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Modelling Fractional Behaviours Without Fractional Models

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This paper first warns about the confusion or rather the implicit link that exists in the literature between fractional behaviours (of physical, biological, thermal, etc. origin) and fractional models. The need in the field of dynamic systems modelling is for tools that can capture fractional behaviours that are ubiquitous. Fractional models are only one class of models among others that can capture fractional behaviours, but with associated drawbacks. Several other modelling tools are proposed in this paper, thus showing that a distinction is needed between fractional behaviours and fractional models.

Keywords: fractional integration, fractional differentiation, fractional models, non singular kernels, volterra equations, nonlinear models, distributed time delay models, diffusion equations

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INTRODUCTION

The domains of fractional calculus and fractional models have grown significantly thanks to the dynamism of the related community. But this community has sometimes tended to generalize existing results to fractional orders without any real interest in or justification of their physical meaning, a tendency that has been called "fractionalization" in the literature (Dokoumetzidis et al., 2010). Consequently, fundamental questions are now catching up with this community. In particular, the emergence of questions and critical analyses on fractional calculus and fractional models is becoming increasingly common in the literature:

- It is shown in (Dokoumetzidis et al., 2010) that physical interpretations can invalidate non-commensurate fractional pseudo state space representations;
- The inability of the Caputo derivative definition to take into account the initial conditions correctly is discussed in (Sabatier et al., 2008; Sabatier et al., 2010; Sabatier and Farges 2018)
- The singularity of fractional calculus operators is questioned and solutions are proposed in (Caputo and Fabrizio, 2015), (Atangana and Baleanu, 2016)
- (Balint and Balint, 2020) explored the mathematical description of the groundwater flow and that of the Impurity Spread and showed that this description is non-objective if Caputo or Riemann–Liouville fractional partial derivatives with integration on a finite interval are used in this description;
- Several drawbacks of fractional models are highlighted in (Sabatier et al., 2020), that mainly result in the doubly infinite dimension of fractional models (Sabatier, 2021).

These studies in fact raise questions about the physical consistency and the limits of fractional models and warrant the questions: what are the needs met by fractional calculus and fractional models? Do fractional calculus and Fractional models solve fundamental problems? Fractional calculus and fractional models are frequently used to capture fractional dynamic behaviours. There is no exact definition to describe a fractional dynamic behaviour. In this paper we therefore assume that a system has a fractional behaviour if its input and output are linked by a function of the form $t^{\gamma-1}$,

$\nu \in \mathbb{R}, \nu < 1$ at least over a time range, or by a function of the form $\omega^{-\nu}$ at least over a frequency range. These kinds of behaviours are ubiquitous (Ionescu and al., 2017; Zhang et al., 2017; Zou et al., 2018; Bonfanti et al., 2020) and it is thus of interest to have efficient models to capture them. But fractional models and fractional behaviours are two different things, contrary to what the abstract of this paper might suggest (Tarasov, 2013) and many others previously cited. Fractional models are only one class of models that can capture fractional behaviours.

As indicated in the title, this paper wants to show that it is important to question the modelling of fractional behaviours (because they are very widespread) and that several modelling tools other than fractional models allow it without the drawbacks of fractional models. This paper is the result of intense research activity which has already led to the publication of several results each relating to an idea or a particular modelling tool. In order to create a self-content but didactic document, some of these papers are not only cited but partially included to allow a detailed description of what is to be demonstrated. Thus, the following alternative models that can generate fractional behaviours are proposed:

- New kernels in convolution operators (that still enable fractional behaviours to be generated but in a limited frequency or time range);
- The Volterra integro-differential equation,
- Distributed time delay models,
- Nonlinear models,
- Time-varying models
- Partial differential equations with spatially variable coefficients.

NEW KERNELS

The fractional integral of order $\nu$of a function $f(t)$, denoted $I^{\nu}[f(t)]$, can be viewed as the convolution of this function with the kernel

$$\eta^{\nu}(t) = \frac{1}{\Gamma(\nu)t^{1-\nu}} \tag{1}$$

and thus

$$I^{\nu}\left[f(t)\right] = \int_0^t \eta^{\nu}(t-\tau) f(\tau) d\tau \tag{2}$$

If the Riemann-Liouville or the Caputo definitions are used, the fractional derivative of order ν of a function $f(t)$also involves kernel **(1)** and a convolution product similar to **(2)** (Samko et al., 1993). Clearly, fractional derivatives and integrals involve a singular kernel, which prompted some authors to introduce new definitions based on non-singular kernels (Caputo and Fabrizio, 2015; Atangana and Baleanu, 2016; Gao and Yang, 2016; Yang et al., 2017). The kernels introduced in these papers are not used in this work, but that clearly highlights that there is a common tendency in the community to introduce new kernels. Although it was not the first time that modifications of fractional derivation and integration operators were proposed (Sandev et al., 2015; Liemert et al., 2017), these publications triggered reactions that were sometimes strong. For example, they have been accused of:

- leading to the definition of operators that are no longer fractional (Tarasov, 2018; Ortigueira and Tenreiro Machado, 2019),
- making it impossible to capture the dynamic behaviour of certain systems (Ortigueira et al., 2019),
- leading to restrictions (Stynes, 2018; Hanyga, 2020; Diethelm et al., 2020).

