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Discussing sources of the β-term in a vorticity equation in rotating coordinates

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In the study of atmospheric dynamics, the vorticity equation in a rotating coordinate system plays a crucial role. However, a paradox arises when one considers the term related to spatial variations in Coriolis parameters known as the "β-term". The β-term should not appear in the vorticity equation because the three-dimensional (3D) planetary vorticity is a constant vector. However, it is always in the vorticity equation. In this article, the source of the β-term in different rotating coordinates are investigated. The results show that in the spherical coordinate system, the β-term comes from the directions changing of one of the unit vectors ($\vec{j}$) with the spatial position and originates from the tilting term. By contrast, in the height coordinate system, the β-term cannot be derived from the tilting term as the individual changes of the coordinate frames with time are omitted, Instead it is proven to be related to the advection term. Although the both coordinate systems are rotating coordinate systems, the sources of their β-terms differ due to the simplification levels of the coordinate systems. Although the 3D planetary vorticity is a constant vector in the spherical coordinate system, the conversions between its components are allowable and spatial derivatives of its components can be observed, eliminating the paradox of the β-term. However, in the height coordinate system, the 3D planetary vorticity vector is not a constant vector in order to maintain the conservation of the absolute angular momentum and mechanical energy. To account for the influence of the earth's curvature on atmospheric motion, the β-term of the Coriolis parameters varying with the latitude appears. So, the origin of the β-term paradox proposed in the height coordinate system comes from a misunderstanding of the physical constraints of the height coordinate system.

KEYWORDS

vorticity equation, rotating coordinates, Coriolis parameters, spherical coordinates, height coordinates

1 Introduction

On daily weather maps, the distribution of the pressure field or wind field in the middle and upper troposphere often exhibits wavelike patterns. In the middle latitudes of the Northern Hemisphere, approximately 3–5 waves can be observed. These waves are known as atmospheric long waves or Rossby waves. They are referred to as slow waves due to their significantly slower propagation speed compared to acoustic waves and gravity waves. The β-parameter is commonly defined as the northward gradient of the vertical component of the three-dimensional (3D) planetary vorticity ($2\vec{\Omega}$). The spherical coordinate system is a rotating frame but not a Cartesian system because the unit vectors are not constant (Martin, 2006). The height coordinate system commonly used in meteorology is one of the Cartesian coordinates, which is a simplified form of the spherical coordinate system, known as the Cartesian rectangular coordinate system. In the height coordinate system, the β-term refers to the change in the planetary vorticity (f) with the latitude resulting from the earth's rotation. When the atmosphere moves north–southward, the inter-transformation between the planetary and relative vorticities can lead to variations in the vortex intensity of synoptic systems. In 1939, Rossby first theoretically studied the properties of atmospheric long waves and identified the β-effect as the cause of such waves in the barotropic atmosphere under the condition of the absolute vorticity conservation (Rossby, 1939). Long waves are closely associated with large-scale weather system evolution and represent the most important wave type in the atmosphere. The development of the long wave theory has significantly contributed to the advancement of the modern dynamic meteorological theory and has provided a theoretical foundation for numerical weather prediction. As a tribute to Rossby's contributions to the establishment of the long wave theory, these atmospheric long waves were named Rossby waves. The formation and propagation of the atmospheric Rossby waves are explained by a β-term in the non-divergent barotropic vorticity equation (Rossby, 1939; Dickinson, 1978; Holton, 2004; Cai and Huang, 2013).

