Blockchain Competition Between Miners: A Game Theoretic Perspective

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We model the competition over mining resources and over several cryptocurrencies as a non-cooperative game. Leveraging results about congestion games, we establish conditions for the existence of pure Nash equilibria and provide efficient algorithms for finding such equilibria. We account for multiple system models, varying according to the way that mining resources are allocated and shared and according to the granularity at which mining puzzle complexity is adjusted. When constraints on resources are included, the resulting game is a constrained resource allocation game for which we characterize a normalized Nash equilibrium. Under the proposed models, we provide structural properties of the corresponding types of equilibrium, e.g., establishing conditions under which at most two mining infrastructures will be active or under which no miners will have incentives to mine a given cryptocurrency.

Keywords: competition, mining, bitcoin, blockchain, game theory

1. INTRODUCTION

The blockchain is a distributed synchronized secure database containing validated blocks of transactions. A block is validated by special nodes called miners and the validation of each new block is done via the solution of a computationally difficult problem, which is called the proof-of-work puzzle. The miners compete against each other and the first to solve the problem announces it, the block is then verified by the majority of miners in this network, trying to reach consensus. After the propagated block reaches the consensus, it is successfully added to the distributed database. The miner who found the solution receives a reward either in the form of cryptocurrencies or in the form of a transaction reward.

Due to the huge energy requirement necessary to be the first to solve a puzzle, blockchain mining is typically executed in specialized hardware. In Zhang et al. (2016) an Edge computing Service Provider (ESP) is introduced to support proof-of-work puzzle offloading by using its edge computing nodes. In Xiong et al. (2017) a game is formulated between the miners in the presence of a single ESP and then a Stackelberg game is used to compute the pricing that maximizes the revenue of the ESP.

Our work addresses the following two questions:

1) Given a single blockchain, how should rational users contribute to the mining process, possibly counting on third-party ESPs or mining pools to offload infrastructure costs?
2) Given multiple blockchains, e.g., in a multi-cryptocurrency ecosystem, how should rational users distribute their monetary and/or computational budget toward mining?

In this paper, we focus on the competition between miners while addressing the two questions above. We model the competition between miners, who have to choose which ESP to use and which blockchains to mine, as a non-cooperative game. Note that each ESP corresponds to a separate mining infrastructure, and each blockchain corresponds to a different cryptocurrency. Then, we specialize our results to two instances of the general game, showing properties of the Nash equilibrium.

In the first game, there is a single blockchain (e.g., cryptocurrency) and any of the M ESPs (or mining pools) can be used by the miners to solve the puzzle. In the second game, we consider K opportunities, each of which corresponding to another blockchain. At each time slot of duration T (which corresponds to a new puzzle to be solved) each of the miners decides which of K puzzles to solve. We formulate both games and establish conditions for the existence of a pure Nash equilibrium for the association problem between miners and ESPs, providing an efficient algorithm for solving it. We summarize our contributions as follows:

Congestion game for mining competition: we model the competition among users searching for a solution to the mining puzzle as a game (section 3). In essence, as the number of users willing to mine increases, the chances that a given user is the first to succeed in solving the mining puzzle and wins a reward decreases (i.e., the system becomes *congested*). In particular, we assume that users can count on third-parties to offload infrastructure costs, and can mine multiple cryptocurrencies. Under the assumption that such third-parties are roughly indistinguishable, we further show that when there is one single cryptocurrency of interest the *congestion game* admits a simple equilibrium accounting for users that must decide whether to mine or otherwise not join the system (section 4).

Analysis of multi-cryptocurrency ecosystem: we analyze the congestion game involving multiple cryptocurrencies. In that case, miners compete against those that decide to mine the same cryptocurrency (section 5) and we show that the proposed game admits a potential.

Continuous actions and physical bounds on resources: we consider two extensions of the proposed games. First, we consider continuous actions, wherein miners can split their budget across multiple ESPs and multiple cryptocurrencies (section 7.1). Second, we allow for physical bounds on resources, such as energy, which can be consumed by the whole system (section 7.2).

Paper outline The remainder of this paper is organized as follows. Sections 2, 3 present background on mining competition and the general game framework considered in this paper to characterize such competition. Then, section 4 specializes to the setup wherein there is only one single cryptocurrency, and section 5 accounts for multiple cryptocurrencies. The general game accounting for multiple ESPs and multiple cryptocurrencies is considered in section 6. Extensions to account for continuous actions and physical bounds on resources are introduced in section 7. Discussion and related work follow in sections 8, 9, and section 10 concludes. Appendices contain supplementary material, including a discussion on positive and negative mining externalities (**Appendix A**), a technical proof (**Appendix B**) and the analysis of the setup

wherein ESPs continuously use their resources at maximum capacity (**Appendix C**).

2. MINING COMPETITION

In this section we discuss two key aspects pertaining mining competition. First, we indicate how the granularity of the adjustment of mining difficulty impacts the nature of the competition (section 2.1). Then, we relate the granularity of the mining difficulty adjustment to the horizon at which competition takes place (section 2.2). In the appendix we further indicate more broadly how competition and cooperation play important roles in blockchain systems.

2.1. Granularity of Adjustment of Mining Complexity

The goal of adjusting the mining complexity is to find a difficulty point at which the network mines a block containing outstanding transactions every x minutes. In Bitcoin, we have $x = 10$ minutes. By decreasing (resp., increasing) difficulty, the Bitcoin protocol also decreases (resp., increases) the amount of time, processing power, and electricity required to solve a block.

Next, we discuss the implications of the granularity at which mining complexity is adjusted. Bitcoin's difficulty adjustment, for instance, is naturally adjusted by the system every 2016 blocks. This adjustment probabilistically averages to 2 week intervals between adjustments.

In this paper, we consider two extreme cases with respect to the granularity of adjustment of mining complexity:

Fine grained adjustment of mining complexity: under the fine grained adjustment of complexity, every time a miner joins or leaves the network the mining complexity is correspondingly adjusted. In this case, the mean time to solve a puzzle by the network is independent of the number of miners. From the perspective of each miner, however, the mean time to solve a puzzle increases as the number of miners grows.

Coarse grained adjustment of mining complexity: under the coarse grained adjustment of complexity, the number of miners may vary inbetween the adjustment of mining complexity. In that case, the mean time to solve a puzzle decreases as the number of miners grows.

In section 3 we introduce the general game, accounting for the two scenarios described above. Then, we present specialized results to the two instances in the upcoming sections.

2.2. Horizon of Competition

The horizon of competition among miners depends on the granularity at which the adjustment of mining complexity takes place. Under the fine grained adjustment of mining complexity, competition occurs both at a short term and long term horizon. This is because as the number of miners increases, the difficulty of the puzzle grows and the competition becomes more aggressive. Under the coarse grained adjustment of mining complexity, in contrast, competition occurs only at the short term horizon. In essence, miners still compete to decide who will be the next to mine the upcoming block (see **Figure 1**).
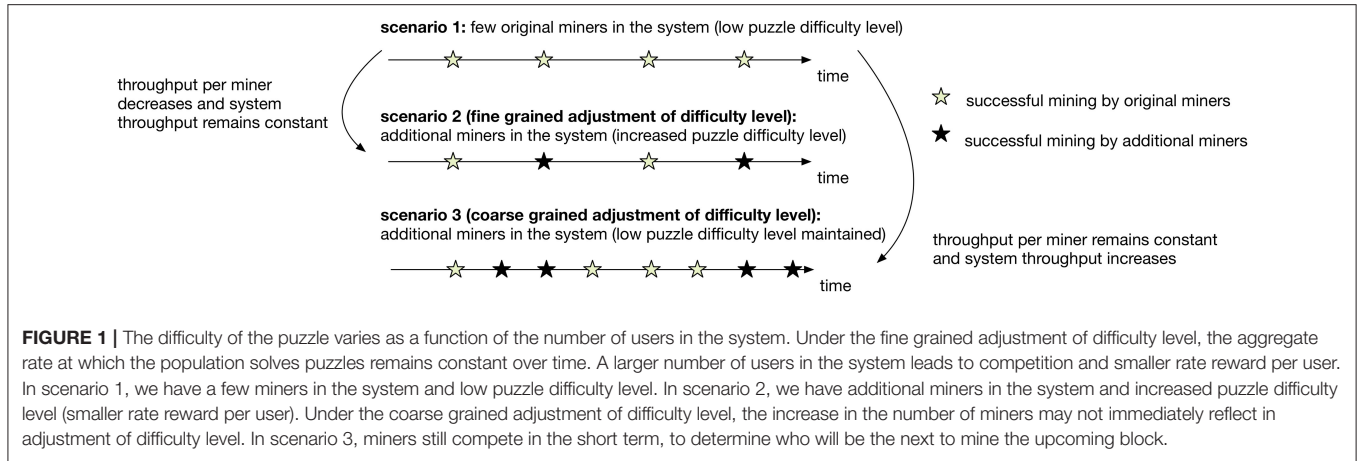
FIGURE 1 | The difficulty of the puzzle varies as a function of the number of users in the system. Under the fine grained adjustment of difficulty level, the aggregate rate at which the population solves puzzles remains constant over time. A larger number of users in the system leads to competition and smaller rate reward per user. In scenario 1, we have a few miners in the system and low puzzle difficulty level. In scenario 2, we have additional miners in the system and increased puzzle difficulty level (smaller rate reward per user). Under the coarse grained adjustment of difficulty level, the increase in the number of miners may not immediately reflect in adjustment of difficulty level. In scenario 3, miners still compete in the short term, to determine who will be the next to mine the upcoming block.

3. BLOCKCHAIN COMPETITION GAME

3.1. Basic Concepts

Miners, mining servers, and puzzles. We consider a population of M ESPs and a set of K cryptocurrencies, where each cryptocurrency is associated to its blockchain. We denote by $\mathcal{N} = \{1, 2, \ldots, N\}$ the set of miners, also referred to as users. There is a finite population of miners, and if a miner changes his strategy this will cause a change in the utilities of other miners. Let $\mathcal{K} = \{1, 2, \ldots, K\}$ be the set of puzzles, each of which associated with a different cryptocurrency that the miners are trying to solve. We assume that each cryptocurrency corresponds to exactly one puzzle. Let $\mathcal{M} = \{1, 2, \ldots, M\}$ denote the ESPs, also referred to as mining servers, that miners can rely on. A special virtual ESP with index 0 corresponds to an always idle ESP, whose service rate is zero. Miners join ESP 0 when they decide not to join the mining game. Notation is summarized in **Table 1**.