However, one can question the validity of some of these conclusions. For instance, the results in (Stynes, 2018; Hanyga, 2020; Diethelm et al., 2020) are not correct, since they are based on a definition of the initial conditions that is not consistent with an intrinsic property of fractional operators: memory. And this memory must also be taken into account at the initial time. This is proved in (Sabatier 2020a) and the proof is reinforced by the interpretation of fractional models described (Sabatier, 2021) and the analysis in (Sabatier and Farges, 2021).

In spite of these reactions, perhaps dictated by a conservative spirit in an area which has prospered greatly, the author believes on the contrary that this is one of the ways to eliminate several drawbacks associated to fractional calculus and models. The kernels introduced in (Caputo and Fabrizio, 2015; Atangana and Baleanu, 2016; Gao and Yang, 2016; Yang et al., 2017) permit to eliminate the singularity of the kernel used in the definition of fractional operators. But it is possible to go further and to solve others drawbacks among those mentioned in the introduction. This is highlighted with the following kernel:

$$\eta^{\nu}(t) = C\frac{(\omega_{max})^{\nu}}{(\omega_{min})^{\nu}}\varphi_2(\nu-1, 1-\nu, 1; \omega_{min}t, \omega_{max}t)$$
$$with \quad C = \frac{\left(\frac{1}{\omega_{min}}+1\right)^{\frac{1-\nu}{2}}}{\left(\frac{1}{\omega_{max}}+1\right)^{\frac{1-\nu}{2}}} \tag{3}$$

where $\phi_2(\beta, \beta', \gamma; x, y)$is a two-variable hypergeometric function defined by (Erdelyi, 1954)

$$\phi_2(\beta, \beta', \gamma; x, y) = \sum_{m,n} \frac{(\beta)_m \left(\beta'\right)_n}{(\gamma)_{m+n} m!n!} x^m y^n. \tag{4}$$

The Laplace transform of this kernel is defined by (Erdelyi, 1954) (p. 238):

$$\eta^{\nu}(s) = C\frac{\left(\frac{s}{\omega_{min}}+1\right)^{1-\nu}}{s\left(\frac{s}{\omega_{max}}+1\right)^{1-\nu}} \tag{5}$$

The Bode diagrams of transfer functions $\eta^{\nu}(s)$ are shown in **Figure 1** with $\omega_{min} = 0.001$ rd/s, $\omega_{max} = 1000$ rd/s, and for various values of ν, showing that this kernel exhibits a fractional behaviour of order ν in the frequency band $[\omega_{min}, \omega_{max}]$.

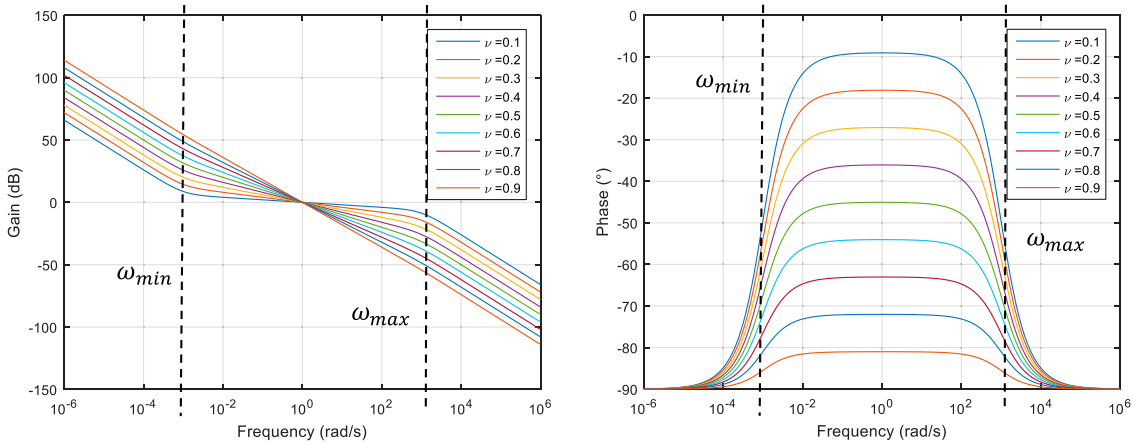

FIGURE 1 | Bode diagrams of transfer functions $\eta^{\nu}(s)$ with ω_{min} = 0.001 rd/s, ω_{max} = 1000 rd/s, for ν varying from 0.1 to 0.9.