When the 3D vorticity equation is derived from the 3D vector motion equation in the rotating coordinates (the spherical coordinate system and the height coordinate system), the 3D planetary vorticity, known as the planetary vorticity vector generated by the Earth's rotation, remains constant and equals twice the angular velocity of the Earth's rotation. Consequently, the 3D planetary vorticity vector is always zero in theory, under the full derivative operation, allowing it to be eliminated from the equation. In the widely used height coordinate system, if the full derivative of the 3D planetary vorticity vector also equals to zero, it implies the absence of the full derivative term of the Coriolis parameter in the vorticity equation or the variation term of the vertical component of the 3D planetary vorticity ($2\vec{\Omega}$) with latitude. This absence suggests that the crucial β-term, which is vital for atmosphere motion would not exist. This scenario is known as the β-term paradox (Viudez, 2003). Viudez (2003) proposed redefining the β-term as the northward component of the planetary vorticity vector. By doing so, the spatial derivative of the planetary vorticity are no longer involved in the vorticity equation, thus resolving the paradox. Viudez (2003) substantiated this redefinition to avoid encountering the derivatives of planetary vorticity by demonstrating the consistency between the northward component of planetary vorticity and the northward derivatives of the vertical component of planetary vorticity in the spherical coordinate system. Viudez (2003) emphasized that the β-term originates from the tilting term of the planetary vorticity, asserting that this conclusion is independent of the coordinate system. Unfortunately, the proof of the source of the β-term in the commonly used vorticity equation in the height coordinates has not been demonstrated. In the vorticity equation derived in the spherical coordinates, the expression of the β-term is not provided, and the derivation in the spherical coordinates cannot be directly extended to the height coordinates. In the vorticity equation in the height coordinate, the β-term represents the advection term of vertical planetary vorticity (Yang et al., 1980; Lv et al., 2004; Martin, 2006). The objective of this study is to investigate the origin of the β-term in the commonly used rotating coordinate system and provide an explanation for the underlying paradox of the β-term. Section 2 will focus on analyzing the origin of the β-term in the spherical coordinate and height coordinate systems. In Section 3, the emphasis will be put on clarifying the β-term paradox in the height coordinate system.

2 Source of the β-term in the rotating coordinate system

2.1 Source of β-term in the spherical coordinates

The Coriolis force is an important "apparent force" and plays a crucial role in rotating coordinate systems. The spherical coordinate system takes into account the complete influence of the Coriolis force. The β-term is fundamentally derived from the curl of the Coriolis force.

After the vector formula, ∇×(A × B) = B(∇·A) - A(∇·B) + (A·∇)B - (B·∇)A is applied, the curl vector of the Coriolis force can be decomposed into

$$\nabla_3 \times \left(2\vec{\Omega} \times \vec{V}_3\right) = -\left(\vec{V}_3 \cdot \nabla_3\right)2\vec{\Omega} - 2\vec{\Omega}\left(\nabla_3 \cdot \vec{V}_3\right) + \vec{V}_3\left(\nabla_3 \cdot 2\vec{\Omega}\right) + \left(2\vec{\Omega} \cdot \nabla_3\right)\vec{V}_3. \tag{1}$$

Since the rotation vector $\vec{\Omega}$ is divergence-free, so $\vec{V}_3\,(\nabla_3 \cdot 2\vec{\Omega}) = 0$. Here, $-(\vec{V}_3 \cdot \nabla_3)2\vec{\Omega}$ is referred to the planetary vorticity advection term, $-2\vec{\Omega}\,(\nabla_3 \cdot \vec{V}_3)$ represents the divergence-related term, and $(2\vec{\Omega} \cdot \nabla_3)\vec{V}_3$ is the tilting-related term, coined as the planetary vorticity tilting term by Viudez (2003).

The expression for the Coriolis force in the spherical coordinates is

$$-2\vec{\Omega} \times \vec{V} = (2\Omega v \sin\phi - 2\Omega w \cos\phi)\vec{i} - 2\Omega u \sin\phi\,\vec{j} + 2\Omega u \cos\phi\,\vec{k}. \tag{2}$$

By applying the curl formula in the spherical coordinates, we obtain the curl of the Coriolis force in the spherical coordinates, as shown in the equations as follows:

