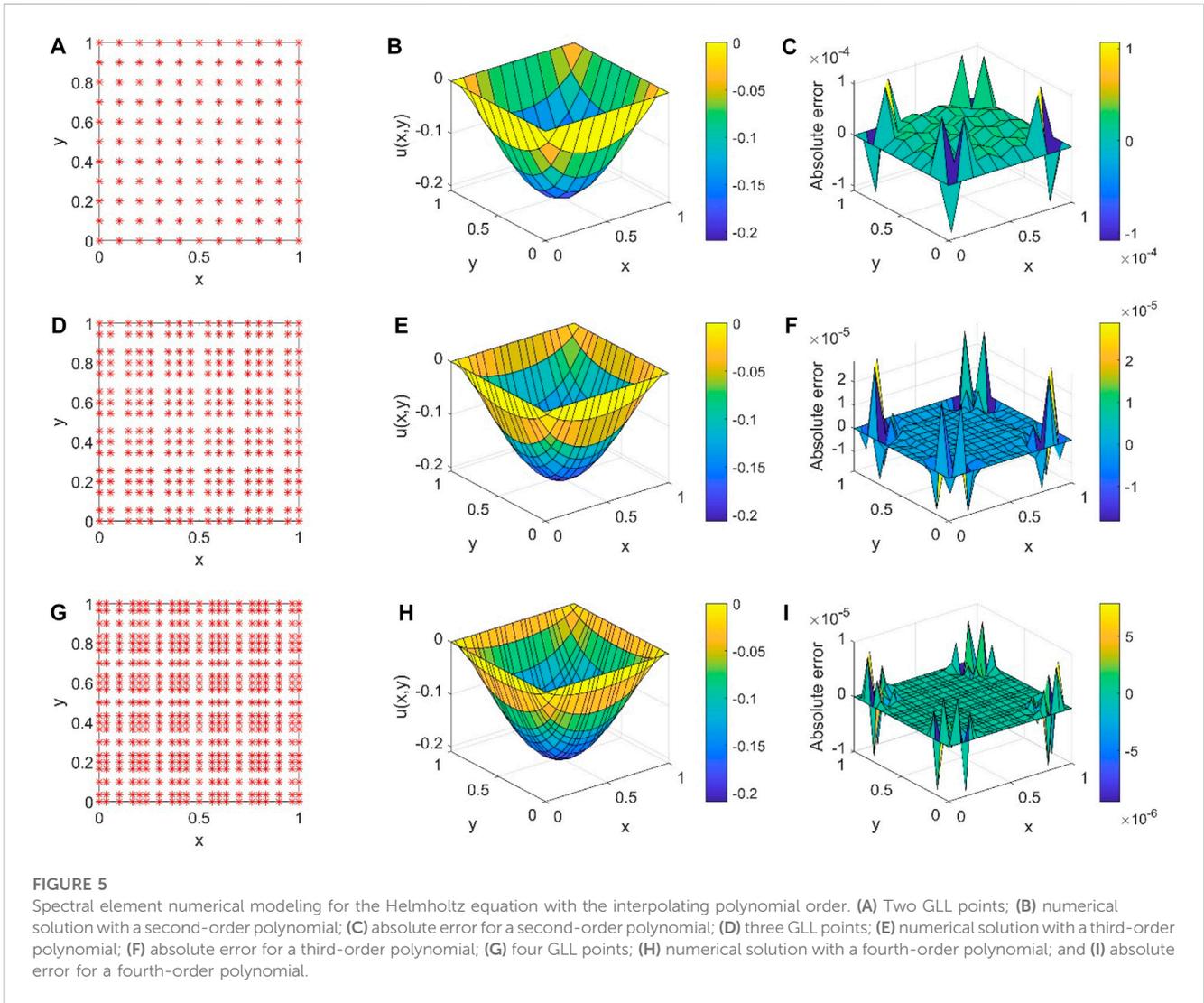
FIGURE 5
Spectral element numerical modeling for the Helmholtz equation with the interpolating polynomial order. **(A)** Two GLL points; **(B)** numerical solution with a second-order polynomial; **(C)** absolute error for a second-order polynomial; **(D)** three GLL points; **(E)** numerical solution with a third-order polynomial; **(F)** absolute error for a third-order polynomial; **(G)** four GLL points; **(H)** numerical solution with a fourth-order polynomial; and **(I)** absolute error for a fourth-order polynomial.

$$\left(\frac{\partial u}{\partial z} + ku\right)\Bigg|_{z=z_{max}} = 0 \ (\text{in CD}) \qquad (14c)$$

where $k = \sqrt{-i\omega\mu\sigma}$.