A property (and advantage) of this kernel is a distribution of its poles on a limited frequency band. To demonstrate such a property, the impulse response of the transfer function **(5)** is computed using Cauchy's theorem as demonstrated in the appendix and in (Sabatier et al., 2019) (Sabatier, 2020b). For $t > 0$, it is given by

$$\eta^{\nu}(t) = C\left(\frac{\omega_{max}}{\omega_{min}}\right)^{1-\nu}\left(1+\frac{\sin((1-\nu)\pi)}{\pi}\int_{\omega_{min}}^{\omega_{max}}\frac{(x-\omega_{min})^{1-\nu}}{x(\omega_{max}-x)^{1-\nu}}e^{-xt}dx\right)H_e(t) \quad (6)$$

where $H_e(t)$ is the Heaviside function. The Laplace transform of relation **(6)** gives:

$$\eta^{\nu}(s) = C\left(\frac{\omega_{max}}{\omega_{min}}\right)^{1-\nu}\left(\frac{1}{s}+\frac{\sin((1-\nu)\pi)}{\pi}\int_{\omega_{min}}^{\omega_{max}}\frac{(x-\omega_{min})^{1-\nu}}{(s+x)x(\omega_{max}-x)^{1-\nu}}dx\right). \quad (7)$$

Relation **(7)** demonstrates that the poles [variable x in relation **(7)**] of the kernel $\eta^{\nu}(t)$ are distributed in a limited frequency band defined by $[\omega_{min}, \omega_{max}]$.

Now, the definition of the fractional integration in relation **(2)** is modified in order to limit its memory. The following modified operator is thus introduced:

$$J^{\nu}_{T_f}[f(t)] = \int_0^t f(\tau)d\tau + \int_{t-T_f}^t \chi^{\nu}(t-\tau)f(\tau)d\tau \quad (8)$$

with

$$\chi^{\nu}(t) = C_4\left(\frac{\omega_{max}}{\omega_{min}}\right)^{1-\nu}\frac{\sin((1-\nu)\pi)}{\pi}\int_{\omega_{min}}^{\omega_{max}}\frac{(x-\omega_{min})^{1-\nu}}{x(\omega_{max}-x)^{1-\nu}}e^{-xt}dx. \quad (9)$$

Using the change of variable $\xi = t - \tau$ and thus, $d\xi = -d\tau$, relation **(8)** becomes

$$J^{\nu}_{T_f}[f(t)] = \int_0^t f(\tau)d\tau + \int_0^{T_f}\chi^{\nu}(\xi)f(t-\xi)d\xi. \quad (10)$$

The Laplace transform of relation **(10)** without considering initial conditions is

$$\mathcal{L}\left\{J^{\nu}_{T_f}[f(t)]\right\} = \frac{F(s)}{s} + F(s)\int_0^{T_f}\chi^{\nu}(\xi)e^{-\xi s}d\xi\mathcal{L}\{f(t)\} = F(s) \quad (11)$$

and thus, the operator corresponding to this modified fractional integrator is:

$$\eta^{\nu}_{T_f}(s) = \frac{1}{s} + \int_0^{T_f}\chi^{\nu}(\xi)e^{-\xi s}d\xi. \quad (12)$$

Figure 2 compares the gain diagram of $\eta^{\nu}_{T_f}(s)$ (computed numerically) with the gain diagram of $\eta^{\nu}(s)$ given by relation **(7)** in the domain of interest. It shows that the frequency response of $\eta^{\nu}_{T_f}(s)$ remains similar to that of $\eta^{\nu}(s)$ under the condition $T_f > 1/\omega_{min}$. In such a situation, parameter T_f can be viewed as the memory length of this kernel. Note that the memory length is connected to the corner frequency below which the kernel frequency response has a pure integrator behaviour.

At first glance, this kind of operator might not solve the problem of objectivity mentioned in (Balint and Balint, 2020). But it must be noticed that the objectivity of the fractional models studied in (Balint and Balint, 2020) can be restored by adding an appropriate initialization function as Lorenzo and Hartley did (Lorenzo and Hartley, 2000). This initialisation should be used to cancel relations **(33)** and **(37)** that appear in (Balint and Balint, 2020), and thus to restore objectivity. However, as shown in (Sabatier, 2021), (Sabatier and Farges, 2021), with a fractional model, the initialisation must take into account all its past from -∞ to 0 (infinite length memory). The interest of the kernel proposed (but also those proposed in (Sabatier 2020b) (Sabatier 2020c), of course with memory, is the limited length of its memory in relation **(12)** that facilitates its initialisation.