$$\begin{aligned}\nabla\times\left(-2\vec{\Omega}\times\vec{V}\right) = &\left[\frac{1}{r}\left(2\Omega\cos\phi\frac{\partial u}{\partial\phi}\right) + 2\Omega\sin\phi\frac{\partial u}{\partial r}\right]\vec{i}\\ &+\left[2\Omega\sin\phi\frac{\partial v}{\partial r} - 2\Omega\cos\phi\frac{\partial w}{\partial r}\right.\\ &\left.+\frac{(2\Omega v\sin\phi - 2\Omega w\cos\phi)}{r} - \frac{2\Omega}{r}\frac{\partial u}{\partial\lambda}\right]\vec{j}\\ &+\left[-\frac{2\Omega\tan\phi}{r}\frac{\partial u}{\partial\lambda}\right.\\ &-\frac{1}{r}\left(2\Omega v\cos\phi + 2\Omega\sin\phi\frac{\partial v}{\partial\phi} - 2\Omega\cos\phi\frac{\partial w}{\partial\phi}\right)\\ &\left.+\frac{(2\Omega v\sin\phi\tan\phi - 4\Omega w\sin\phi}{r}\right]\vec{k}.\end{aligned} \tag{3}$$

Considering $f = 2\Omega\sin\phi$ and $\partial y = r\partial\phi$, then

$$-\frac{v}{r}(2\Omega\cos\phi)\vec{k} = -\frac{v}{r}\frac{\partial(2\Omega\sin\phi)}{\partial\phi}\vec{k} = -v\frac{\partial f}{\partial y}\vec{k} = -v\beta\vec{k}. \tag{4}$$

This implies that the β-term originates from the decomposition of the curl of the Coriolis force in the $\vec{k}$-direction.

Using the formula for the advection derivative term in spherical coordinates (referring to Formula A1 in the Appendix A), we can obtain

$$-\left(\vec{V}_3\cdot\nabla_3\right)2\vec{\Omega}=0. \tag{5}$$

Furthermore, it is worth noting that the 3D planetary vorticity in spherical coordinates remains a constant vector, confirming the validity of Eq. 5.

By applying the spherical coordinate divergence formula (refer to Formula A2 in the Appendix A), we obtain

$$\begin{aligned}&-2\vec{\Omega}\left(\nabla_3\cdot\vec{V}_3\right)\\&=\left[-\frac{2\Omega}{r}\frac{\partial u}{\partial\lambda}-\frac{2\Omega\cos\phi}{r}\frac{\partial v}{\partial\phi}-2\Omega\cos\phi\frac{\partial w}{\partial r}+\frac{v2\Omega\sin\phi}{r}-\frac{4w\Omega\cos\phi}{r}\right]\vec{j}\\&\quad+\left[-\frac{2\Omega\tan\phi}{r}\frac{\partial u}{\partial\lambda}-\frac{2\Omega\sin\phi}{r}\frac{\partial v}{\partial\phi}-2\Omega\sin\phi\frac{\partial w}{\partial r}+\frac{v2\Omega\tan\phi\sin\phi}{r}\right.\\&\quad\left.-\frac{4w\Omega\sin\phi}{r}\right]\vec{k}.\end{aligned} \tag{6}$$

From Eq. 6, it is evidently seen that $-2\vec{\Omega}(\nabla_3\cdot\vec{V}_3)$ represents the divergence-related term without involving the β-term.

Similarly, by utilizing the advection derivative term formula in the spherical coordinates (referring to Formula A3 in the Appendix A), we derive

$$\begin{aligned}\left(2\vec{\Omega}\cdot\nabla_3\right)\vec{V}_3=&\left[\frac{1}{r}\left(2\Omega\cos\phi\frac{\partial u}{\partial\phi}\right)+2\Omega\sin\phi\frac{\partial u}{\partial r}\right]\vec{i}\\&+\left[\frac{2\Omega\cos\phi}{r}\frac{\partial v}{\partial\phi}+2\Omega\sin\phi\frac{\partial v}{\partial r}+\frac{2w\Omega\cos\phi}{r}\right]\vec{j}\\&+\left[\frac{2\Omega\cos\phi}{r}\frac{\partial w}{\partial\phi}+2\Omega\sin\phi\frac{\partial w}{\partial r}-\frac{1}{r}(2\Omega v\cos\phi)\right]\vec{k}.\end{aligned} \tag{7}$$