Strategies. Set $\mathcal{S}_i \subset \mathcal{K} \times \mathcal{M}$ denotes the set of ordered pairs (puzzle, ESP), corresponding to ESPs that miner i can rely on to solve puzzles of a given type. The set $\mathcal{S}_i$ can differ across miners due to political or economic restrictions. For instance, certain countries do not allow investment in certain cryptocurrencies. Alternatively, the set of available ESPs for two different miners may not be the same. A strategy for miner i is denoted by $s_i \in \mathcal{S}_i$, corresponding to the puzzle (cryptocurrency) which the miner intends to solve using a given infrastructure. Strategy $s_i = (k, m)$ corresponds to user i using ESP infrastructure m to mine cryptocurrency k. A strategy vector $s = (s_i)_{i \in \mathcal{N}}$ produces a load vector $\ell = (\ell_{k,m})_{k,m}$, where $\ell_{k,m}$ denotes the number of miners using ESP m to mine cryptocurrency k.

Mining complexity. We denote by $\mu_{k,m,i}$ the service rate from ESP m requested by miner i to solve puzzle k. We assume $\mu_{k,m,i} > 0$ when $m \neq 0$, and $\mu_{k,0,i} = 0$, for $k = 1, \ldots, K$ and $i = 1, \ldots, N$. For convenience, the service rate is measured:

- In rate of hashes computed per time unit (trials to solve the puzzle per time unit), when accounting for the fine grained adjustment of mining complexity, wherein the average number of puzzles solved per time unit for the whole population is fixed and given, and

TABLE 1 | Table of notation.

Variable	Description
K	Number of blockchains (cryptocurrencies)
M	Number of edge service providers (ESPs)
N	Number of miners (willing to mine using ESPs)
$U_{k,m}(\ell)$	Utility of user mining blockchain k at ESP m
$\gamma_{k,m}$	Mining cost associated to blockchain k at ESP m
$\mu_{k,m}$	Service rate from ESP m requested by each miner to solve puzzle k
Action space and corresponding variables	
$\mathcal{S}_i \subset \mathcal{K} \times \mathcal{M}$	Set of ordered pairs (puzzle, ESP),
	Corresponding to ESPs that miner i Can use to mine k
$\ell_{k,m}$	Number of users mining blockchain k at ESP m
ℓ	Strategy profile, $\ell = (\ell_{1,1}, \ell_{1,2}, \ldots, \ell_{k,m}, \ldots, \ell_{K,M})$
	(discrete action space, all sections except section 7)
Control variables	
s_i	$s_i = (k, m)$ if user i mines blockchain k at ESP m
	(discrete action space, all sections except section 7)
x_m	Amount bid by ESP m, proportional to the load invested by ESP m for mining
	(continuous action space, section 7)
Metrics	
$p_{k,m}(\ell)$	Probability that user is first to mine a block

- In rate of puzzles successfully solved per time unit, when accounting for the coarse grained adjustment, so as to simplify notation[1].

Let η_k be the load of miners across all ESPs toward cryptocurrency k. Then,

$$\eta_k = \sum_{m' \in \mathcal{M}} \sum_{i' \in \mathcal{N}} \mu_{k,m',i'}. \tag{1}$$

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[1] Alternatively, the service rate could be uniformly set in units of hashes per time unit, but in that case one would need to introduce and additional parameter to relate the number of hashes computed per time unit and the fraction of those that translate into successful mining.
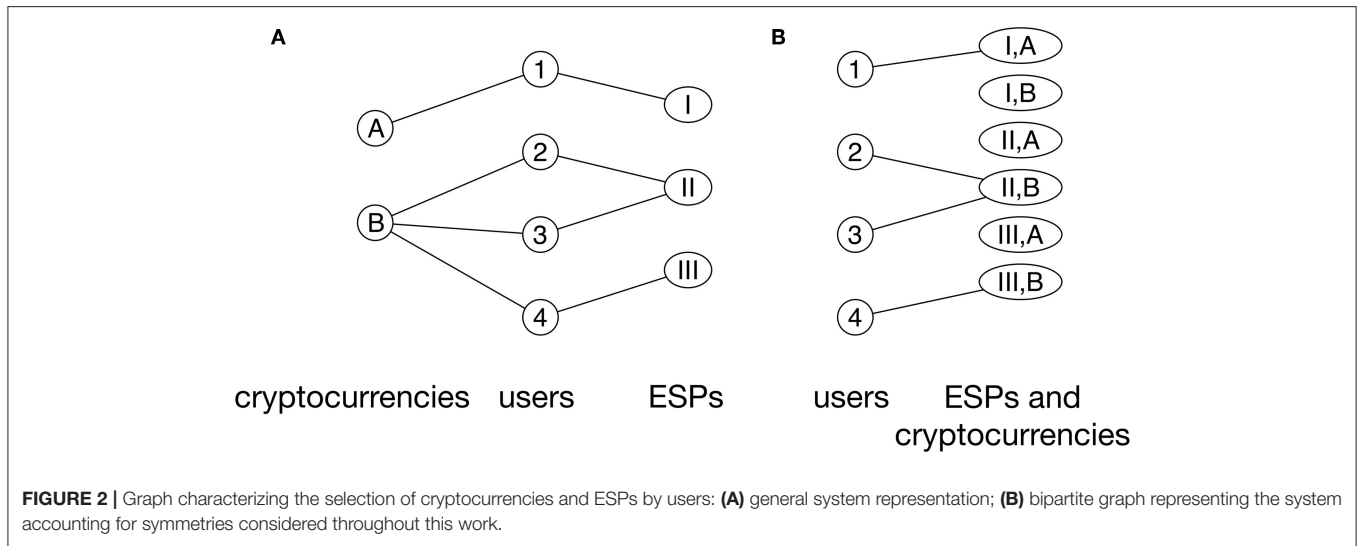
FIGURE 2 | Graph characterizing the selection of cryptocurrencies and ESPs by users: **(A)** general system representation; **(B)** bipartite graph representing the system accounting for symmetries considered throughout this work.

TABLE 2 | Granularity of difficulty adjustment.

	Average time between two blocks mined, for whole population ($\mathcal{T}_k$)	Average time between two blocks mined, per miner	Probability of success by time T (q_k)	Time horizon to grant rewards to given player (T)
Fine grained adjustment of mining complexity	Fixed and given	Variable, depends on η_k, for fixed $\mu_{k,m}$	≈ 1	Large, compared against puzzle complexity adjustment
Coarse grained adjustment of mining complexity	Variable, depends on η_k, for fixed $\mu_{k,m}$	Fixed and given, for fixed $\mu_{k,m}$	$1 - \exp(-T\eta_k)$	Small, compared against puzzle complexity adjustment

In the remainder of this paper, except otherwise noted, we assume that a user who selects a given (ESP, cryptocurrency) pair is allocated a given hash power by the ESP[2]. **Figure 2** illustrates the considered setup. Then, (1) simplifies to:

$$\eta_k = \sum_{m' \in \mathcal{M}} \ell_{k,m'} \mu_{k,m'}. \qquad (2)$$

Note that (2) is obtained from (1) by lumping the state space: for symmetric users it suffices to track the *number* of users selecting each of the available (ESP, cryptocurrency) pairs rather than their identities.

Let $\mathcal{T}_k$ be the time it takes for the first miner, across all ESPs, to solve puzzle k. Let q_k be the probability that puzzle k is solved by time T since the last puzzle of cryptocurrency k was solved. Note that under the fine grained adjustment of mining complexity, $\mathcal{T}_k$ and q_k are functionally independent of η_k, as far as the number of miners of cryptocurrency k is positive. Without loss of generality,

we assume that the time horizon of interest, T, is set to a large enough value, independent of η_k, so that $q_k \approx 1$. Under the coarse grained adjustment of mining complexity, in contrast, $\mathcal{T}_k$ and q_k depend on η_k as the time it takes for a block of cryptocurrency k to be successfully mined is a function of the load of miners toward k (**Table 2**).

Under the coarse grained adjustment of mining complexity, $\mathcal{T}_k$ depends on the number of miners in the system. In that case, we denote by $R_{k,m,i}$ the amount of service time from ESP m required by miner i to solve puzzle k. As we assume that users are symmetric, the random variables $R_{k,m,i}$ are independent and identically distributed, for $i = 1, \ldots, N$, with each $R_{k,m,i}$ being exponentially distributed with rate $\mu_{k,m}$. Thus, if there are $\ell_{k,m}$ miners associated to ESP m mining currency k, the time it takes for the fastest of them to solve the puzzle corresponding to currency k is exponentially distributed with rate $\eta_k = \sum_m \mu_{k,m} \ell_{k,m}$. Then,

$$\mathcal{T}_k \sim \text{Exp}(\sum_m \mu_{k,m} \ell_{k,m}) \qquad (3)$$

$$q_k = 1 - \exp(-T\eta_k). \qquad (4)$$

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[2]To account for non-symmetric users, one may add additional virtual users and/or virtual (ESP, cryptocurrency) pairs representing different service level agreements offered by a given ESP to users.

Note that in this case if T is set to a large enough value, dependent on η_k, we also have $q_k \approx 1$ as in the previous paragraph.

Rewards and costs. Let $\tilde{p}_{k,m}$ denote the probability that a miner using ESP m is the first to solve puzzle k at state ℓ. Then,

$$\tilde{p}_{k,m}(\ell) = q_k \frac{\ell_{k,m}\mu_{k,m}}{\eta_k}. \tag{5}$$

Throughout this paper, $0/0 = 0$. In the expression of $\tilde{p}_{k,m}$, for instance, if $\eta_k = 0$ and $\ell_{k,m} = 0$, then $\tilde{p}_{k,m} = 0/0 = 0$.

Under strategy profile ℓ, the probability that a given miner using ESP m is the first to solve puzzle k is:

$$p_{k,m}(\ell) = 1_{\ell_{k,m}>0} \frac{q_k\mu_{k,m}}{\eta_k}, \tag{6}$$

where 1_c equals 1 if condition c holds and 0 otherwise.

We denote by $\gamma_{k,m}$ the cost of mining blockchain k at ESP m. Under the fine grained adjustment of puzzle complexity, $\gamma_{k,m}$ is measured in cost per time unit. Under the coarse grained adjustment of puzzle complexity, $\gamma_{k,m}$ is the cost incurred by users to reserve mining resources during the time horizon T of interest.