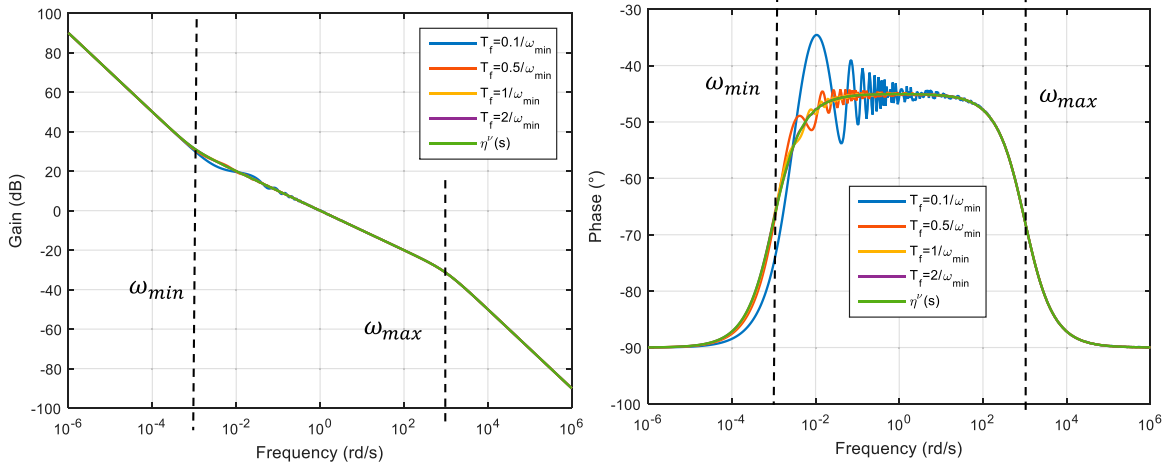

FIGURE 2 | A comparison of Bode diagrams of $\eta^{\nu}_{T_f}(s)$ and of $\eta^{\nu}(s)$ in the domain of interest for $\nu = 0.5$, $\omega_{min} = 0.001$ *rd*/s, and $\omega_{max} = 1000$ rd/s and various values of T_f. Note that with kernels $\eta^{\nu}(t)$ or $\eta^{\nu}_{T_f}(t)$ previously defined and for a given function $f(t)$, an integral fractional-like operator can be obtained with the relations.

$$\int_0^t f(z)\eta^{\nu}(t-z)dz \tag{13}$$

$$\int_0^t f(z)\eta^{\nu}_{T_f}(t-z)dz \tag{13}$$

.

Also, a derivative fractional-like behaviour is obtained using the relations

$$\frac{d}{dt}\int_0^t f(z)\eta^{\nu}(t-z)dz \;\; or \;\; \int_0^t \frac{df(z)}{dz}\eta^{\nu}(t-z)dz \tag{14}$$

and

$$\frac{d}{dt}\int_0^t f(z)\eta^{\nu}_{T_f}(t-z)dz \;\; or \;\; \int_0^t \frac{df(z)}{dz}\eta^{\nu}_{T_f}(t-z)dz. \tag{15}$$

It thus can be concluded that the proposed non-singular kernel and the new integration/derivation operator introduced make it possible to solve some of the drawbacks described in (Sabatier et al., 2020) (double infinite dimension, infinite memory, distribution of time constants and poles on an infinite domain) and can be used to model fractional behaviours.

VOLTERRA EQUATIONS

The kernel previously studied define operators that exhibit a fractional behaviour but can also be used to define new models involving a Volterra equation of the first kind. This class of equations, associated to appropriate kernels, is more general than the fractional differential equation or pseudo state space model very frequently used in the literature, and can be described by:

$$\begin{cases} D^{\nu}[x(t)] = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{16}$$

where $x(t) \in \mathbb{R}^n$ is the pseudo state vector, $\nu$is the fractional order of the system and $A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nx1}$, $C \in \mathbb{R}^{1xn}$and $D \in \mathbb{R}$ are constant matrices. D^{ν} denotes the fractional differentiation operators of orders $\nu \in \mathbb{R}$.

According to (Samko et al., 1993) (p. 46) (if the fractional integral of order ν of each component of vector $x(t)$ exists) and as shown in (Sabatier 2020c), and after first-order integration of both sides of the first equation in relation **(16)**, the following equation can be obtained:

$$\int_0^t D^{\nu}[x(t)]_{t=\tau}d\tau = I^{1-\nu}[x(t)] = \int_0^t \frac{(t-\tau)^{-\nu}}{\Gamma(1-\nu)}x(\tau)d\tau$$
$$= \int_0^t [Ax(\tau) + Bu(\tau)]d\tau \tag{17}$$

where the kernel $(t)^{-\nu}/\Gamma(1-\nu)$ multiplies each component of vector $x(t)$.

Representation **(16)** can thus be rewritten under the form of a Volterra equation of the first kind:

$$\int_0^t \eta(t-\tau)x(\tau)d\tau = \nu(t)$$

with

$$\nu(t) = \int_0^t Bu(\tau)d\tau, \qquad y(t) = Cx(t) + Du(t),$$
$$\eta(t) = \left(\frac{t^{-\nu}}{\Gamma(1-\nu)}I_n - A\right) \tag{18}$$

where I_n denotes an identity matrix of dimension *nxn*. Relation **(17)** demonstrates that a pseudo state space description is a particular case of a Volterra equation of the first kind, as the kernel $\eta(t)$ in **(17)** has a fixed structure. Using a Volterra equation with a more general kernel, it is thus possible to generalise a pseudo state space description in two ways:

- By adapting the kernel $\eta(t)$ in relation **(18)**, it is possible to produce, with the same kind of equation, power law behaviours of various types (denoted explicit, implicit), but also many other long memory behaviours;
- In relation **(18)**, if $x(t) \in \mathbb{R}^n$, then $\eta(t)$ is a matrix of kernels such that $\eta(t) = [\eta^{\nu}_{i,j}(t)\ ]$, thus permitting great flexibility in the tuning of relation **(18)**.