As seen in Eq. 7, $\frac{1}{r}(2\Omega v\cos\phi)$ is incorporated on the right-hand side of the equation, which is the β-term, and it is derived from the curl vector decomposition of the Coriolis force. In other words, the β-term is from the tilting term. This conclusion aligns with the findings of Viudez (2003). The planetary vorticity tilting term can also be decomposed as follows:

$$\begin{aligned}\left(2\vec{\Omega}\cdot\nabla_3\right)\vec{V}_3=&\vec{i}\left(2\vec{\Omega}\cdot\nabla u\right)+\vec{j}\left(2\vec{\Omega}\cdot\nabla v\right)+\vec{k}\left(2\vec{\Omega}\cdot\nabla w\right)+u\left(2\vec{\Omega}\cdot\nabla\right)\vec{i}\\&+v\left(2\vec{\Omega}\cdot\nabla\right)\vec{j}+w\left(2\vec{\Omega}\cdot\nabla\right)\vec{k}.\end{aligned} \tag{8}$$

The last three terms on the right-hand side of Eq. 8 are due to the spatial variation in the unit vectors along the three coordinate axes in the spherical coordinate system. In the spherical coordinate system, the formula for the spatial derivatives of the three-unit vectors is given by

$$\begin{aligned}&\frac{\partial\vec{i}}{\partial\lambda}=\sin\phi\vec{j}-\cos\phi\vec{k};\quad\frac{\partial\vec{i}}{\partial\phi}=\frac{\partial\vec{i}}{\partial r}=0,\\&\frac{\partial\vec{j}}{\partial\lambda}=-\sin\phi\vec{i};\quad\frac{\partial\vec{j}}{\partial\phi}=-\vec{k};\quad\frac{\partial\vec{j}}{\partial r}=0,\\&\frac{\partial\vec{k}}{\partial\lambda}=\cos\phi\vec{i};\quad\frac{\partial\vec{k}}{\partial\phi}=\vec{j};\quad\frac{\partial\vec{k}}{\partial r}=0.\end{aligned} \tag{9}$$

Regarding the β-term which is in direction $\vec{k}$, let us focus on the term in the direction $\vec{k}$, as mentioned previously. Combined with Eq. 9, the expressions for the third, fourth, and fifth terms on the right-hand side of Eq. 8 can be deduced as follows:

$$\left(2\vec{\Omega}\cdot\nabla w\right)\vec{k}=\left(2\Omega\sin\phi\frac{\partial w}{\partial r}+2\Omega\cos\phi\frac{\partial w}{r\partial\phi}\right)\vec{k}, \tag{10}$$

$$u\left(2\vec{\Omega}\cdot\nabla\right)\vec{i}=u\left(2\Omega\cos\phi\frac{\partial\vec{i}}{r\partial\phi}+2\Omega\sin\phi\frac{\partial\vec{i}}{\partial r}\right)=0, \tag{11}$$

$$v\left(2\vec{\Omega}\cdot\nabla\right)\vec{j}=v\left(2\Omega\cos\phi\frac{\partial\vec{j}}{r\partial\phi}+2\Omega\sin\phi\frac{\partial\vec{j}}{\partial r}\right)=-2\Omega\cos\phi\frac{v}{r}\vec{k}. \tag{12}$$

In Eq. 12, the right-hand side of Eq. 12 corresponds to the β-term, indicating the β-term originates from $v(2\vec{\Omega}\cdot\nabla_3)\vec{j}$ in the spherical coordinates and is caused by one of the unit vectors ($\vec{j}$) of the three coordinate frames in the spherical coordinate system, which varies with the spatial position.

Even though our approach differs from Viudez's (2003) method, we achieve the same conclusion. It can be stated that the β-term in the spherical coordinate system is derived from the tilting term of the curl decomposition of the Coriolis force. Furthermore, we provide further clarification that the β-term in the spherical coordinate system arises from the variation in the coordinate axis. Although the planetary vorticity vector remains constant, the change in the coordinate axis direction allows components in different directions to be transformed into each other. This implies that even if the vector is zero, its components can still have spatial differential quotients. The Taylor–Proudman constraint, which asserts that the tilting vector of the planetary vorticity in the spherical coordinate system is zero, indicates a balance between the different components of the tilting term of the planetary vorticity (Viudez, 2003). This provides a physical explanation for the constant vector of the planetary vorticity in the spherical coordinate system, where its vertical component can have spatial differential quotients. Consequently, the northward differential quotient of planetary vorticity is not necessarily zero. In other words, the β-term can exist in the spherical coordinate system.