Utilities. Let $U_{k,m}(\ell)$ denote the utility to a miner who tries to find the solution of puzzles associated to cryptocurrency k, using ESP m. The utility is given by rewards minus costs. Thus,

$$U_{k,m}(\ell) = \begin{cases} p_{k,m}\rho - \gamma_{k,m} & \text{if } m > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Under the fine grained adjustment of puzzle complexity, ρ is the rate reward granted to successful miners, which is fixed and given. Therefore, to simplify presentation we let $\rho = 1$, and $\gamma_{k,m}$ is adjusted accordingly. Under the coarse grained adjustment of puzzle complexity, in contrast, if users are still interested in the long term rewards they need to account for a rate reward that is a function of the users actions. This is because under the coarse grained adjustment of puzzle complexity, the larger the number of users mining a given cryptocurrency, the larger the rate at which blocks are mined. Alternatively, motivated by Bissias et al. (2019) we assume that users under the coarse grained adjustment of puzzle complexity are greedy and myopic, as detailed next.

Whereas under the fine grained adjustment of puzzle complexity users are interested in maximizing a long term average rate reward, under the coarse grained adjustment they are interested in maximizing the reward collected by time T, assuming that during that time horizon the chances that more than one user collects rewards are negligible. In that case, users are granted a reward if and only if they are the first to successfully mine by the time horizon of interest, T. We let $\rho = 1$ and $\gamma_{k,m}$ characterizes the cost of reserving mining resources to mine during slot T.

Note that under the fine grained adjustment of puzzle complexity, a new mining interval starts immediately after a successful mining event occurs. Under the coarse grained adjustment of puzzle complexity, in contrast, we assume that mining resources are reserved for a mining interval T, and even

if a success occurs before T miners pay for resources allocated up until T. In both cases, the utility reduces to:

$$U_{k,m}(\ell) = \begin{cases} p_{k,m} - \gamma_{k,m} & \text{if } m > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

User i utility is $\tilde{U}_i(s_i, s_{-i}) = \sum_{(k,m)\in\mathcal{S}_i} 1_{s_i=(k,m)} U_{k,m}(\ell)$, where $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$ is the vector of strategies of all miners except miner i. Given the ingredients above, the blockchain competition game is characterized by $C = \langle \mathcal{N}, \mathcal{K} \times \mathcal{M}, (\mathcal{S}_i)_{i\in\mathcal{N}}, (U_{k,m})_{(k,m)\in\mathcal{K}\times\mathcal{M}} \rangle$. In sections 4 and 5 we analyze two special instances of this game.

3.2. Summary of Terminology

We summarize the basic terminology used throughout this paper.

Edge Service Providers (ESPs) continuously try to solve blockchain puzzles, by allocating hash power for that purpose.

Hash power dedicated to a given blockchain by a given ESP is the number of hashes computed per time unit by that ESP to solve puzzles from the corresponding blockchain.

Service rate dedicated to a given blockchain by a given ESP equals the corresponding hash power. Under the coarse grained adjustment of puzzle complexity, it is more convenient to measure the service rate in number of successful puzzles solved per time unit, noting that in this case the number of successful puzzles solved per time unit equals the hash power times a constant multiplicative factor smaller than one.

Miners pay to ESPs in order to solve blockchain puzzles.

Active miners participate in the mining game, by paying a strictly positive amount to ESPs in order to solve blockchain puzzles. **Inactive miners** decide not to actively join the mining game. They receive no rewards, and incur zero costs.

Rewards are granted to miners once the contracted ESP solves the corresponding puzzle.

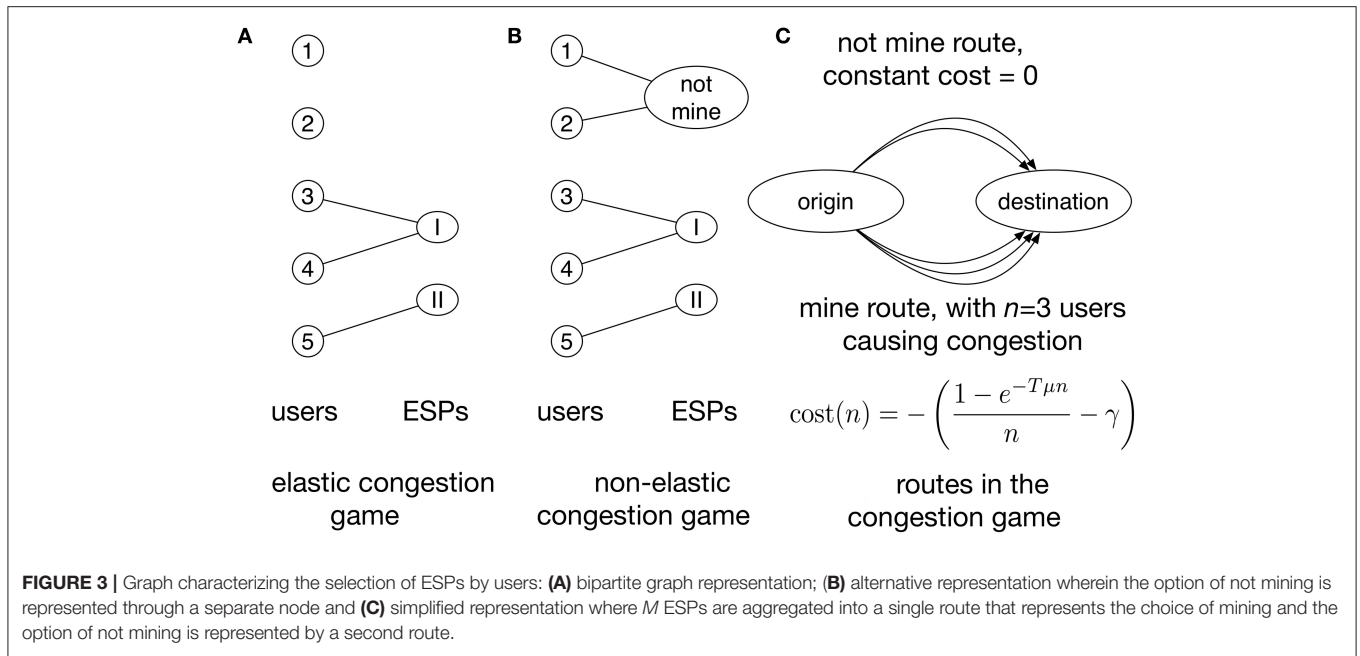**Costs** are incurred by miners to contract ESPs.

Revenue corresponds to rewards minus costs incurred by each miner.

3.3. Congestion Games and Potentials

Next, we briefly introduce some basic background on congestion games, crowding games and potentials. Such background is instrumental in the analysis of the blockchain competition game that follows.

Congestion games were introduced by Monderer and Shapley (1996) and are equivalent to routing over an arbitrary graph, when all routed objects have the same size, and are non splittable. The cost of using an edge is the same for all players. Crowding games proposed by Milchtaich (1996) are congestion games with more restricted topology (parallel links) but more general costs (user dependent).

In our setup, the routed object is the mining power. The network has a bipartite topology, where one side consists of mining users (end users) and the other side consists of ESPs that mine according to mining users requests. A virtual ESP corresponds to the option of not mining. The cost incurred by a user who decides to mine through a given ESP is the cost of an edge between the user and the ESP (see **Figure 3**).

FIGURE 3 | Graph characterizing the selection of ESPs by users: **(A)** bipartite graph representation; **(B)** alternative representation wherein the option of not mining is represented through a separate node and **(C)** simplified representation where *M* ESPs are aggregated into a single route that represents the choice of mining and the option of not mining is represented by a second route.

A congestion game without player specific payoff functions is guaranteed to admit a standard potential and a pure equilibrium (see Monderer and Shapley, 1996). A game that does not admit a standard potential may still admit an ordinal potential. A game with an ordinal potential can have any finite subset of actions available to a player, still admitting a pure equilibrium.

Milchtaich (1996) proves the existence of a pure Nash equilibrium given user dependent costs in crowding games. In this paper, we are interested in user dependent strategy sets. Nonetheless, one can show an equivalence between user dependent costs and user dependent strategy sets, and henceforth we use interchangeably the two notions.

4. ESP CONNECTION GAME

In this section, we introduce the ESP connection game and analyze some properties of its equilibria. We consider the special case where we have only one cryptocurrency, which we denote by $\star$.

4.1. Coarse Grained Adjustment of Mining Difficulty

In this section we consider the coarse grained adjustment of mining complexity under a scenario wherein there is a single cryptocurrency. First, we consider the simplest setting wherein all miners are symmetric (section 4.1.1). Then, we relax our assumptions and show conditions under which the mapping between ESP connection games and potential games still holds, posing a conjecture on the extent at which the assumptions can be further relaxed (section 4.1.2).

4.1.1. To Mine or Not to Mine? A Simple Congestion Game Accounting for Symmetric ESPs

In this section, our goal is to illustrate the relationship between the games considered in this work and congestion games. To that aim, we assume ESPs are symmetric, i.e., $\mu_{\star,m} = \mu_\star$ and $\gamma_{\star,m} = \gamma_\star$ for all m. Although the scenario is very simple, it already serves to appreciate the sort of analysis considered in the remainder of this work. In the following section, we relax those assumptions.

Let $\ell_\star$ be the number of miners that decide to associate to an ESP,

$$\ell_\star = \sum_{m=1}^{M} \sum_{i \in \mathcal{N}} 1_{s_i^* = (\star,m)}. \tag{9}$$

Then, $N - \ell_\star$ is the number of users that decide not to mine.

When all $\mu_{\star,m}$ are equal we denote them by $\mu_\star$. Then, Equation (6) reduces to:

$$p_\star(\ell_\star) = 1_{\ell_\star > 0} \frac{1 - \exp(-T\mu_\star \ell_\star)}{\ell_\star}, \tag{10}$$

where $p_\star$ is the probability that a user that decides to connect to an ESP is the first to solve the puzzle. The utility for a miner associating to ESP m is given by (8).

THEOREM 1 (no player-specific strategies). *If for all i and j, $\mathcal{S}_i = \mathcal{S}_j$, the Nash equilibrium is given by the solution of the following optimization problem,*

$$argmax_{\ell_\star} \sum_{l=1}^{\ell_\star} (p_\star(l) - \gamma_\star) \tag{11}$$

$$subject\ to:\ \ell_\star \leq N, \quad \ell_\star \geq 0, \tag{12}$$

where $\ell_\star$ solution of (11)-(12) is the number of users that decide to mine. Equation (11) is the game potential function. The optimization problem (11)-(12) is equivalent to a bin-packing problem with concave costs. Therefore existence and uniqueness is guaranteed.