Description **(18)** has another important advantage. With a more general kernel $\eta^{\nu}(t)$, model memory can be limited by introducing a parameter T_f in the integral bounds such that

$$\int_{t-T_f}^{t} \eta^{\nu}(t-\tau)x(\tau)d\tau = v(t). \tag{19}$$

Using the change of variable $\xi = t - \tau$, relation **(19)** becomes

$$\int_{0}^{T_f} \eta^{\nu}(\xi)x(t-\xi)d\xi = v(t). \tag{20}$$

Relation **(19)** explicitly shows that knowledge of the model state $x(t)$ is required on $[0, T_f]$ to compute its future. Modification of the lower bound of relation **(18)** to produce relation **(20)** is thus of interest in the initialisation problem. The initialisation of relation **(20)** only requires knowledge of the past of the variable $x(t)$ on the interval $[t_0 - T_f, t_0\ ]$ if t_0 denotes the initial time, while knowledge of the past on $[-\infty, t_0\ ]$ is required for the fractional model **(16)** for instance.

For all these reasons, in a modelling approach it is better to work with model **(20)** than with models **(16)**, since it is more general as previously demonstrated. This can be done by searching the kernel $\eta^{\nu}(t)$ directly without any assumption on its structure.

By choosing an appropriate kernel $\eta^{\nu}(t)$ in relation **(20)**, it is possible to produce fractional behaviours of various kinds. To highlight this, the following model involving a Volterra equation of the first kind is considered:

$$\int_{0}^{t} \eta^{\nu}(t-\tau)x(\tau)d\tau = v(t) \quad with \quad v(t) = \int_{0}^{t} u(\tau)d\tau,\ y(t) = x(t). \tag{21}$$

It is assumed that $u(t) \in \mathbb{R}$ denotes the input of the model and that $y(t) \in \mathbb{R}$ is its output. The Laplace transform of relation **(21)** is, without considering initial conditions,

$$\eta^{\nu}(s)x(s) = v(s) \quad with \quad v(s) = \frac{1}{s}u(s),\ y(s) = x(s), \tag{22}$$

and from an input–output point of view, the following transfer function is thus obtained:

$$\frac{y(s)}{u(s)} = \frac{1}{s\eta^{\nu}(s)}\ . \tag{23}$$

Several kernels producing fractional behaviours for transfer function **(23)** were proposed in (Sabatier, 2020c).

DISTRIBUTED TIME DELAY MODELS

Volterra equations or more generally integro-differential equations are one of the ways to define models for fractional behaviours without the drawbacks cited in (Sabatier et al., 2020). Other solutions exist and for instance, the following class of distributed time delay model (Gouaisbaut, 2005) was considered in (Sabatier, 2020d).

$$\frac{d}{dt}x(t) = A_0x(t) + A_1\int_{0}^{T_f} \eta^{\nu}(\tau)x(t-\tau)d\tau + Bu(t). \tag{24}$$

This class of model is particularly interesting to make interpretation of fractional behaviours produced by adsorption phenomena (Tartaglione et al., 2020), (Tartaglione et al., 2021) as it will be shown soon by the author. The Laplace transform of relation **(24)** is given by

$$sx(s) = A_0x(s) + A_1\int_{0}^{T_f} \eta^{\nu}(\tau)x(s)e^{-\tau s}d\tau + Bu(s) \tag{25}$$

$$and\ thus \quad sx(s) = A_0x(s) + A_1x(s)\int_{0}^{T_f} \eta^{\nu}(\tau)e^{-\tau s}d\tau + Bu(s). \tag{26}$$

The transfer function linking the system input $u(t)$ to the system output $x(t)$ is thus defined by:

$$\frac{x(s)}{u(s)} = \frac{B}{s - A_0 - A_1\int_0^{T_f} \eta^{\nu}(\tau)e^{-\tau s}d\tau}. \tag{27}$$

As an example, among the infinity of possible functions, some of which are given in (Sabatier 2020d), kernel $\eta^{\nu}(t)$ in **(24)** is assumed to be defined by:

$$\eta^{\nu}(t) = C_0\left(\frac{\omega_{min}^{\nu}}{\Gamma(\nu)}t^{\nu-1}e^{-\omega_{min}t} - \frac{\omega_{min}^{\nu}}{\Gamma(\nu)}t^{\nu-1}e^{-\omega_{max}t} + \omega_{min}^{\nu}\omega_{max}^{1-\nu}e^{-\omega_{max}t}\right) \tag{28}$$

$$with \quad C_0 = \left|\left(\frac{1}{\left(\frac{j}{\omega_{min}}+1\right)^{\nu}} - \left(\frac{\omega_{min}}{\omega_{max}}\right)^{\nu}\frac{1}{\left(\frac{j}{\omega_{max}}+1\right)^{\nu}} + \left(\frac{\omega_{min}}{\omega_{max}}\right)^{\nu}\frac{1}{\frac{j}{\omega_{max}}+1}\right)^{-1}\right|. \tag{29}$$

As t tends towards 0, the following relation holds [using Taylor expansions of the exponential function in relation **(28)**]:

$$\eta^{\nu}(t) \underset{t\to 0}{\approx} C_0\left(\frac{\omega_{min}^{\nu}}{\Gamma(\nu)}t^{\nu-1}(-\omega_{min}t+\omega_{max}t)+\omega_{min}^{\nu}\omega_{max}^{1-\nu}\right) \quad (30)$$

and thus
$$\lim_{t\to 0}\eta^{\nu}(t) = C_0\omega_{min}^{\nu}\omega_{max}^{1-\nu}. \quad (31)$$