2.2 Source of β-term in the height coordinates

During the simplification process of converting the equation of motion from the spherical coordinates to the height coordinates, certain terms of the Coriolis force are neglected in order to satisfy the "constraints of absolute angular momentum conservation and mechanical energy conservation.", which means implies that the height coordinate system only partially accounts for the influence of the Coriolis force and can be considered an approximate inertial coordinate system. The Coriolis force in the height coordinates can be expressed as

$$-2\vec{\Omega}\times\vec{V}_3=2\Omega\sin\phi v\vec{i}-2\Omega\sin\phi\, u\vec{j}, \tag{13}$$

$$\nabla\times\left(-2\vec{\Omega}\times\vec{V}_3\right)=\frac{\partial(2\Omega\sin\phi u)}{\partial z}\vec{i}+\frac{\partial(2\Omega\sin\phi v)}{\partial z}\vec{j}-\left[\frac{\partial(2\Omega\sin\phi u)}{\partial x}+\frac{\partial(2\Omega\sin\phi v)}{\partial y}\right]\vec{k}. \quad (14)$$

By introducing the variable $f=2\Omega\sin\phi$, Eq. 14 can be written as

$$\nabla\times\left(-2\vec{\Omega}\times\vec{V}_3\right)=\left[-f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-v\frac{\partial f}{\partial y}\right]\vec{k}+\frac{\partial(fv)}{\partial z}\vec{j}+\frac{\partial(fu)}{\partial z}\vec{i}. \quad (15)$$

Based on Eq. 15, the β-term is derived from the decomposition of the curl of the Coriolis force in direction $\vec{k}$.

Here, we can investigate the origin of the β-term through the vector decomposition in the height coordinate system, as given by the following equations:

$$-\left(\vec{V}_3\cdot\nabla_3\right)2\vec{\Omega}=-\left[u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}\right]\vec{k}=-v\frac{\partial f}{\partial y}\vec{k}, \quad (16)$$

$$-2\vec{\Omega}\left(\nabla\cdot\vec{V}\right)=-f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\vec{k}, \quad (17)$$

$$\left(2\vec{\Omega}\cdot\nabla_3\right)\vec{V}_3=f\frac{\partial u}{\partial z}\vec{i}+f\frac{\partial v}{\partial z}\vec{j}+f\frac{\partial w}{\partial z}\vec{k}. \quad (18)$$

From Eq. 16, it is evident that the β-term arises from the advection term of the curl vector decomposition of the Coriolis force, rather than the tilting term. As mentioned previously, in the spherical coordinate system, the β-term is derived from the tilting term. However, in the height coordinate system, the β-term is contributed by the advection term. In the spherical coordinate system, since the planetary vorticity is a constant vector, the advection term is zero, making it impossible to obtain the β-term from the advection term. When the spherical coordinate is simplified to a height coordinate, it is approximately regarded as $\frac{d\vec{i}}{dt}=\frac{d\vec{j}}{dt}=\frac{d\vec{k}}{dt}=0$, and the individual changes of the three coordinate frames over time are neglected. Consequently, in the height coordinates, the β-term cannot be derived from the tilting term. If planetary vorticity is a constant vector in the height coordinate system, $-(\vec{V}_3\cdot\nabla_3)2\vec{\Omega}$ should be zero. Accordingly, the β-term cannot be obtained from the vorticity equation. In the following discussion, we aim to demonstrate that the planetary vorticity in the height coordinate system is not a constant vector.

3 The β-term paradox in the height coordinates

The paradox surrounding the β-term arises from the physical law that the planetary vorticity is a constant vector in a rotating coordinate system. As discussed in Section 2.2, in the height coordinate system, the source of the β-term is originated from the advection term of the planetary vorticity vector, However this term will be zero if the physical law or physical constraint is obeyed. It is evident that the 3D planetary vorticity vector is not a constant vector (referring to Appendix B) in the height coordinate system, and there is no inherent paradox regarding the β-term. In the following discussion, we will briefly explore the reasons why the 3D planetary vorticity vector is not a constant vector in the height coordinate system.