PROOF: This is a congestion game in the sense of Rosenthal (1973) and therefore has a potential. Indeed, in this game each player can decide to associate or not with an ESP. Thus all connections to the M ESPs can be aggregated to a single route that represents the choice of mining and the option of not associating represents the second route (see **Figure 3**). □

THEOREM 2 (player-specific strategies). *If S_i depends on the identity of user i, the game may not admit a standard potential, but still admits pure Nash equilibria.*

PROOF: The game is a crowding game, and the result follows from Milchtaich (1996, 1998). □

4.1.2. Existence of Equilibrium Under General Conditions

Next, our goal is to illustrate results on the existence of equilibria. To that aim, we generalize the conditions considered in the previous section, allowing for multiple non-symmetric ESPs, and indicate how the considered game still relates to congestion games.

THEOREM 3 (existence). *If $\gamma_{\star,m} = \gamma_{\star,m'}$ for all m' and m, $\mu_{\star,m} \neq \mu_{\star,m'}$ for all m and m' such that $m \neq m'$, and $S_i = S_j$ then:*

1. *a pure Nash equilibrium exists*
2. *miners will only rely on ESP m^*, with $m^* = \max\{m : \mu_{\star,m} \geq \mu_{\star,m'} \; \forall m'\}$ and*
3. *the Nash equilibrium is given by the solution of the following optimization problem,*

$$\text{argmax}_{\ell_{\star,m^*}} \sum_{l=1}^{\ell_{\star,m^*}} (p_{\star,m^*}(l) - \gamma) \qquad (13)$$

$$\text{subject to:} \quad \ell_{\star,m^*} \leq N, \quad \ell_{\star,m^*} \geq 0. \qquad (14)$$

PROOF: Let $l'_{\star,m}$ be the number of users, except one, mining the unique cryptocurrency using ESP m. $l'_{\star,m}$ needs not to be at Nash Equilibrium. The player that did not take his decision is facing the following optimization problem:

$$\max\left\{ \max_m \left\{ \frac{\mu_{\star,m}(1 - \exp(-T(\mu_{\star,m} + \sum_{m'} l'_{\star,m'}\mu_{\star,m'})))}{\mu_{\star,m} + \sum_{m'} l'_{\star,m'}\mu_{\star,m'}} \right\}, \gamma \right\}. \qquad (15)$$

Let us define the function f such that:

$$f(x) = \frac{x(1 - \exp(-T(x + \sum_{m'} l'_{\star,m'}\mu_{\star,m'})))}{x + \sum_{m'} l'_{\star,m'}\mu_{\star,m'}}. \qquad (16)$$

$f(x)$ is strictly increasing for $x > 0$. Therefore, for all $\sum_{m'} l'_{\star,m'}\mu_{\star,m'}$:

$$\max_x f(x) = f(\mu_{\star,m^*})$$

$$= \frac{\mu_{\star,m^*}(1 - \exp(-T(\mu_{\star,m^*} + \sum_{m'} l'_{\star,m'}\mu_{\star,m'})))}{\mu_{\star,m^*} + \sum_{m'} l'_{\star,m'}\mu_{\star,m'}}, \qquad (17)$$

with $m^* = \max\{m : \mu_{\star,m} \geq \mu_{\star,m'} \; \forall m'\}$. It follows that the utility of a player at equilibrium will be:

$$\max\left\{ \frac{\mu_{\star,m^*}(1 - \exp(-T(\mu_{\star,m^*} + l'_{\star,m^*}\mu_{\star,m^*})))}{\mu_{\star,m^*} + l'_{\star,m^*}\mu_{\star,m^*}} - \gamma, 0 \right\}. \qquad (18)$$

To summarize, the best-response of any player to any $l'_{\star,m}$ is such that miners will only rely on ESP m^*, with $m^* = \max\{m : \mu_{\star,m} \geq \mu_{\star,m'} \; \forall m'\}$. Moreover, let us assume that each player is now only focusing on the ESP m^*. In this case, the ESP connection game is a congestion game, in the sense of Rosenthal (1973). The rest of the proof follows as a special case of Theorem 1. In a network congestion game, the time it takes to travel (expected number of trials to be the first to mine) on any road (ESP) is an increasing (payoff decreasing) function of the number of people (miners) selecting that road (ESP). Then, by the theorem 1 from Rosenthal (1973) there exists pure Nash equilibrium. □

4.1.2.1. Illustrative examples

Consider 4 miners and 3 ESPs, $N = 4$ and $M = 3$. Let $\mu_{\star,m}$ equal 0, 0.2, 0.4 and 0.6 for $m = 0, 1, 2, 3$, respectively. Let $T = 1$ and $\gamma = 0.3$. Then, the game admits 6 pure equilibria, where $6 = \binom{4}{2}$. In each equilibrium, two of the players adopt strategy 0 and the other two players adopt strategy 3. The players adopting strategies 3 and 0 have corresponding utilities of 0.049 and 0, respectively, where $m^* = 3$. In addition, $p_{\star,m^*}(l) - \gamma$ equals 0.15, 0.049, –0.02, and –0.09 for $l = 1, 2, 3, 4$, indicating that $\sum_{m=1}^{\ell_{\star,m^*}} (p_{\star,m^*}(l) - \gamma)$ is maximized for $\ell_{\star,m^*} = 2$ which is in agreement with the fact that 2 users are active in equilibrium [see (13)].

Consider now the following additional example, which is out of the scope of Theorem 3, wherein 4 miners compete over 3 ESPs, $N = 4$ and $M = 3$. Let $\mu_{\star,m}$ equal 0, 0.24, 0.45, and 0.6 for $m = 0, 1, 2, 3$, respectively. Let $\gamma_{\star,m}$ equal 0, 0.147, 0.26, and 0.46 for $m = 0, 1, 2, 3$, respectively. Note that Theorem 3 assumes $\gamma_{\star,m}$ to be the same across all ESPs, which is not the case in the current setup. This game admits 19 pure Nash equilibria: 12 equilibria correspond to permutations of the strategy profile $(0,1,1,2)$, 6 equilibria correspond to permutations of the strategy profile $(0,0,2,2)$ and the last equilibrium equals $(1,1,1,1)$. Note that strategy 3, which corresponds to the highest rate, does not appear in any of the equilibrium profiles. This is in stark contrast with the previous setup, wherein the strategy with highest rate was the only candidate to be an element in the equilibrium. In addition, note that users adopting different strictly positive rates may together comprise the equilibrium. This motivates the following conjecture.

CONJECTURE 4. *If (i) $\mu_{\star,m} \neq \mu_{\star,m'}$ whenever $m \neq m'$, (ii) $\gamma_{\star,m} \geq \gamma_{\star,m'}$ implies that $\mu_{\star,m} \geq \mu_{\star,m'}$, and (iii) $S_i = S_j$ then:*

Section	ESPs	Symme-tric ESPs	Users can decide not to mine	Puzzle complexity adjustment	Multiple ESPs	Multiple crypto	Atomic miners	Conti-nuous actions
4.1.1	One or more	Yes	Yes	Coarse	Yes	No	Yes	No
4.1.2	One or more	No	Yes	Coarse	Yes	No	Yes	No
4.2	One or more	No	Yes	Fine	Yes	No	Yes	No
5	One	No	Yes	Coarse	No	Yes	Yes	No
6	One or more	No	Yes	Fine or coarse	Yes	Yes	No	No
7	One or more	No	Yes	Fine	Yes	No	Yes	Yes

1. *a pure Nash equilibrium exists*
2. *at equilibrium, across the set of active miners there will be connections to at most two ESPs, denoted by m' and m'' and*
3. *when $m' \neq m''$, the Nash equilibrium is given by the solution to the following optimization problem,*

$$argmax_{(\ell_{\star,m'},\ell_{\star,m''})} \sum_{l'=1}^{\ell_{\star,m'}} \sum_{l''=1}^{\ell_{\star,m''}} p_{\star,m'}(\ell) - \gamma_{\star,m'} + p_{\star,m''}(\ell) - \gamma_{\star,m''} \quad (19)$$

$$subject\ to: \quad \ell_{\star,m'} + \ell_{\star,m''} \leq N, \quad \ell_{\star,m'} \geq 1, \quad \ell_{\star,m''} \geq 1, (20)$$

where $\ell = (N - l' - l'', l', l'')$ denotes a strategy profile wherein $N - l' - l''$ miners are inactive, l' miners adopt ESP m' and l'' miners adopt ESP m''.

To illustrate the last part of the conjecture above, consider again the previous numerical result. Let $m' = 1$ and $m'' = 2$, and let the lumped strategy profile be a vector (n_0, n_1, n_2, n_3) which corresponds to a profile wherein n_i users adopt ESP i. Then, the lumped strategy profiles $(2, 1, 1, 0), (1, 1, 2, 0), (0, 1, 3, 0), (1, 2, 1, 0), (0, 2, 2, 0),$ and $(0, 3, 1, 0)$ evaluate the objective function (19) to 0.0914, 0.0961, 0.0345, 0.1336, 0.1055 and 0.1333 indicating that the equilibrium with strategy profile $(0, 1, 1, 2)$ found in the previous paragraph, which corresponds to the lumped strategy profile $(1, 2, 1, 0)$, is in agreement with the conjecture. In section 6.2 we prove a result inspired by the conjecture above, under the setup of non-atomic games.

4.2. Fine Grained Adjustment of Mining Difficulty

Next, we consider the fine grained adjustment of mining difficulty. To that aim, we assume $q_k = 1$, i.e., we do not include the exponential term in the definition of q_k (Equation 4). Recall that the exponential term captures the probability that the puzzle is not solved by time T, which we assume to be negligible (i.e., much smaller than 1), for large enough T (see **Table 3**).

4.2.1. Best Response Dynamics and Convergence Under *M*-concave Potential

Consider any better response learning scheme. In particular, the best response learning scheme is one of such schemes. Note that for a player to update its response it only needs to have access to the total load across all ESPs. Note also that for a player to

compute its response, without previous knowledge of historical responses, it needs to know the overall load generated by all the miners over each ESP.