This highlights the non-singularity of kernel **(28)** as time t tends towards 0. Integral

$$I(s) = \int_0^{T_f}\eta^{\nu}(\tau)e^{-\tau s}d\tau \quad (32)$$

is defined by:

$$\begin{aligned} I(s) \;=\; & K\Big(\big((s+\omega_{\min})T_f\big)^{-\frac{\nu}{2}}e^{-\frac{(s+\omega_{\min})T_f}{2}}T_f^{\nu}(s+\omega_{\max}) \\ & W_M\Big(\frac{\nu}{2},\frac{\nu}{2}+\frac{1}{2},(s+\omega_{\min})T_f\Big) \\ & -\big((s+\omega_{\max})T_f\big)^{-\frac{\nu}{2}}e^{-\frac{(s+\omega_{\max})T_f}{2}}T_f^{\nu}(s+\omega_{\max}) \\ & W_M\Big(\frac{\nu}{2},\frac{\nu}{2}+\frac{1}{2},(s+\omega_{\max})T_f\Big) - (1+\nu)\Big(\big(\omega_{\max}^{1-\nu}\Gamma(\nu)\nu \\ & + T_f^{\nu}(s+\omega_{\max})\big)e^{-(s+\omega_{\max})T_f} \\ & +T_f^{\nu}(s+\omega_{max})e^{-(s+\omega_{\min})T_f} - \omega_{\max}^{1-\nu}\Gamma(\mathrm{v})\Big)\Big) \end{aligned} \quad (33)$$

$$with\;\; K = \frac{C_0\left(s^2\omega_{\min}^{\nu}+2s\omega_{\min}^{1+\nu}+\omega_{\min}^{2+\nu}\right)}{(s+\omega_{\max})\Gamma(\nu)(s+\omega_{\min})^2\nu(1+\nu)} \quad (34)$$

where $W_M(a,\ b,\ z)$ is the Whittaker M function defined by

$$\begin{aligned} W_M(a,b,z) &= z^{b+1/2}e^{-z/2}\sum_{n=0}^{\infty}\frac{\left(b-a+\frac{1}{2}\right)_n}{n!\,(2b+1)_n}z^n \\ &= z^{b+1/2}e^{-z/2}\,{}_1F_1\Big(\frac{1}{2}+b-a,\ 1+2b,z\Big). \end{aligned} \quad (35)$$

Figure 3 shows the gain and phase diagrams of integral $I(s)$ for various values of ν and with $\omega_{min} = 10^{-3}$ rd/s, $\omega_{max} = 10^4$ rd/s and $T_f = 10/\omega_{min}$.

Parameters A_0, A_1 and B in model **(24)**, can be used to control the frequency band on which the fractional behaviour takes place. As an example, **Figure 4** shows the Bode diagram of the transfer function $x(s)/u(s)$ given by relation **(27)** under the following conditions:

- $\nu = 0.7, \omega_{min} = 10^{-3}$ rd/s, $\omega_{max} = 10^4$ rd/s, $T_f = 10/\omega_{min}$, $A_0 = -36969$, $A_1 = -10^8$, $B = 10^8$
- $\nu = 0.3, \omega_{min} = 10^{-3}$ rd/s, $\omega_{max} = 10^4$ rd/s, $T_f = 10/\omega_{min}$, $A_0 = -3686$, $A_1 = -10^5$, $B = 10^5$

These Bode diagrams exhibit a fractional behaviour of order $1-\nu$ in a limited frequency band.

NONLINEAR MODELS

The idea of using non-linear models to model systems with fractional behaviours came from analysing the evolution of the top of a pile of sand $h(t)$ under a constant flow $Q(t)$ of sand, and assuming the angle of repose $\alpha(t)$ constant (see **Figure 5**). The radius of the heap $r(t)$ is deduced from $\alpha(t)$ and $h(t)$: $tan(\alpha(t)) = h(t)/r(t)$.

The kinetics of the evolution of this heap is of the form $Kt^{1/3}$, and can be very well fitted by a fractional transfer function model $Ks^{\frac{1}{3}-1}$. But such a model is no longer suitable if the flow of falling sand is doubled or if this flow is stopped (the fractional model relaxes while the height of the top remains constant). By analysing this system a little more closely (Sabatier et al., 2020), we can analytically demonstrate that a model of this system is of the form

$$\frac{dh(t)}{dt} = \frac{K}{h(t)^2}Q(t), \quad (36)$$

that is to say a non-linear input affine (or distributional) model form:

$$\frac{dx(t)}{dt} = f(x(t))u(t), \quad (37)$$

where $x(t)$ is the state and the output of the system, and where $u(t)$ denotes its input.

Such a modelling approach solves a large part of the problems described in **section 2** and has been applied to the modelling of adsorption/desorption phenomena (Tartaglione et al., 2020) which are used in many fields and in particular for the design of Love wave based sensors (Hallil et al., 2009).