The planetary vorticity is not a constant vector in the height coordinate system, which can be understood by examining the principles that must be satisfied when simplifying the motion equations from spherical coordinates to height coordinates. In the process of applying the thin-layer approximation, it is necessary to adhere to the principles of absolute angular momentum conservation and mechanical energy conservation. Consequently, certain Coriolis force terms related to the 3D planetary vorticity are neglected. For instance, the vertical Coriolis force $\tilde{f}u$ and the term involving $\tilde{f}w$ in the horizontal Coriolis force are omitted, thus violating the physical constraint that the 3D planetary vorticity vector is a constant vector. In other words, the simplification of the motion equation from the spherical coordinates to the height coordinates compromises the physical constraint that the angular velocity of the Earth's rotation (i.e., the 3D planetary vorticity vector) is a constant vector. Because the height coordinate system fails to fully incorporate the planetary vorticity vector, the planetary vorticity differential quotient term appears in the vertical vorticity equation. The spherical coordinate system fully incorporates the Coriolis force resulting from the earth's rotation.

4 Conclusion

Using a different approach from Viudez (2003), we have deduced the origins of the β-term in both spherical and height coordinate systems, clarifying the misconception surrounding the β-term paradox. The following conclusions are arawn as follows:

The β-term in the spherical coordinate system arises from the tilting term of the planetary vorticity and is caused by one of the unit vector directions that vary with the spatial position, specifically a polar-pointing unit vector that changes with the latitude. The β-term in the height coordinates originates from the planetary vorticity advection term.

The spherical coordinate system fully incorporates the Coriolis force resulting from the earth's rotation. In this system, the 3D planetary vorticity vector is a constant vector; but, there is a mutual transformation among its components, resulting in the existence of spatial differential quotients. Therefore, the existence of the β-term in the spherical coordinate system does not contradict with the constant planetary vorticity vector. On the other hand, the height coordinate system is a simplified coordinate system that only partially considers the influence of the Coriolis force, so the planetary vorticity is not a constant vector. To uphold the conservations of absolute angular momentum and mechanical energy, the constant vector of the planetary vorticity cannot be maintained. The physical constraint that the vorticity vector of the Earth's rotation is a constant vector fails to be support for the height coordinate system, allowing for the existence of spatial differential quotients of planetary vorticity. Therefore, the β-term representing the spatial differential quotient of the planetary vorticity vector can exist in both of the spherical coordinate and the height coordinate systems, which resolves the paradox of the β-term.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

XW contributed to the conception and design of the study and wrote the first draft of the manuscript. SL revised the formula derivations in the attachments, as well as the manuscript. HT contributed to revise the manuscript. HL contributed to polish the manuscript. All authors contributed to the article and approved the submitted version.

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Conflict of interest

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Appendix A Formula of the advection derivative term in spherical coordinates

The curl formula in the spherical coordinates can be expressed as

$$\nabla\times\vec{A}=\left(\frac{1}{r}\frac{\partial A_r}{\partial\phi}-\frac{\partial A_\phi}{\partial r}-\frac{A_\phi}{r}\right)\vec{i}+\left(\frac{\partial A_\lambda}{\partial r}+\frac{A_\lambda}{r}-\frac{1}{r\cos\phi}\frac{\partial A_r}{\partial\lambda}\right)\vec{j}+\left(\frac{1}{r\cos\phi}\frac{\partial A_\phi}{\partial\lambda}-\frac{1}{r}\frac{\partial A_\lambda}{\partial\phi}+\frac{A_\lambda\tan\phi}{r}\right)\vec{k}. \quad \text{(A1)}$$

The divergence formula in the spherical coordinates is given by

$$\nabla\cdot\vec{A}=\frac{1}{r\cos\phi}\frac{\partial A_\lambda}{\partial\lambda}+\frac{1}{r}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_r}{\partial r}-\frac{A_\phi\tan\phi}{r}+\frac{2A_r}{r}. \quad \text{(A2)}$$