Since the utility is concave we may expect the potential to converge to a global optimum in finite time under any standard best response strategy or better response policy. However, the concave function is defined only on integers, which is not a convex compact set. In this case, some modifications of the definition of concavity and convex sets are needed in order to guarantee that any local extremal point of the function is a global extremal point. These modifications are called *M*-concavity and *M*-convex set, respectively (see Lebeau et al., 2019 and references therein). Then, the key result of this section follows.

THEOREM 5. *The ESP competition game under fine grained adjustment of mining difficulty admits a potential.*

PROOF: It is shown in Lebeau et al. (2019) that the social medium selection game is a congestion game. We have already shown that the ESP selection game is a congestion game and that there exists a potential. The potential function for the social medium selection game is also a potential function for the ESP selection game. Moreover, Theorem 2 from Lebeau et al. (2019) shows that the potentials are *M*-concave functions defined over an *M*-convex set. □

5. CRYPTOCURRENCY ASSOCIATION GAME

In this section, we introduce the multiple cryptocurrencies game and derive structural properties of the associated set of equilibria. As in section 4.1, we assume a coarse grained adjustment of difficulty level. In addition, we assume that there are K cryptocurrencies. We consider a single ESP, and drop subscript m from all variables.

For a given load vector ℓ, the time it takes till the fastest puzzle to be solved is exponentially distributed with expectation $1/(\mu_k \ell_k)$. Thus, the probability that a miner is the first to solve the puzzle is:

$$p_k(\ell_k) = \frac{1 - \exp(-T\mu_k\ell_k)}{\ell_k}. \quad (21)$$

Note that $p_k = 0$ if $\ell_k = 0$ (recall that we assume $0/0 = 0$ throughout this paper). The utility of a tagged miner to mine

a cryptocurrency k when there are ℓ_k miners associated with the same cryptocurrency (including the tagged miner) is given by (8), where:

$$U_k(\ell_k) = p_k - \gamma_k. \tag{22}$$

We add to it the constraint that a miner does not participate in solving the puzzle if its utility is negative. In that case the equilibrium is characterized by the condition $\sum_k \ell_k^* \leq N$, with $\ell_k^* \geq 0$, for $k = 1, \ldots, K$. This game is referred to as an *elastic game*. Alternatively, we can consider an additional cryptocurrency, indexed by 0, with corresponding utility being constant equal to 0. Then,

$$U_k(\ell_k) = \begin{cases} p_k - \gamma_k, & \text{if } k > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

This game is referred to as the *non-elastic game* equivalent to the elastic game above (**Figure 3** illustrates the elastic and non-elastic instances of the ESP connection game). If the equilibrium vector ℓ^* saturates the constraint in the elastic game ($\sum_k \ell_k^* = N, \ell_k^* \geq 0, k = 1, ..., K$) or, alternatively, if $\ell_0^* = 0$ in the non-elastic game, then for each k for which $\ell_k^* > 0$, and each k', $U_k(\ell_k^* - 1) \geq U_{k'}(\ell_{k'}^* + 1)$.

Similar theorems as those presented in the previous section establishing the existence of pure Nash equilibria and characterizing the equilibria still hold under the blockchain association game. The statements of the theorems and the proofs are similar to those in the previous section, and are omitted for conciseness. Recall that in Theorem 1 for all i and j, $\mathcal{S}_i = \mathcal{S}_j$. Then, in this case the number of miners associated to each cryptocurrency $\ell_k^* = \sum_{i \in \mathcal{N}} 1_{s_i^* = k}$ is now the solution of the following optimization problem,

$$\operatorname{argmax}_{\ell} \sum_{k \in \mathcal{K}} \sum_{l=1}^{\ell_k} (p_k(l) - \gamma_k) \tag{24}$$

$$\text{subject to: } \sum_{k \in \mathcal{K}} \ell_k \leq N, \quad \ell_k \geq 0. \tag{25}$$

Theorem 1 holds replacing (11)-(12) by the equations above.

6. NON-ATOMIC MINERS FOR THE MULTIPLE ESP'S AND MULTIPLE CRYPTOCURRENCIES GAME

We will now study a mean-field approximation of the cryptocurrencies game. This approximation is instrumental to provide additional insight concerning the multiple ESPs/multiple cryptocurrencies game.

6.1. Wardrop Equilibrium Basics
6.1.1. Problem Formulation
We assume that the miners are non-atomic. In this case, for a given load vector ℓ, a miner solves the following optimization problem:

$$\max \left\{ \max_{k,m} \left\{ \mathcal{U}_{k,m}(\ell) \right\}, 0 \right\}, \tag{26}$$

where,

$$\mathcal{U}_{k,m}(\ell) = \frac{\mu_{k,m}}{\sum_{m'} \mu_{k,m'} \ell_{k,m'}} q_k \left(\sum_{m'} \mu_{k,m'} \ell_{k,m'} \right) - \gamma_{k,m}. \tag{27}$$

If the miners were atomic, the mining decision of a given miner (assuming that the rest of the miners will not modify their strategies) would impact the load vector ℓ. However, under the assumption that the miners are non-atomic, the deviation of one miner will not modify the load vector ℓ and therefore the miner's best-response to a given load ℓ is given by the arg max of (26).

This assumption is valid in two cases: (1) when miners do not realize that their mining decisions will impact utilities $\mathcal{U}_{k,m}(\ell)$ for all k and all m or (2) when the number of miners is large and $\gamma_{k,m}$ and $\mu_{k,m}$ are small. Haurie and Marcotte (1985) were the first to prove that the non-atomic equilibrium (also known as Wardrop equilibrium) is the limit of many players of Nash equilibrium, under strict diagonal concavity conditions established by Rosen (1965). Altman et al. (2011, 2019) showed that for a game equivalent to the one considered in the section, under the fine grained adjustment of puzzle complexity, i.e., when $q_k = 1$, the strict diagonal concavity conditions hold. In this case, the assumption that miners do not account for the impact of their decisions on the actions of the others is referred to as a *mean-field* approximation.

6.1.2. Equilibrium Characterization
A non-atomic equilibrium load vector ℓ^* satisfies:

$$\mathcal{U}_{k,m}(\ell^\star) = \mathcal{U}_{k'',m''}(\ell^\star), \tag{28a}$$
$$\quad \text{if } \ell_{k,m}^* > 0, \ \ell_{k'',m''}^* > 0, \ \forall m, m'', k, k'',$$
$$\mathcal{U}_{k,m}(\ell^\star) \geq \mathcal{U}_{k'',m''}(\ell^\star), \tag{28b}$$
$$\quad \text{if } \ell_{k,m}^* > 0, \ell_{k'',m''}^* = 0, \ \forall m, m'', k, k'',$$
$$\mathcal{U}_{k,m}(\ell^\star) \leq 0, \text{ if } \ell_{k,m}^* = 0, \ \forall m, k, \tag{28c}$$
$$\sum_{k,m} \ell_{k,m}^\star \leq N. \tag{28d}$$

Before studying the properties of such equilibria, we provide some intuition for the rationale behind Equations (28a)–(28d). For an in depth analysis of non-atomic equilibria, we refer the reader to Roughgarden (2005) and Wan (2012).

Can a miner be interested in deviating from ℓ^*? To answer that question, let $S_A(\ell^*)$ be the set of all pairs (k, m) corresponding to active miners under equilibrium ℓ^*,

$$S_A(\ell^*) := \{(k, m) \mid \ell_{k,m}^* > 0, \ \ell_{k,m}^* \text{ solution of } (28a) - (28d)\}. \tag{29}$$

Equation (28a) implies that for all pairs in $S_A(\ell^*)$ the utility is the same. Therefore, if a miner is interested in deviating from ℓ^*, arg max of (26) must be a pair $(k'', m'') \notin S_A(\ell^*)$. However, an investment, say, in $(k'', m'') \notin S_A(\ell^*)$, must be suboptimal

according to (28b). Therefore, a miner will always choose $(k, m) \in S_A(\ell^*)$, which naturally implies that ℓ^* satisfying (28a)-(28d) is an equilibrium strategy.

By studying (28a)-(28d), we will give some basic insights about the structure of any equilibrium. For now, we will assume that an equilibrium ℓ^* always exists. Later in this section, we will study the existence and uniqueness of ℓ^*.

6.2. Miners Invest at Maximum in Two ESPs for a Given Cryptocurrency

Next, we show that under equilibrium miners invest at maximum in two ESPs for a given cryptocurrency. To that aim, we start with the following definition.

DEFINITION 6. Two pairs of ESPs (m, m') and (m'', m'''), such that $\mu_{k,m} < \mu_{k,m'}$ and $\mu_{k,m''} < \mu_{k,m'''}$, are said to be colinear with respect to cryptocurrency k if:

$$\frac{\mu_{k,m} - \mu_{k,m'}}{\gamma_{k,m} - \gamma_{k,m'}} = \frac{\mu_{k,m''} - \mu_{k,m'''}}{\gamma_{k,m''} - \gamma_{k,m'''}}. \tag{30}$$

Intuitively, two pairs of ESPs are colinear when their difference in capacities and costs can be linearly aligned. The following theorem establishes our main structural result for this section.

THEOREM 7. If no two pairs of ESPs are colinear with respect to cryptocurrency k, then at equilibrium miners invest at maximum in two ESPs for that cryptocurrency.

PROOF: The proof of this statement is based on a contradiction argument. Let us assume, without loss of generality, that $\ell_{k,m}, \ell_{k,m'}, \ell_{k,m''} > 0$. Then we have the following system:

$$\frac{\mu_{k,m} - \mu_{k,m'}}{\sum_n \mu_{k,n} \ell_{k,n}^*} q_k (\sum_n \mu_{k,n} \ell_{k,n}^*) = \gamma_{k,m} - \gamma_{k,m'} \tag{31}$$

$$\frac{\mu_{k,m} - \mu_{k,m''}}{\sum_n \mu_{k,n} \ell_{k,n}^*} q_k (\sum_{m'} \mu_{k,n} \ell_{k,n}^*) = \gamma_{k,m} - \gamma_{k,m''} \tag{32}$$

which leads to the following contradiction $\frac{\mu_{k,m} - \mu_{k,m''}}{\gamma_{k,m} - \gamma_{k,m''}} = \frac{\mu_{k,m} - \mu_{k,m'}}{\gamma_{k,m} - \gamma_{k',m'}}$, concluding the proof. □

6.2.1. When Will Miners Invest in Only One ESP for a Given Cryptocurrency?

Next, we further establish sufficient conditions for miners to invest in only one ESP for a given cryptocurrency.