3 Spectral element formulation

3.1 Discretization of a variational problem

The magnetotelluric fields can be simulated by the Helmholtz equation of Eq. 12 under the boundary conditions of Eq. 14. Using the variational principle (Pozrikidis, 2014), the boundary value problem of the partial differential form displayed in Eq. 12 and Eq. 14 can be written as the variational problem of the integral form:

$$\begin{cases} F(u) = \iint \left\{\frac{1}{2}\tau\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] - \frac{1}{2}\lambda u^2\right\}dydz + \int_{CD}\frac{1}{2}\tau ku^2 dl = \min \\ u|_{y=y_{min}} = 0. \end{cases}$$

$$(15)$$

Within spectral element approximation, the magnetotelluric field can be expanded with two-dimensional interpolation basis functions:

$$\boldsymbol{u} = \sum_{i=1}^{N_r} N_i(\xi, \eta)u_i \qquad (16)$$

where N_r is the number of primary procedures for an element and $N_i(\xi, \eta)$ are the essential functions.

The integral of all elements, Eq. 15, can be rewritten as

$$F(u) = \sum \iint_e \left\{\frac{1}{2}\tau\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] - \frac{1}{2}\lambda u^2\right\}dydz$$
$$+ \sum \int_{CD}\frac{1}{2}\tau ku^2 dl = \min \qquad (17)$$

This will lead to a discrete linear equation as follows:

$$\boldsymbol{Ku} = 0 \qquad (18)$$

where u represents the values of the unknown magnetic field or electric field.
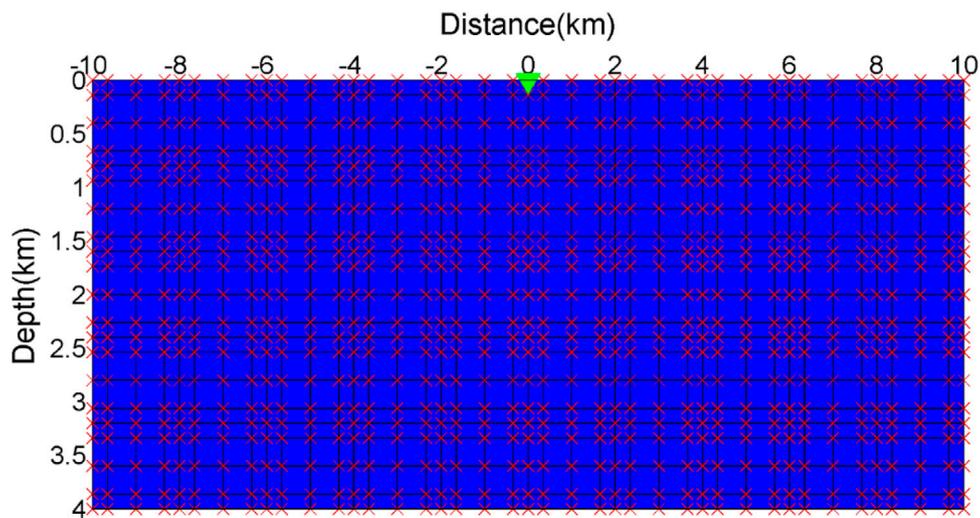
FIGURE 6
Homogeneous half-space model meshed with four GCL points per element.
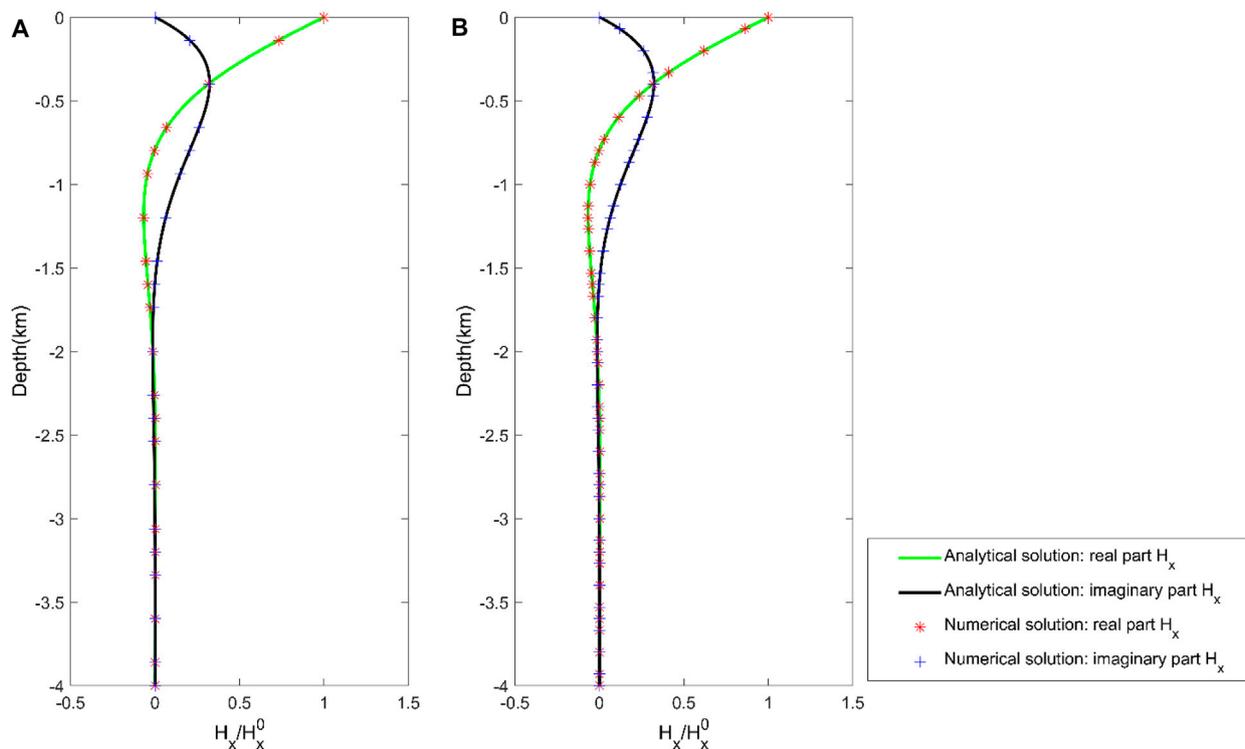
FIGURE 7
Spectral element numerical solution of magnetic field H_x for the frequency f=10 Hz in the half-space resistivity model. The number of elements in the depth direction for **(A)** Nz = 5 and **(B)** Nz = 10.