TIME VARYING MODELS

Without referring to fractional models, the Avrami model is often used to model the kinetics of crystallization, as well as other phase changes or chemical reactions (Avrami, 1939; Fanfoni and Tomellini, 1998). This model is described by the relation

$$x(t) = K\left(1-e^{-(at)^{\nu}}\right). \quad (38)$$

The Laplace transform of relation **(38)** is given by:

$$x(s) = K\left(\frac{1}{s}-\frac{1}{a}\sum_{r=0}^{\infty}\frac{(-a)^r}{r!}\frac{\Gamma(1+\nu r)}{\left(\frac{s}{a}\right)^{1+\nu r}}\right). \quad (39)$$

Figure 6 shows the frequency response of $x(s)$ and demonstrates a fractional behaviour.

Function $x(t)$ in relation **(38)** is also a solution of the differential equation:

$$\frac{dx(t)}{dt} = -\nu a^{\nu}t^{\nu-1}x(t) + K\nu a^{\nu}t^{\nu-1} \quad (40)$$

This equation involves singular coefficients. Such a matter can be solved by a differential equation of the form

$$\frac{dx(t)}{dt} = -\nu a^{\nu}(t+c)^{\nu-1}x(t) + K\nu a^{\nu}(t+c)^{\nu-1} \quad (41)$$

which also exhibits a fractional behaviour and thus shows that time varying models are possible models to capture fractional behaviours.

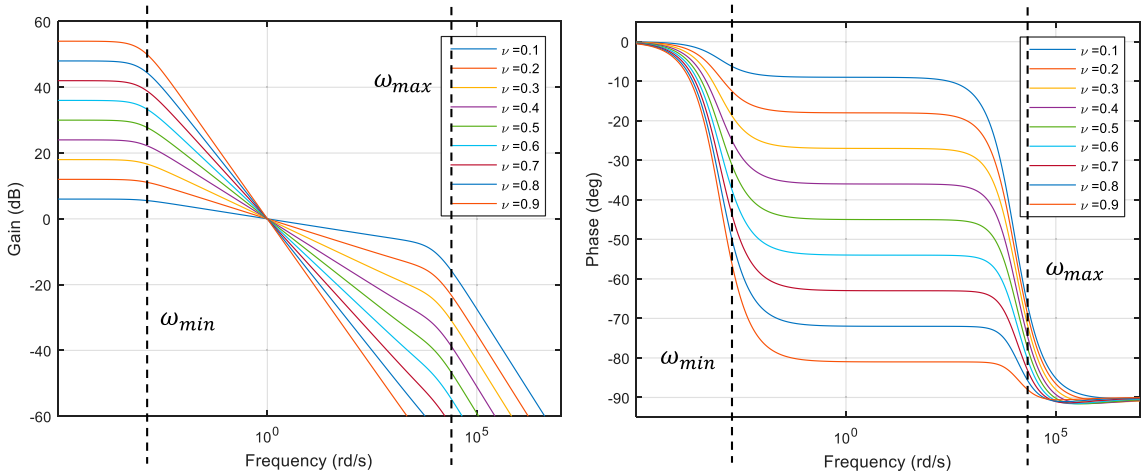

FIGURE 3 | Frequency response of integral $I(s)$ for ν varying from 0.1 to 0.9 and with $\omega_{min} = 10^{-3}$ rd/s, $\omega_{max} = 10^{4}$ rd/s and $T_I = 10/\omega_{min}$.

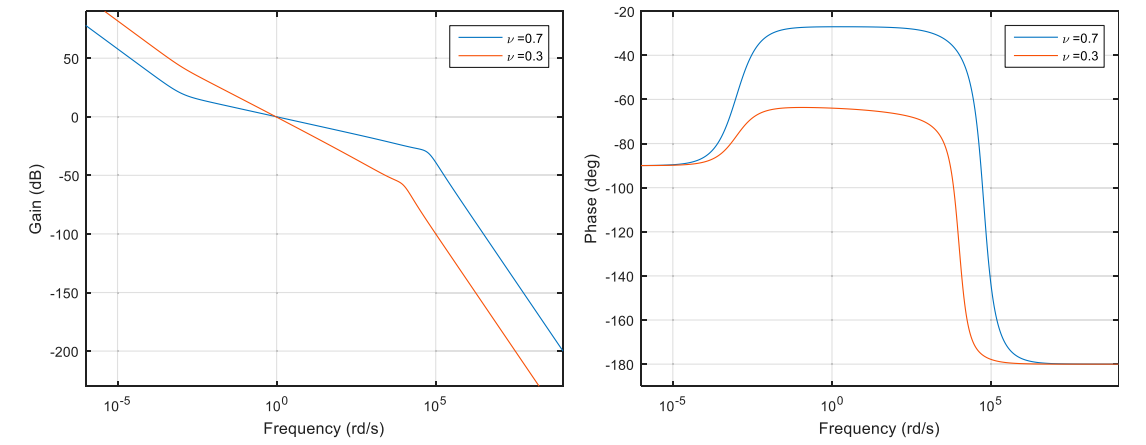

FIGURE 4 | Frequency response of transfer function $x(s)/u(s)$ for two sets of parameters.