THEOREM 8. If, for a given cryptocurrency, the costs are the same across all ESPs ($\gamma_{k,m} = \gamma_{k,m'}$ for all m, m'), and service rates are different from each other ($\mu_{k,m} \neq \mu_{k,m'}$ for any $m \neq m'$), then: (1) only one ESP will be used, and (2) the ESP that will be used will be the one with the highest service rate.

PROOF: Let us assume that for a given cryptocurrency, say k, the cost for using each ESP is the same ($\gamma_{k,m} = \gamma_{k,m'}$ for all m, m'), and the service rate associated to each ESP is different ($\mu_{k,m} \neq \mu_{k,m'}$ for all m, m'). Let us assume that there exists at equilibrium ℓ^* two elements $\ell_{k,m}$ and $\ell_{k,m'}$ such that $\ell_{k,m}^* > 0$

and $\ell_{k,m'}^* > 0$, for a pair of ESPs (m, m'), with $m \neq m'$. Then, according to (28a) $\mu_{k,m} = \mu_{k,m'}$, which by a contradiction argument implies (1). Moreover, if miners invest only in one ESP, then according to (28b), that ESP will be the one with the highest $\mu_{k,m}$, establishing (2). □

Note that Theorems 7 and 8 are the mean field results equivalent to Conjecture 4 and Theorem 3. It is often the case that structural results are easier to be derived under the mean field approximation, as further illustrated through the following additional structural results.

6.3. Blockchain Mining Collapse

Next, we characterize conditions under which the mining costs preclude miners from investing their computational resources into the mining game.

DEFINITION 9. A given cryptocurrency k dies under equilibrium ℓ^* if $\ell_{k,m}^* = 0$ for $1 \leq m \leq M$.

THEOREM 10. If no two pairs of ESPs are colinear with respect to cryptocurrency k, and:

$$\max_{m : 1 \leq m \leq M} \left\{ \frac{q_k (\mu_{k,m} N)}{N} - \gamma_{k,m} \right\} < 0 \tag{33}$$

then cryptocurrency k dies under all equilibria.

PROOF: Equations (28a)-(28d) imply that if:

$$\mathcal{U}_{k,m}(\ell) < 0, \text{ for all } m \in \{1, \ldots, M\} \text{ and } \ell \text{ such that } \sum_m \ell_{k,m} \leq N \tag{34}$$

then cryptocurrency k dies. Condition (34) is satisfied if:

$$\max_m \left\{ \max_{\sum_{m'} \ell_{k,m'} \leq N} \mathcal{U}_{k,m}(\ell) \right\} < 0. \tag{35}$$

Next, we further characterize the solution of the fractional pseudo-concave optimization problem:

$$\max_{\sum_{m'} \ell_{k,m'} \leq N} \mathcal{U}_{k,m}(\ell) \tag{36}$$

We denote by $\ell^\star(k, m')$ the optimal load vector for the previously defined optimization problem, for a given pair (k, m'), where $1 \leq m' \leq M$. The first order optimality conditions that must be satisfied by the solution $\ell^\star(k, m')$ of the problem above entail the existence of $\lambda(k, m') \in \mathbb{R}$ such that:

$$\mu_{k,m'} q_k \left(\sum_{m''} \mu_{k,m''} \ell_{k,m''}^\star(k, m') \right)$$
$$= \lambda(k, m'), \text{ if } \ell_{k,m''}^\star(k, m') > 0, \tag{37}$$

$$\mu_{k,m} q_k \left(\sum_{m''} \mu_{k,m''} \ell_{k,m''}^\star(k, m') \right)$$
$$\leq \lambda(k, m'), \text{ if } \ell_{k,m''}^\star(k, m') = 0. \tag{38}$$

Equations (37) and (38) together with the fact that no two pairs of ESPs are colinear with respect to cryptocurrency k imply that the optimal load is given by,

$$\ell^{\star}_{k,m''}(k,m') = \begin{cases} N, & \text{if } m'' = m' \\ 0, & \text{otherwise.} \end{cases} \quad (39)$$

Therefore, $\mathcal{U}_{k,m'}(\ell^{\star}(k,m')) = \frac{q_k(\mu_{k,m'}N)}{N} - \gamma_{k,m'}$, which together with (35) concludes the proof. □

Then, we consider the most extreme scenario, wherein miners have no incentives to mine any of the existing cryptocurrencies.

DEFINITION 11. Blockchain mining collapses if there is an equilibrium under which all cryptocurrencies die.

COROLLARY 12. *If, for each k, there are no two pairs of ESPs that are colinear with respect to cryptocurrency k, and if for all m and k,*

$$1 - \exp(-T_k\mu_{k,m}N)) - N\gamma_{k,m} < 0, \quad (40)$$

then blockchain mining collapses. Note that there exists an $\overline{N}$ such that for every $N > \overline{N}$ the condition above is satisfied. Moreover, if $N\gamma_{k,m} > 1$ for all k and m, the condition above also holds.

PROOF: The proof follows directly from Theorem 10. Indeed, blockchain mining collapses if:

$$\max_m \left\{ \frac{q_k(\mu_{k,m}N)}{N} - \gamma_{k,m} \right\} < 0, \forall k. \quad (41)$$

The condition above is equivalent to:

$$\frac{q_k(\mu_{k,m}N)}{N} - \gamma_{k,m} < 0, \forall(m,k), \quad (42)$$

which concludes the proof. □

6.4. Existence and Uniqueness of Equilibrium

Concerning the existence and the uniqueness of the equilibrium, we will restrict to the scenario wherein for each cryptocurrency k, the cost across all ESPs are the same ($\gamma_{k,m} = \gamma_{k,m'}$ for all m, m'), and the service rate associated to each ESP is different ($\mu_{k,m} \neq \mu_{k,m'}$ for all m, m'). As shown in Theorem 8, under equilibrium, for each cryptocurrency, at most one ESP will be used to actively mine. Let $m(k) := \max_{m'} \mu_{k,m'}$. ESP $m(k)$ is the only candidate to be actively used for mining cryptocurrency k. Therefore, the equilibrium conditions (28a)-(28d) simplify to:

$$\mathcal{U}_{k,m(k)}(\ell^{\star}) = \mathcal{U}_{k'',m(k'')}(\ell^{\star}), \quad (43a)$$
$$\quad \text{if } \ell^*_{k,m(k)} > 0, \ \ell^*_{k'',m(k'')} > 0, \ \forall k, k'',$$

$$\mathcal{U}_{k,m(k)}(\ell^{\star}) \geq \mathcal{U}_{k'',m(k'')}(\ell^{\star}), \quad (43b)$$
$$\quad \text{if } \ell^*_{k,m(k)} > 0, \ \ell^*_{k'',m(k'')} = 0, \ \forall k, k'',$$

$$\mathcal{U}_{k,m(k)}(\ell^{\star}) \leq 0, \quad (43c)$$
$$\quad \text{if } \ell^*_{k,m(k)} = 0, \ \forall k,$$

$$\sum_k \ell^{\star}_{k,m(k)} \leq N. \quad (43d)$$

THEOREM 13. *The non-atomic game under symmetric costs considered in this section admits at most one interior equilibrium, which is the solution to the following optimization problem:*

$$\operatorname{argmax} \sum_{k=1}^{K} \int_{\epsilon}^{\ell_{k,m(k)}} \frac{q_k(\mu_{k,m(k)}x)}{x} dx - \gamma_{k'} \ell_{k,m(k)}, \quad (44)$$

$$\text{subject to } \sum_k \ell_{k,m(k)} \leq N, \quad \ell_{k,m(k)} \geq \epsilon \quad (45)$$

PROOF: First, note that if there exists an interior solution to the optimization problem (44)-(45), i.e., if each load is strictly greater than ϵ, then the first-order optimality conditions of the posed optimization problem are given by (43a)-(43d), which implies a one-to-one correspondence between the solution to the optimization problem and an equilibrium of the non-atomic game. In addition, note that for all k and m, $\frac{q_k(\mu_{k,m(k)}\ell_{k,m(k)})}{\ell_{k,m(k)}}$ is a decreasing function in $\ell_{k,m(k)} > 0$. Therefore the function $\int_{\epsilon}^{\ell_{k,m(k)}} \frac{q_k(\mu_{k,m(k)}x)}{x} dx - \gamma_{k'} \ell_{k,m(k)}$ is strictly concave and the optimization problem posed above has a unique solution, as all the functions are strictly concave. □

7. PARALLEL COMPUTATIONS: AUCTIONS AND CONTINUOUS ACTIONS

The models studied so far assumed that a puzzle to be solved by a miner is sent entirely to a single ESP both in the context of competition over ESPs (section 4) as well as for the competition over cryptocurrencies (section 5). In this section we consider a game in which each miner can decide how much to bid for the computation power proposed by the ESP. The load on an ESP need not be a multiple of its service rate anymore.

Assumptions In this section we assume $q_k = 1$. This corresponds to a fine grained adjustment of puzzle complexity (see section 2.1 and **Table 1**). In addition, we consider a one-to-one correspondence between miners and ESPs and a single cryptocurrency, i.e., $K = 1$. Then, $\ell_{\star,m} = 1$ for $m = 1, \dots, M$.

Let x_m denote the value bid by the miner corresponding to ESP m. We have a minimum constraint $x_m \geq \epsilon$ for all m. We also assume that the service rate from ESP m requested by miner m, $\mu_{\star,m}$, equals the value bid by miner m, x_m, i.e., $\mu_{\star,m} = x_m$. Then, (2) reduces to:

$$\eta = \sum_{j=1}^{M} x_j. \quad (46)$$

7.1. Basic Model

The probability that miner m is the first to solve the puzzle is:

$$P_m = \frac{x_m}{\eta} = \frac{x_m}{\sum_{j=1}^{M} x_j}, \quad (47)$$

which is the miner *expected gain* that can be contrasted against (6). The total cost for miner m is $x_m\gamma$, where γ is a

constant. The utility for player m is thus:

$$U_m(x) = \frac{x_m}{\sum_{j=1}^{M} x_j} - x_m \gamma. \qquad (48)$$

The utility above gives rise to the following UNCONSTRAINED GAME,

$$\text{UNCONSTRAINED GAME: } \max_{x_m} \frac{x_m}{\sum_{i=1}^{M} x_i} - x_m \gamma \qquad (49)$$

The main result of this section establishes the uniqueness of the Nash equilibrium of the UNCONSTRAINED GAME.