TABLE 1 RMS errors of the magnetotelluric responses for the half-space resistivity model.

Frequency (Hz)	TM mode		TE mode	
	Apparent resistivity	Phase	Apparent resistivity	Phase
0.01	7.73e-9	3.11e-8	7.71e-9	3.09e-8
0.1	1.83e-6	1.74e-7	1.81e-6	1.72e-7
1	1.69e-4	5.32e-5	1.62e-4	5.24e-5
10	1.26e-2	1.25e-2	1.21e-2	1.22e-2
100	0.39	1.42	0.36	1.39

3.2 Spectral basis functions

Its spectral accuracy characterizes the spectral element, i.e., the numerical error depends on the order of the basis functions (Lee and Liu, 2005). We choose Gauss–Lobatto–Legendre (GLL) element discretization for the magnetotelluric forward problem. The Nth-order GLL basis functions in a one-dimensional reference element $\xi \in [-1, 1]$ can be expressed by

$$\phi_i(\xi) = \frac{1}{N(N+1)L_N(\xi_i)} \frac{\xi^2 - 1}{\xi - \xi_i} L_N'(\xi) \tag{19}$$

for $i = 1, 2, \cdots, N+1$, where $L_N(\xi)$ is the Nth-order Legendre polynomial and $L_N'(\xi)$ is its derivative. On a reference element with $\xi, \eta \in [-1, 1] \times [-1, 1]$, the two-dimensional basis function can be written as

$$N_{ij}(\xi, \eta) = \phi_i(\xi)\phi_j(\eta) \tag{20}$$

For example, if order $p = 4$, there are 25 basis functions to the interpolation nodes. Figure 2 shows two-dimensional basis functions of the 4th order in part, and four of the nodal basis functions corresponding to $N_{12}(\xi, \eta) = \phi_1(\xi)\phi_2(\eta)$, $N_{22}(\xi, \eta) = \phi_2(\xi)\phi_2(\eta)$, $N_{34}(\xi, \eta) = \phi_3(\xi)\phi_4(\eta)$, and $N_{45}(\xi, \eta) = \phi_4(\xi)\phi_5(\eta)$ are represented.

3.3 Spectral element equation

In the spectral element method, a physical sub-element needs to be mapped into a reference parent element and the element coefficient matrix can be achieved in the reference element. Figure 3 shows a mapping example of a two-dimensional spectral element (y, z)-coordinate and the normalized $\xi, \eta \in [-1, 1] \times [-1, 1]$ reference coordinate.

The derivatives and the volume in the (y, z)-coordinate system in Eq. 17 can be transformed to the (ξ, η)-coordinate system as follows:

$$\mathrm{d}y\mathrm{d}z = \begin{vmatrix} \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \end{vmatrix} \mathrm{d}\xi\mathrm{d}\eta = |J|\mathrm{d}\xi\mathrm{d}\eta \tag{21}$$
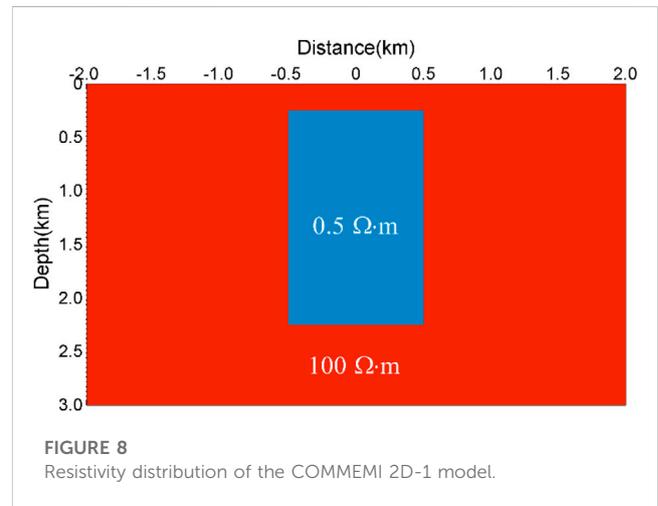
FIGURE 8
Resistivity distribution of the COMMEMI 2D-1 model.

where J is the Jacobian matrix.