The formula of the advection derivative term in the spherical coordinates is as follows:

$$\vec{A}\cdot\nabla\vec{B}=\left[\frac{A_\lambda}{r\cos\phi}\frac{\partial B_\lambda}{\partial\lambda}+\frac{A_\phi}{r}\frac{\partial B_\lambda}{\partial\phi}+A_r\frac{\partial B_\lambda}{\partial r}+\frac{A_\lambda B_r}{r}-\frac{\tan\phi}{r}A_\lambda B_\phi\right]\vec{i}+\left[\frac{A_\lambda}{r\cos\phi}\frac{\partial B_\phi}{\partial\lambda}+\frac{A_\phi}{r}\frac{\partial B_\phi}{\partial\phi}+A_r\frac{\partial B_\phi}{\partial r}+\frac{A_\phi B_r}{r}+\frac{\tan\phi}{r}A_\lambda B_\lambda\right]\vec{j}+\left[\frac{A_\lambda}{r\cos\phi}\frac{\partial B_r}{\partial\lambda}+\frac{A_\phi}{r}\frac{\partial B_r}{\partial\phi}+A_r\frac{\partial B_r}{\partial r}-\frac{A_\phi B_\phi}{r}-\frac{A_\lambda B_\lambda}{r}\right]\vec{k}. \quad \text{(A3)}$$

Appendix B Derivation of conservation of the three-dimensional planetary vorticity vector in a rotating coordinate system

a. The 3D planetary vorticity vector in the spherical coordinate system

In meteorology, in the spherical coordinate system, the planetary vorticity vector is a constant vector, and it can be represented as

$$\vec{\Omega}=\Omega\cos\phi\,\vec{j}+\Omega\sin\phi\,\vec{k}, \quad \text{(B1)}$$

$$\frac{d\vec{\Omega}}{dt}=\frac{d(\Omega\cos\phi)}{dt}\vec{j}+\frac{d(\Omega\sin\phi)}{dt}\vec{k}+\Omega\cos\phi\frac{d\vec{j}}{dt}+\Omega\sin\phi\frac{d\vec{k}}{dt}. \quad \text{(B2)}$$

In the spherical coordinates

$$\frac{d\vec{j}}{dt}=-\frac{utg\phi}{r}\vec{i}-\frac{v}{r}\vec{k}, \quad \text{(B3)}$$

$$\frac{d\vec{k}}{dt}=\frac{u}{r}\vec{i}+\frac{v}{r}\vec{j}. \quad \text{(B4)}$$

So,

$$\frac{d\vec{\Omega}}{dt}=0. \quad \text{(B5)}$$

Consequently, it can be inferred that the 3D planetary vorticity vector is a constant vector in the spherical coordinate system.

b. The 3D planetary vorticity vector in the height coordinate system

The direction of the coordinate axis is the same as the spherical coordinate; however, the unit vector is assumed to be constant in space, denoted as

$$\frac{d\vec{i}}{dt}=\frac{d\vec{j}}{dt}=\frac{d\vec{k}}{dt}=0. \quad \text{(B6)}$$

Similarly, we have

$$\frac{d\vec{\Omega}}{dt}=\frac{d(\Omega\cos\phi)}{dt}\vec{j}+\frac{d(\Omega\sin\phi)}{dt}\vec{k}, \quad \text{(B7)}$$

$$\frac{d(\Omega\cos\phi)}{dt}=-\frac{v\Omega\sin\phi}{a}. \quad \text{(B8)}$$

By extension, it follows that

$$\frac{d(\Omega\sin\phi)}{dt}=\frac{v\Omega\cos\phi}{a}, \quad \text{(B9)}$$

then

$$\frac{d\vec{\Omega}}{dt}=-\frac{v\Omega\sin\phi}{a}\vec{j}+\frac{v\Omega\cos\phi}{a}\vec{k}. \quad \text{(B10)}$$

The right-hand side of Eq. B10 is not always zero, indicating that the 3D geostrophic vector is not a constant vector in the height coordinate system.