THEOREM 14 (continuous actions). *(i) For any strictly positive value of γ, the above game has a unique Nash equilibrium and (ii) U_m is concave in x_m.*

PROOF: This was established in Altman et al. (2016) using a modification of the diagonal strict concavity property.   □

The game presented above was introduced and studied in Dimitri (2017). In what follows, we extend the results from Dimitri (2017) to account for physical constraints on the resources consumed by the population of miners.

7.2. Normalized Equilibrium: Physical Bounds on Resources and Shadow Prices

The games we have seen so far involved orthogonal constraints. By that we mean that the actions that a miner can use do not depend on the actions of other miners. We next introduce a capacity constraint. Formally, for some constant V which bounds the total service rate from all ESPs, we introduce the following game with capacity constraints. For each player m,

$$\text{CONSTRAINED GAME: } \max_{x_m} \frac{x_m}{\sum_{i=1}^{M} x_i} \qquad (50)$$

$$\sum_{j=1}^{M} x_j \leq V \qquad (51)$$

Note that in the game above we assume that each player maximizes the probability of being the first to successfully solve the puzzle, P_m, under constraints on the total amount bid by all players. Recall that the amount bid by a player is proportional to the amount of resources invested by that player to mine.

Capacity constraints may represent physical bounds on resources, such as bounded power, or resources that are bounded by regulation. For example, legislation may impose bounds on the power consumption. With the additional capacity constraints, the Nash equilibrium is no more unique and there may in fact be an infinite number of equilibria. We call this the *game with capacity constraints*.

Let y be an equilibrium of the CONSTRAINED GAME and let $y_{[-m]}$ denote the action vectors of all miners other than m. Note that for each m, U_m is concave in y_m. Then, by the KKT theorem,

there is a Lagrange multiplier $\lambda_m(y_{[-m]})$ such that y_m maximizes the Lagrangian:

$$\mathcal{L}_m(y_m) = \frac{y_m}{\sum_{j=1}^{M} y_j} - \lambda_m(y_{[-m]}) \left(\sum_{j=1}^{M} y_j - V \right) \qquad (52)$$

and

$$\lambda_m(y_{[-m]}) \left(\sum_{j=1}^{M} y_j - V \right) = 0. \qquad (53)$$

The last condition is referred to as *complementarity property*. We call the game with utilities given by Lagrangians $\mathcal{L}_m$ as the *relaxed game* or *Lagrangian game*.

GENERAL RELAXED GAME:

$$\max_{x_m} \quad \mathcal{L}_m(x_m) = \frac{x_m}{x_m + \sum_{j=1, j \neq m}^{M} y_j}$$

$$-\lambda_m(y_{[-m]}) \left(x_m + \sum_{j=1, j \neq m}^{M} y_j - V \right) \qquad (54)$$

A simplified version of the GENERAL RELAXED GAME will be instrumental in the upcoming section to prove properties about the CONSTRAINED GAME.

7.2.1. Shadow Prices and Normalized Equilibrium

The Lagrange multipliers can be interpreted as shadow prices: if a price is set on miner m such that when other players are at equilibrium, the miner pays $y_m \lambda_m(y_{[-m]})$ for its use of cryptocurrency, then y is an equilibrium in the game with capacity constraints. Yet this pricing is not scalable since for the same use of the resources the price may vary from user to user and it further depends on the chosen equilibrium. For billing purposes one would prefer λ_m not to depend on y nor on m, but to be a constant.

Does there exist a constant Lagrange multiplier λ independent of strategies of the payers and of the identity m of the player, along with an associated equilibrium y for the corresponding relaxed game? If the answer is positive then y is called a *normalized equilibrium* (Rosen, 1965; Ghosh et al., 2015; Altman et al., 2019). Then, λ is the Lagrange multiplier corresponding to the normalized equilibrium.

The CONSTRAINED GAME admits an infinite number of equilibria. Nonetheless, as will be shown in the sequel, it admits a unique normalized equilibrium. To prove that claim, we translate global constraints from the CONSTRAINED GAME into local penalties (associated to the Lagrange multipliers) in a simpler version of the GENERAL RELAXED GAME, referred to as the RELAXED GAME.

RELAXED GAME: $\max_{x_m} \quad \mathcal{L}_m(x_m) = \frac{x_m}{x_m + \sum_{j=1, j \neq m}^{M} y_j}$

$$-\lambda \left(x_m + \sum_{j=1, j \neq m}^{M} y_j - V \right) \qquad (55)$$

Whereas the actions of the players are coupled through hard constraints in the CONSTRAINED GAME, the local penalties (and

corresponding Lagrange multipliers) allow us to decouple the actions of the players in the RELAXED GAME.

DEFINITION 15. A symmetric game is a game wherein the functional dependency of the utility with respect to the actions is the same for all players.

Note that the RELAXED GAME is a symmetric game, whereas the GENERAL RELAXED GAME is not. In the former, the constant λ that appears in the utility function is fixed and given, whereas in the latter it is player-dependent.

Let $\mathcal{E}_0$ and $\mathcal{E}_1$ be the set of equilibria of the GENERAL RELAXED GAME and of the RELAXED GAME, respectively (see **Table 4** and **Figure 4**). As mentioned earlier, $\mathcal{E}_0$ in general contains multiple elements, i.e., the CONSTRAINED GAME admits multiple equilibria. Then, our initial aim was to establish necessary and sufficient conditions for $\mathcal{E}_1$ to be a singleton, i.e., for the CONSTRAINED GAME to admit a single normalized equilibrium. However, we were only able to establish those conditions for a symmetric normalized equilibrium to the CONSTRAINED GAME. For this reason, in the upcoming section we restrict to symmetric equilibria of the RELAXED GAME and the corresponding symmetric normalized equilibria of the CONSTRAINED GAME, and refer to the corresponding set as $\mathcal{E}_2$. We will show that $\mathcal{E}_2$ is a singleton, and we leave the necessary and sufficient conditions for $\mathcal{E}_1$ to be a singleton as subject for future work.

7.2.2. Existence and Uniqueness of Symmetric Normalized Equilibrium

Next, we establish the existence and uniqueness of the normalized equilibrium. We start by showing a condition under which the game admits a symmetric equilibrium.

DEFINITION 16. A symmetric equilibrium $\tilde{y}^\star$ is an equilibrium wherein $y_i = \tilde{y}^\star$ for all i, $i = 1, \ldots, M$.

THEOREM 17 (symmetric equilibrium). *If*

$$\gamma = \frac{M-1}{MV} \tag{56}$$

where V is a constant, fixed and given, then

$$\tilde{y}^\star = \frac{V}{M} = \frac{M-1}{M^2\gamma} \tag{57}$$

is the unique symmetric equilibrium to the UNCONSTRAINED GAME.

PROOF: We replace x_m by $\tilde{y}^\star$ in equation (48) to obtain:

$$U_m(\tilde{y}^\star) = \frac{\tilde{y}^\star}{\tilde{y}^\star + \sum_{j=1,j\neq m}^{M} x_j} - \tilde{y}^\star\gamma. \tag{58}$$

The symmetric equilibrium is obtained by differentiating (58) with respect to $\tilde{y}^\star$ and equating the resulting expression to 0,

$$\frac{1}{\tilde{y}^\star + \sum_{j=1,j\neq m}^{M} x_j} - \frac{\tilde{y}^\star}{\left(\tilde{y}^\star + \sum_{j=1,j\neq m}^{M} x_j\right)^2} - \frac{M-1}{MV} = 0 \tag{59}$$

$$\left(\sum_{j=1,j\neq m}^{M} x_j\right) - \frac{M-1}{MV}\left(\tilde{y}^\star + \sum_{j=1,j\neq m}^{M} x_j\right)^2 = 0 \tag{60}$$

Noting that the same argument holds for all players, we conclude that $\tilde{y}^\star = V/M$ is a symmetric equilibrium. Indeed, setting $x_m = V/M$ for $m = 1, \ldots, M$ equation (60) is satisfied. □

It is worth noting that (57) corresponds to a special case of equation (4) in Dimitri (2017), and the proposition above follows from the main proposition in Dimitri (2017). Indeed, starting from equation (4) in Dimitri (2017) and replacing R, n, $c_{(n)}$ and c_i by 1, M, γM and γ we obtain (57). In what follows, we extend the analysis of Dimitri (2017), which encompasses unconstrained games, to the setup wherein constraints are active.

COROLLARY 18 (normalized equilibrium). *If*

$$\lambda = \frac{M-1}{MV} \tag{61}$$

where V is a constant, fixed and given, determining the system constraints, then:

$$\tilde{y}^\star = \frac{V}{M} = \frac{M-1}{M^2\lambda} \tag{62}$$

is an equilibrium to the RELAXED GAME and a normalized equilibrium to the CONSTRAINED GAME.

PROOF: The proof follows by noting that the equilibrium corresponding to (58) in the proof of Theorem 17 is also an equilibrium corresponding to the utility function (52) of the RELAXED GAME. This is because the utility of the RELAXED GAME can be obtained from (58) replacing γ, $\tilde{y}^\star$ and x_j by λ, y_m and y_j, respectively, for $j \neq m$, and adding a term $-\lambda(\sum_{j\neq m} y_j - V)$. Note that after adding this term, the equilibrium of the original UNCONSTRAINED GAME is also an equilibrium of the modified game since the new utility differs from the previous one by terms that do not depend on $y^\star = y_m$, the action of player m. □

Next, we establish the main result of this section. **Figure 5** summarizes the proof strategy. The proof follows by relating the symmetric equilibrium to the UNCONSTRAINED GAME into an equilibrium to the RELAXED GAME and a normalized equilibrium to the CONSTRAINED GAME.

THEOREM 19 (normalized equilibrium). *There exists a unique symmetric normalized equilibrium to the CONSTRAINED GAME, i.e., the set $\mathcal{E}_2$ is a singleton.*

TABLE 4 | Normalized and symmetric equilibria.