The first-term integral in Eq. 17 is

$$\iint_e \frac{1}{2}\tau\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right]\mathrm{d}y\mathrm{d}z = \frac{1}{2}\boldsymbol{u}_e^T \boldsymbol{K}_{1e}\boldsymbol{u}_e \tag{22}$$

where

$$\boldsymbol{K}_{1e} = \iint_e \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right]\mathrm{d}y\mathrm{d}z$$

$$= \int_{-1}^1 \int_{-1}^1 \tau\left(\frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial N_j}{\partial \xi}\frac{\partial \xi}{\partial y}\right)|J|\mathrm{d}\xi\mathrm{d}\eta + \int_{-1}^1 \int_{-1}^1 \tau\left(\frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial N_j}{\partial \eta}\frac{\partial \eta}{\partial z}\right)|J|\mathrm{d}\xi\mathrm{d}\eta$$

The second-term integral in Eq. 17 is

$$\iint_e \frac{1}{2}\lambda u^2 \mathrm{d}y\mathrm{d}z = \frac{1}{2}\boldsymbol{u}_e^T \boldsymbol{K}_{2e}\boldsymbol{u}_e \tag{23}$$

where $\boldsymbol{K}_{2e} = \iint_e \lambda u^2 \mathrm{d}y\mathrm{d}z = \int_{-1}^1 \int_{-1}^1 \lambda N_i N_j |J|\mathrm{d}\xi\mathrm{d}\eta$.

The third-term integral in Eq. 17 is

$$\int_{CD} \frac{1}{2}\tau k u^2 \mathrm{d}l = \frac{1}{2}\boldsymbol{u}_e^T \boldsymbol{K}_{3e}\boldsymbol{u}_e \tag{24}$$

where $\boldsymbol{K}_{3e}$ can be obtained with a one-dimensional line integral.

Considering the Dirichlet boundary condition at $z = z_{\min}$, we can derive the linear equations with the spectral element approach to build the two-dimensional magnetotelluric forward modeling:
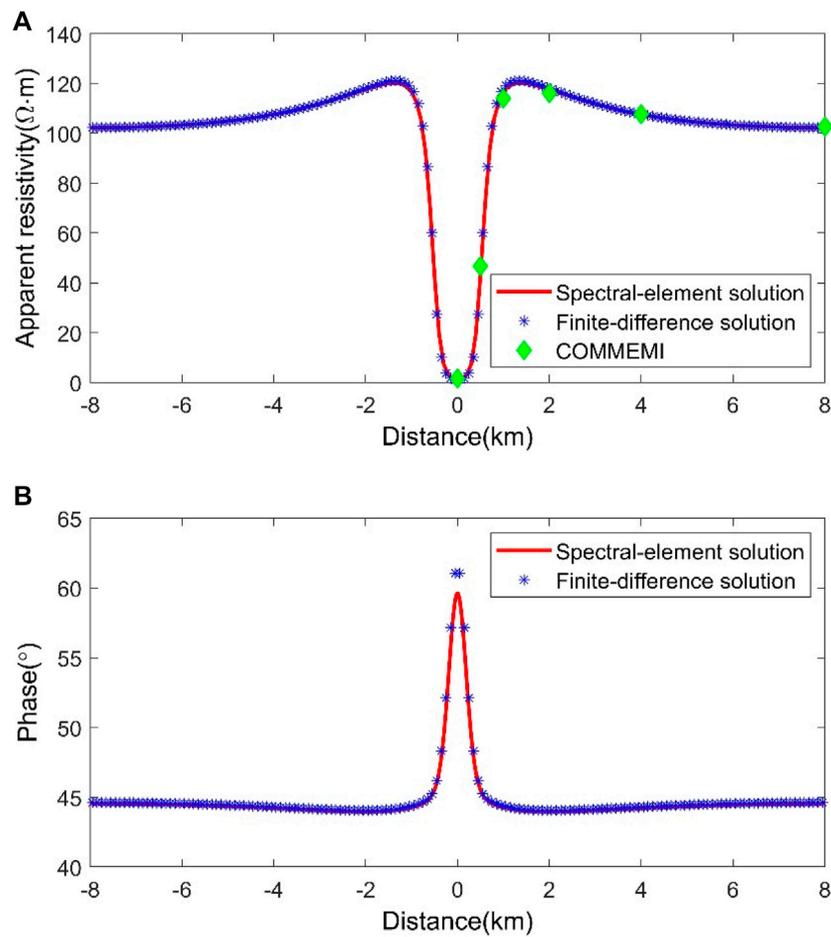
FIGURE 9
Comparison of numerical results for the COMMEMI 2D-1 model in the TM mode. **(A)** Apparent resistivities and **(B)** phases.

$$Ku = p \qquad (25)$$

where $K = \sum_{i=1}^{N_e} K_{1e} - \sum_{i=1}^{N_e} K_{2e} + \sum_{i=1}^{N_e} K_{3e}$ is the coefficient matrix containing conductivity parameters σ, and N_e is the number of all elements. The right-side vector p is interrelated to the Dirichlet boundary conditions. The coefficient matrix K is a sparse, positive-defined, symmetric matrix. Figure 4 shows the sparse elements' distribution of the coefficient matrix for a 3×3 grid with the 4th polynomial order (just for illustration purposes). We use the biconjugate gradient-stabilized algorithm (van der Vorst, 1992; Chen et al., 2002), preconditioned with incomplete LU factorization (Pan et al., 2022), for our forward problem.