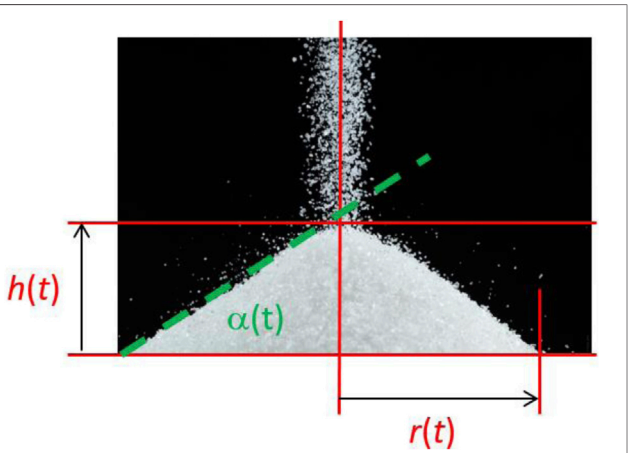

FIGURE 5 | Notations for the characterisation of sand heap growth.

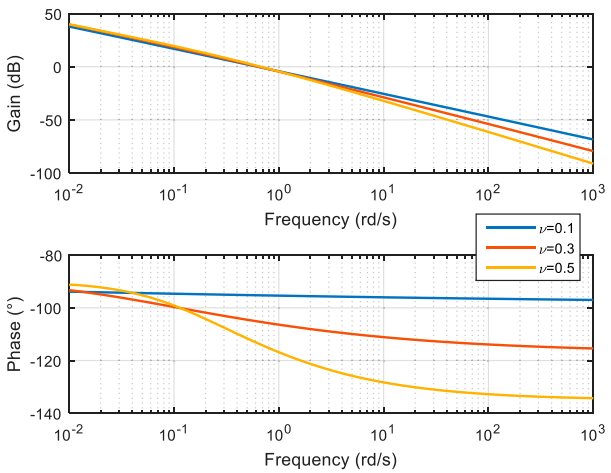

FIGURE 6 | Frequency response of $x(s)$ for $a = 1$, $\nu = 0.1$, $\nu = 0.3$. $\nu = 0.5$

PARTIAL DIFFERENTIAL EQUATIONS (HEAT EQUATIONS) WITH SPATIALLY VARIABLE COEFFICIENTS

With a judicious choice of the spatial functions $\gamma(z)$and $\beta(z)$, it is shown in (Sabatier, 2020e) that the following heat equation

$$\frac{\partial T(z,t)}{\partial t} = \gamma(z)\frac{\partial}{\partial z}\left(\beta(z)\frac{\partial T(z,t)}{\partial z}\right) \; y(t) = \left.\frac{\partial T(z,t)}{\partial z}\right|_{z=0} \; u(t) = T(0,t) \tag{42}$$

generates fractional behaviours.

An infinite number of combinations are possible for the functions $\gamma(z)$ and $\beta(z)$. This leaves considerable room for theoretical investigations in the search for these functions, then for the analysis of the properties of **Eq. 42**, and also for the development of methods to identify the functions $\gamma(z)$ and $\beta(z)$ on real data. It is also possible to generalize this result to partial differential equations other than diffusion equations (Sabatier, 2020e). These results seem promising for modelling phenomena in fractal environments, in order to relate the geometry with the spatially variable coefficients of the equation.

CONCLUSION

This paper demonstrates that several other modelling tools than fractional models can be used to produce fractional behaviours that are ubiquitous and that can be encountered in many domains (physical, chemical, biological, electrical, etc.). These alternative models, i.e.

- new kernels in convolution operators (that still enable fractional behaviours to be generated but in a limited frequency or time range).
- the Volterra integro-differential equation.
- distributed time delay models.
- nonlinear models.
- time-varying models.
- partial differential equations with spatially variable coefficients,

Enable the drawbacks and limitations suffered by fractional models to be overcome. Either way, fractional models remains interesting fitting tools for quickly capturing the input-output behaviour of physical systems. But we must not try to make them say more. Due to their doubly infinite dimension (Sabatier, 2021) they are not adapted to study their internal properties.

Several applications of the proposed models have been presented in the literature. In particular, the application of distributed delay systems to lithium ion battery modelling (Sabatier 2020d) or the application of non-linear models to CO2 adsorption (Tartaglione et al., 2020) can be cited. It has been shown that the models used made it possible to obtain a modelling accuracy similar to that obtained with fractional models, with an equivalent number of parameters, but without the drawbacks of fractional models (no infinite length memory and thus no initialization matter, no singular kernels, ...). Other applications are currently in progress.

This paper also demonstrates that the confusion or rather the implicit link that exists in the literature between fractional behaviours and fractional models should be avoided because it is reductive. Fractional models are only one class of models among others that can capture fractional behaviours. Such a point of view is not a sterile criticism of fractional calculus and models, but suggests avenues for novel developments in the field of fractional behaviour modelling and analysis of the resulting models.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/**Supplementary Material**, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

SUPPLEMENTARY MATERIAL

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fcteg.2021.716110/full#supplementary-material

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