After obtaining E_x, the corresponding magnetic field component H_y can be solved by Eq. 4 for the TE mode. Since we acquired H_x, the corresponding electric component E_y can be solved by Eq. 7 for the TM mode. Then, we can obtain the two-dimensional impedance tensor from

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} 0 & Z_{xy} \\ Z_{yx} & 0 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \end{bmatrix} \qquad (26)$$

The impedance can be used to calculate apparent resistivities

$$\rho_a^{xy} = \frac{1}{\omega\mu}\left|Z_{xy}\right|^2, \rho_a^{yx} = \frac{1}{\omega\mu}\left|Z_{yx}\right|^2 \qquad (27)$$

and impedance phases

$$\phi^{xy} = \arctan\frac{\mathrm{Im}\left[Z_{xy}\right]}{\mathrm{Re}\left[Z_{xy}\right]}, \phi^{yx} = \arctan\frac{\mathrm{Im}\left[Z_{yx}\right]}{\mathrm{Re}\left[Z_{yx}\right]} \qquad (28)$$

4 Accuracy of the method

For all the spectral element numerical approaches, the numerical solution of the boundary value problem depends on two parameters: (1) the size of each spectral element and (2) the interpolating polynomial order. To verify our spectral element method numerically, we consider the Dirichlet boundary for a Helmholtz equation

$$\nabla^2 u - u = 3 \qquad (29)$$

with the exact solution

FIGURE 10
Comparison of numerical results for the COMMEMI 2D-1 model in the TE mode. **(A)** Apparent resistivities and **(B)** phases.
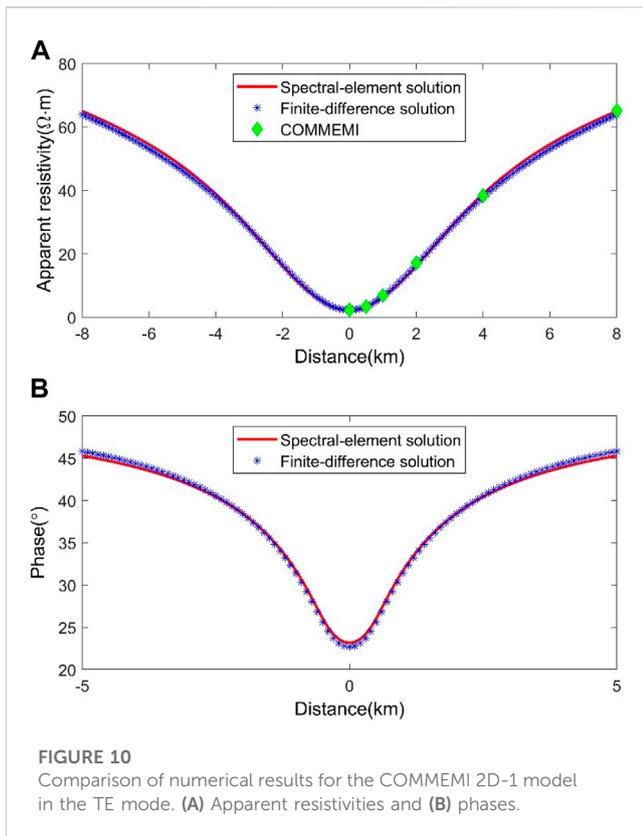
$$u_e\left(x, y\right) = \sum_{m=1,3,\cdots}^{\infty} \sum_{n=1,3,\cdots}^{\infty} -\frac{48}{\left(1 + \pi^2 m^2 + \pi^2 n^2\right)} \frac{1}{mn\pi^2} \sin\left(n\pi x\right) \sin\left(m\pi y\right) \quad (30)$$

The physical domain $\Omega = [0, 1] \times [0, 1]$ is discretized using a uniform mesh with 5×5 elements. Figure 5 shows the spectral element solutions obtained using different interpolating polynomial
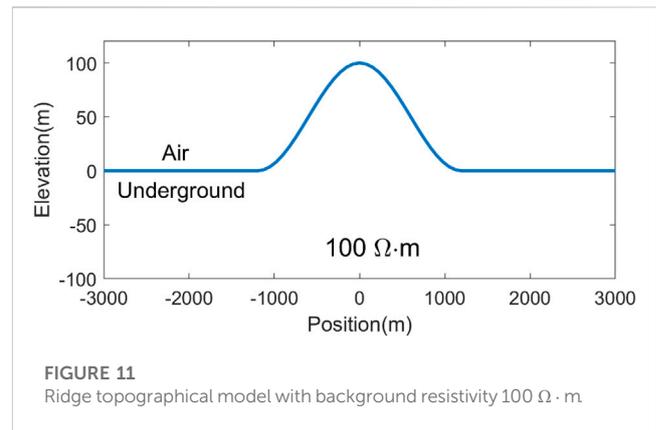
orders. The maximum absolute errors of spectral element solutions for 2, 3, and 4th polynomial orders are 1.06E-4, 2.89E-5, and 7.84E-6, respectively. The numerical results computed by our spectral element approach are generally consistent with the series solutions.

5 Model studies and discussion

5.1 Homogeneous half-space

We developed a half-space resistivity model to test the high-accuracy benchmark of our spectral element scheme. The half-space resistivity is designed as $10\ \Omega \cdot m$ and the computational domain is set as $20\ km \times 4\ km$. During numerical simulation, each spectral element includes 4 GLL points in the y-direction and z-direction, as shown in Figure 6. In addition, we assume that only one measuring point is located on the ground, marked by a green triangle. The numerical magnetic field H_x is simulated at $f = 10$ Hz.

We set the number of elements in the horizontal direction to 10 (i.e., $N_y = 10$), while the number of elements in the depth direction is designed to 5 and 10, respectively (i.e., $N_z = 10$ and 5). Figure 7

TABLE 2 Apparent resistivities simulated by the spectral element code compared to the COMMEMI results.

	0 m	500 m	1,000 m	2,000 m	4,000 m
p_c (T M)					
SEM, $p=1$	1.34	41.49	120.21	117.38	108.20
SEM, $p=2$	1.44	44.56	114.84	115.57	107.23
SEM, $p=3$	1.48	45.76	114.64	116.07	107.40
SEM, $p=4$	1.51	46.02	114.25	116.37	107.81
COMMEMI	1.60 ± 0.27	46.70 ± 3.64	114.01 ± 3.69	116.11 ±2.67	107.62 ± 2.25
p_a (TE)					
SEM, $p=1$	2.27	3.20	6.33	16.20	37.89
SEM, $p=2$	2.28	3.29	6.60	16.59	38.83
SEM, $p=3$	2.30	3.34	6.71	17.06	38.26
SEM, $p=4$	2.30	3.35	6.76	17.12	38.29
COMMEMI	2.31 ± 0.12	3.39 ± 0.36	6.86 ± 0.30	17.19 ± 1.09	38.35 ± 1.96
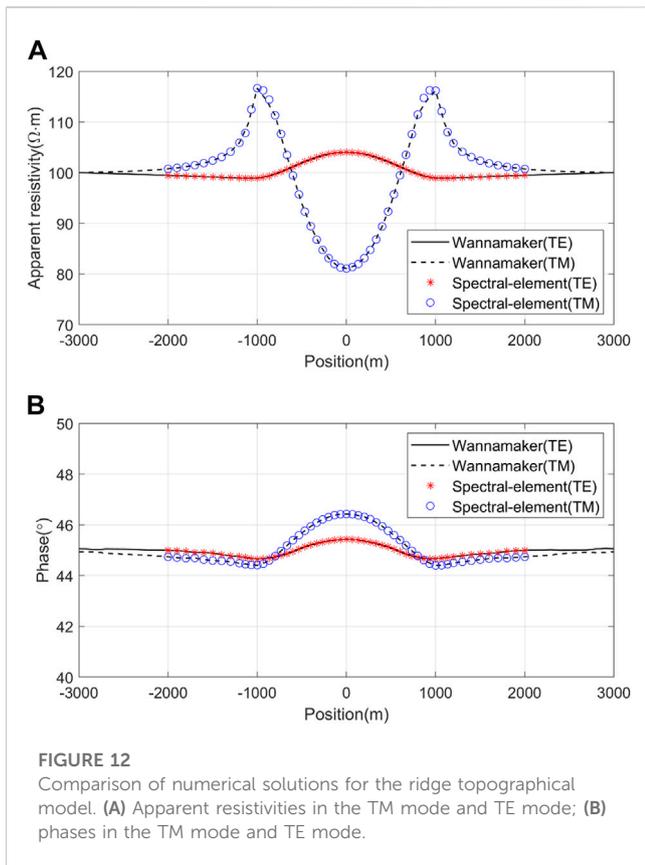
FIGURE 12
Comparison of numerical solutions for the ridge topographical model. **(A)** Apparent resistivities in the TM mode and TE mode; **(B)** phases in the TM mode and TE mode.

shows the numerical solution of the magnetic field H_x for the homogeneous half-space frequency of $f = 10$ Hz. They also offer that the real part and imaginary part of the magnetic field H_x calculated by the spectral element method agree with the analytical solution. This phenomenon also shows the correlation between the number of discrete elements and computational accuracy. Furthermore, it indicates that the number of discrete components does not affect the computational accuracy under high-polynomial order conditions. These results suggest that the spectral element approach can improve the accuracy for the two-dimensional magnetotelluric forward modeling.

To further verify the applicability of our spectral element approach, we increase the number of elements in horizontal and depth directions to 20. We then calculated the magnetotelluric response, including apparent resistivity and phase, at $f = 0.1, 1.1, 10, 10,$ and 100 Hz frequencies in the TM mode and TE mode. The computing time of our code is about 1.6 s for each frequency. The apparent resistivity for each frequency is identical to the true resistivity 10 $\Omega \cdot$ m, and the phase was equal to 45°. The RMS errors of the magnetotelluric responses are given in Table 1. It is clear that the RMS error is proportional to the frequency, with the minimum error at $f = 0.01$ Hz and the maximum error at $f = 100$ Hz. In general, the RMS error value falls within the frequency band that we set, which is also the acceptable error range. The results show that our spectral element approach has high computational accuracy.

5.2 COMMEMI 2D-1 model

We conducted a numerical experiment to compare with the finite difference method. This numerical experiment coincides with the COMMEMI 2D-1 example (Zhdanov et al., 1997), which can test the accuracy and reliability of the spectral element forward algorithm. The COMMEMI 2D-1 model is shown in Figure 8. A symmetrical, rectangular, low-resistivity body is inserted in a homogeneous conductive half-space. The rectangular anomaly body has a width of 1,000 m, a length of 2,000 m, and a burial depth of 250 m from the ground surface. The resistivity of the anomaly is set as = 0.5 $\Omega \cdot$ m, and the half-space resistivity is designed as $\rho = 100$ $\Omega \cdot$ m. The frequency we applied in the next two experiments was $f = 0.1$ Hz.

First, we simulated the numerical solutions for the COMMEMI 2D-1 model using the spectral element algorithm and the finite difference method (Tong et al., 2018). In this example, the uniform meshes of the model for the whole calculation area are set to $\Delta y \times \Delta z = 100\text{m} \times 50\text{m}$. For spectral element numerical modeling, each spectral element includes two GLL points in the y-direction and z-direction, and has nine points for each element. The numerical apparent resistivities and phases in the TM mode are shown in Figure 9. The results show that the numerical results calculated by the spectral element method match well with those of
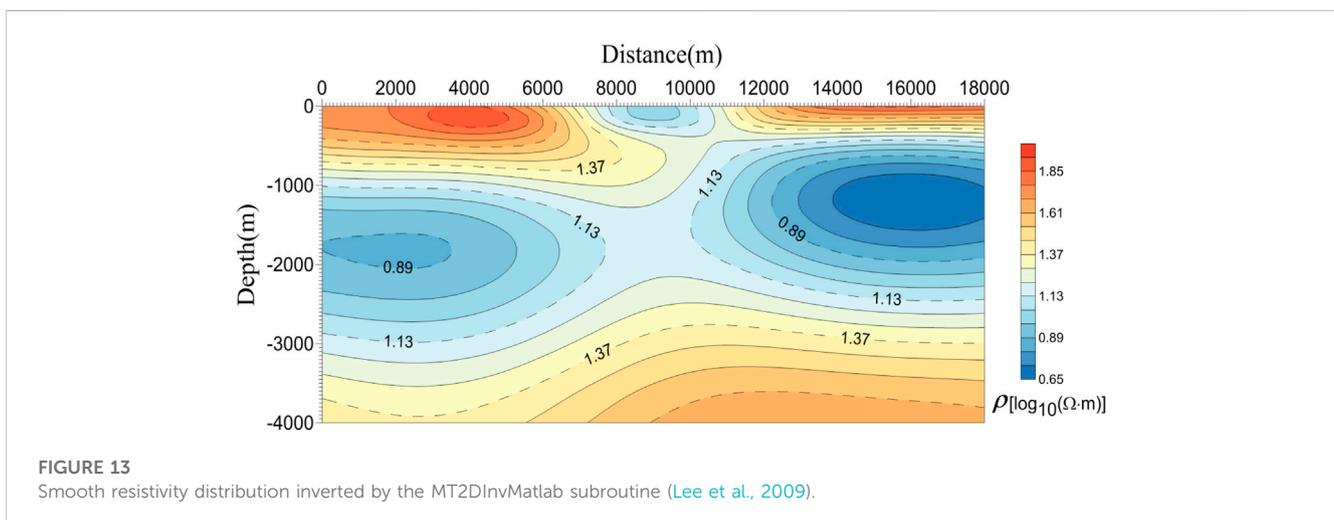

FIGURE 13
Smooth resistivity distribution inverted by the MT2DInvMatlab subroutine (Lee et al., 2009).

FIGURE 14
Comparison of numerical results for the smooth model. **(A)** TM apparent resistivity pseudo-section and **(B)** TM phase pseudo-section simulated by the spectral element method; **(C)** TM apparent resistivity pseudo-section and **(D)** TM phase pseudo-section simulated by the finite element method (Lee et al., 2009).

FIGURE 15
Comparison of numerical results for the smooth model. **(A)** TE apparent resistivity pseudo-section and **(B)** TE phase pseudo-section simulated by the spectral element method; **(C)** TE apparent resistivity pseudo-section and **(D)** TE phase pseudo-section simulated by the finite element method (Lee et al., 2009).

the finite difference method. Compared to finite difference results, the spectral element method shows a clear and continuous downward smoothness in TM mode calculations. The numerical results for the TE mode are shown in Figure 10. These results also matched well for the numerical magnetotelluric responses in the TE mode. The computing time of the COMMEMI 2D-1 model is about 0.5 s for the finite difference algorithm and about 1.8 s for the spectral element algorithm.

We also compare the numerical apparent resistivities calculated by the spectral element scheme and the finite difference approach with the

averaged numerical solutions of the COMMEMI project (Zhdanov et al., 1997), showing that the numerical apparent resistivities of the spectral element scheme agree well with the averaged numerical solutions of the COMMEMI project compared to those measured by the finite difference approach (Figure 9 top; Figure 10 top). It means that the modeling precision of the spectral element scheme is higher than that of the finite difference method in calculating the magnetotelluric responses using the same mesh size.

In the second example, we only simulate the spectral element solutions with different polynomial orders in the COMMEMI locations. In this experiment, we designed a non-uniform grid of the model over the entire computational domain. To make a more precise comparison with the resistivity values published by the committee experiments, in Table 2, we list the standard deviation from Table B.11 (Zhdanov et al., 1997) along with the numerical resistivity values simulated by the spectral element approach. From Table 2, the values produced by the spectral element method match well with the numerical results published in the COMMEMI experiments. The accuracy of the apparent resistivity simulated by the spectral element might depend on the polynomial order.

5.3 Topographical model

To compute the magnetotelluric responses of the two-dimensional undulating terrain, we applied our spectral element code to a ridge topographical model, as shown in Figure 11, which is the same as that used by other researchers (Wannamaker et al., 1986; Liang et al., 2021). The ridge model has a width of 2,400 m with a height of 100 m, and its resistivity value of half-space is = 100 $\Omega \cdot$ m. The calculated magnetotelluric responses obtained by the finite element subroutine (Wannamaker et al., 1986) are the reference for this simulation.

In this study, the non-uniform meshes in the TM mode and TE mode are set as 15×10 and 20×10, respectively (in which 10 km is the air media and its resistivity is equal to 10^{15} $\Omega \cdot$ m), and each spectral element has four GLL points in the y-direction and z-direction. The measurement profile along the atmospheric grounding interface varies from −2,000 to 2,000 m. The frequency to be tested is only 10 Hz. Figure 12 shows the comparison of the finite element results from Wannamaker et al. (1986) and our spectral element forward code, and the results match well. The maximum relative apparent resistivity error between the two forward schemes is equal to 0.15% in the TM mode and 0.07% in the TE mode, respectively. The maximum relative phase error is equal to 0.04% in the TM mode and 0.05% in the TE mode, respectively. As shown in Figure 12A, the topographic resistivity distortion of the TE mode is smaller than that of the TM mode.

5.4 Smooth resistivity model

In this numerical example, a smooth resistivity model is set to 18 km × 4 km, as shown in Figure 13. We calculate the response of a two-dimensional magnetotelluric model with a smooth resistivity distribution. The least-square iterative algorithm calculated the inversion of the resistivity distribution for this model with the MT2DInvMatlab subroutine (Lee et al., 2009) for a fault model tested by Sasaki (1989).

We chose nine frequencies to test this model, which are 0.1, 0.2, 0.5, 1.0, 2.0, 5.0, 10.0, 20.0, and 50.0 Hz (Lee et al., 2009). The computational domain was set as 200 km × 100 km, and the resistivity value for the extended region was designed as 50 $\Omega \cdot$ m. We chose four GLL points for each element in the y-direction and z-direction, and 25 points per element. Figures 14, 15 show the pseudo-sections of the apparent resistivity and phase for this smooth model in the TM mode and the TE mode. By comparing the spectral element results and the finite element results, we found that the accuracy of the two ways is almost the same, and the results agree well.

6 Conclusion

The spectral element method combined with the GLL point interpolating scheme has been developed for the first time to solve the two-dimensional magnetotelluric forward problem. We presented the spectral element formulas and implemented this algorithm. Compared with the finite difference scheme and the finite element technique, our spectral element approach requires fewer elements and produces accurate results. In the first investigation, we apply the spectral element strategy on a simple half-space geo-electric model to test its high accuracy. We presented the comparison results of the finite difference algorithm and the spectral element algorithm for the COMMEMI 2D-1 model. The accuracy of our spectral element method is better than that of the finite difference approach. We compare the numerical results from Wannamaker et al. (1986) and our spectral element scheme for a ridge topographical model, and they agree well. These results demonstrate the effectiveness and flexibility of the spectral element forward algorithm. We also applied the spectral element method to a model with a smooth resistivity structure and compared the simulation results with those of the finite element code (Lee et al., 2009). This shows that the calculation results of the spectral element algorithm are as smooth and accurate as those of the finite element method. These measurements and comparative results suggest that the spectral element method can provide another effective scheme for the two-dimensional magnetotelluric forward problem.

Data availability statement

The raw data supporting the conclusion of this article will be made available by the authors, without undue reservation.

Author contributions

All authors participated in editing and reviewing the manuscript. XT derived the linear equation system of the spectral element approach and developed the numerical simulation code. YS performed the numerical experiments and result analysis. BZ plotted some of the figures. All authors read and agreed to the submitted version of the manuscript.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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