Set of equilibria	Description	Specific description
$\mathcal{E}_0$	General equilibria to general game	General equilibria to GENERAL RELAXED GAME, satisfying complementarity conditions, i.e., general equilibria to CONSTRAINED GAME
$\mathcal{E}_1$	General equilibria to symmetric game	General equilibria to RELAXED GAME, satisfying complementarity conditions, i.e., normalized equilibria to CONSTRAINED GAME
$\mathcal{E}_2$	Symmetric equilibria to symmetric game	Symmetric equilibria to RELAXED GAME, satisfying complementarity conditions, i.e., symmetric normalized equilibria to CONSTRAINED GAME
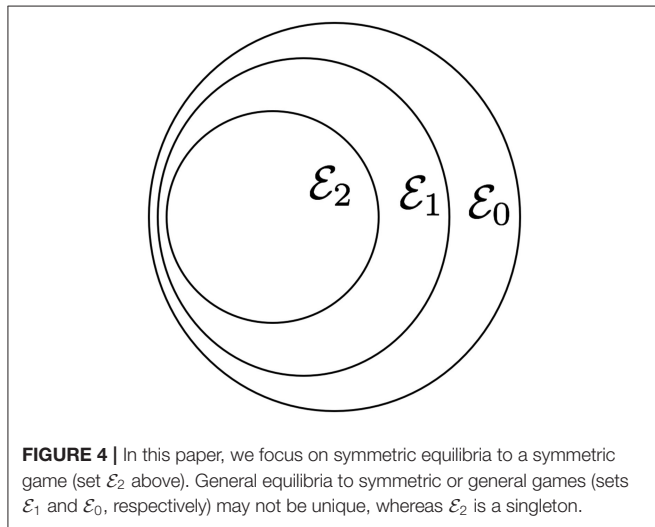
FIGURE 4 | In this paper, we focus on symmetric equilibria to a symmetric game (set $\mathcal{E}_2$ above). General equilibria to symmetric or general games (sets $\mathcal{E}_1$ and $\mathcal{E}_0$, respectively) may not be unique, whereas $\mathcal{E}_2$ is a singleton.

Proof idea. The proof is presented in **Appendix B**, and the proof idea is summarized in **Figure 5**. We know that for any γ there is a unique Nash equilibrium $y(\gamma)$ to the UNCONSTRAINED GAME (Theorem 14). We show that this defines a unique symmetric equilibrium to the RELAXED GAME with Lagrange multiplier $\lambda(\gamma)$. We further show that there is a unique $\gamma^\star$ for which the capacity constraints hold with equality. This implies that $y(\gamma^\star)$ is a normalized equilibrium to the CONSTRAINED GAME where $\lambda(\gamma^\star)$ is the corresponding Lagrange multiplier. □

We have just shown that the symmetric equilibrium to the CONSTRAINED GAME is unique, i.e., $\mathcal{E}_2$ is a singleton. It remains to show the conditions under which the general equilibrium (symmetric or asymmetric) to the CONSTRAINED GAME is unique. *What are the necessary and sufficient conditions under which $\mathcal{E}_1$ is also a singleton?*

The fact that there does not exist asymmetric equilibria to certain class of symmetric games was shown in Orda et al. (1993). If we were able to establish conditions under which the symmetric CONSTRAINED GAME admits only symmetric equilibrium, we would also be able to guarantee uniqueness across general equilibra. However, the conditions of Orda et al. (1993) to show that certain symmetric games admit only symmetric equilibria do not hold in our games. In particular, the sufficient conditions
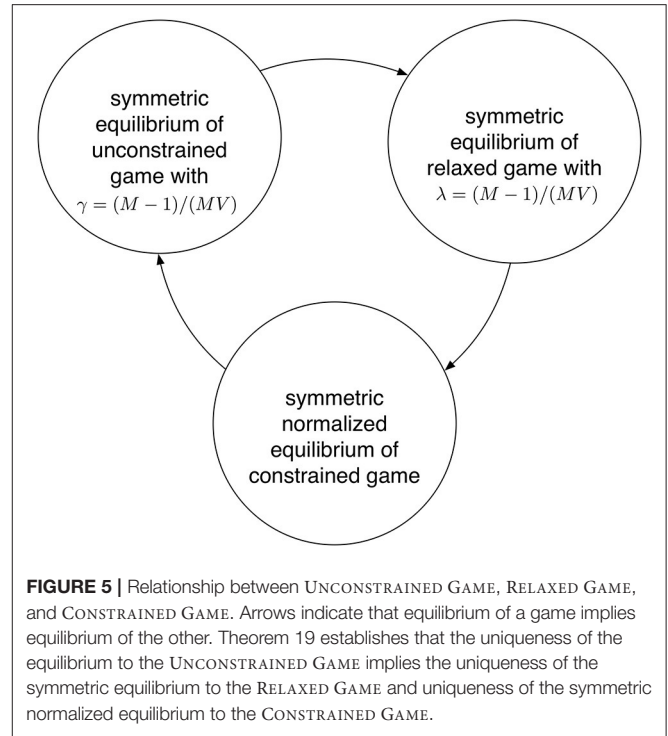

FIGURE 5 | Relationship between UNCONSTRAINED GAME, RELAXED GAME, and CONSTRAINED GAME. Arrows indicate that equilibrium of a game implies equilibrium of the other. Theorem 19 establishes that the uniqueness of the equilibrium to the UNCONSTRAINED GAME implies the uniqueness of the symmetric equilibrium to the RELAXED GAME and uniqueness of the symmetric normalized equilibrium to the CONSTRAINED GAME.

established by Orda et al. (1993) state that the utility must be decreasing in the aggregated actions of all players and in the action of each of the players. In the CONSTRAINED GAME, in contrast, given player j, the utility is decreasing in the actions of the others players, but increasing in the action of player j. The probability that j is the first to solve the puzzle is given by,

$$P_j = \frac{y_j}{y_j + \sum_{i \neq j} y_i} \qquad (63)$$

As y_j increases, the probability that j is the first to solve the puzzle increases, as the numerator increases, and the utility of player j correspondingly increases. Correspondingly, the probability that any other player i is the first to solve the puzzle decreases, as the denominator in P_i increases as y_j grows.

The analysis in this section implies that players have less incentives to invest in blockchain mining when constraints are more stringent (BBC News, 2019). Indeed, as V decreases, i.e., as constraints are more stringent, shadow prices $\gamma^\star$ grow and the investments in blockchain mining, reflected by $y^\star$, decrease (see equation (57)). We further discuss general aspects related to the blockchain ecosystem in section 8.

7.2.3. Global Constraints and Local Costs

In section 7.2.2 we considered the setup wherein each player m maximized a utility whose value increases as its bid, x_m, increases. The players were restricted by global constraints.

Next, we consider the general setup wherein players are constrained both by global constraints, as in the previous section, as well as local constraints. Then, players face the CONSTRAINED GAME WITH LOCAL COSTS. The constrained

game with local costs accounts for both global constraints (through a hard constraint) and local costs (through a term that penalizes large values of x_m in the utility function),

CONSTRAINED GAME WITH LOCAL COSTS:
$$\max_{x_m} \frac{x_m}{\sum_{i=1}^{M} x_i} - \gamma x_m \qquad (64)$$

$$\sum_{j=1}^{M} x_j \leq V \qquad (65)$$

The Lagrangian of the CONSTRAINED GAME WITH LOCAL COSTS is given by:

$$\mathcal{L}_m(x_m) = \frac{x_m}{\sum_{j=1}^{M} x_j} - \gamma x_m - \lambda \left(\sum_k x_k - V \right) \qquad (66)$$

In particular, note that if $\gamma = (M - 1)/(MV)$ the equilibrium presented in Corollary 18 is also a normalized equilibrium to the CONSTRAINED GAME WITH LOCAL COSTS. Nonetheless, for arbitrary values of γ the conditions for existence and uniqueness of the normalized equilibrium to the CONSTRAINED GAME WITH LOCAL COSTS remain open. We envision that the argument presented in the previous section regarding existence and uniqueness of normalized equilibrium can be adapted to this setup, but leave the proof as subject for future work.

8. DISCUSSION

Positive and negative externalities. In the models proposed in this paper, we assumed that users who contribute to the system by mining cryptocurrencies generate negative externalities toward their mining peers. Indeed, the competition among miners is a very fundamental aspect of the mining process (Dimitri, 2017). Nonetheless, by incorporating more miners, the blockchain becomes more robust (Garay et al., 2015). Such robustness, in turn, may translate into an increase in the real value of the cryptocurrency under consideration (Shah and Zhang, 2014; Prat and Walter, 2018; Raval, 2018; Biais et al., 2019). Therefore, by increasing the pool of miners, each miner is also contributing with positive externalities toward the system, and we leave such aspect as subject for future work (Pashigian and Gould, 1998; Pretty et al., 2001; Hassin and Haviv, 2003; Morris, 2018).

Mining pools. Mining pools play a key role in todays' public blockchain systems (Eyal, 2015; Wang et al., 2019)[3]. The competition analyzed in this paper applies to mining pools under two scenarios. First, from the perspective of the mining pool, it can use cloud resources for mining purposes. Therefore, the mining pools assume the role of players as considered in this work. Alternatively, the players are the end users, who contract mining pool services. Then, mining pools assume the role of ESPs. In the first case, we consider competition among mining pools, at the macro level, and in the latter case, we consider the micro-competition among end-users.
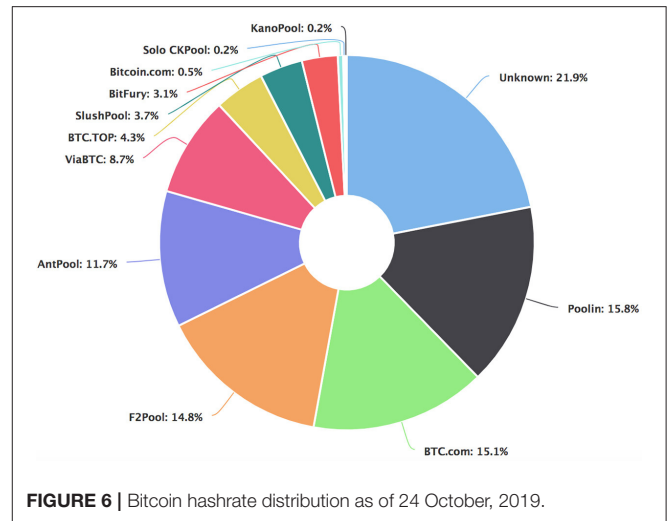

FIGURE 6 | Bitcoin hashrate distribution as of 24 October, 2019.

Figure 6 illustrates the hashrate distribution over Bitcoin, as of 24 October 2019[4]. Note that a significant portion of the hashrate is originated from four mining pools. According to Conjecture 4 and Theorem 7, the proposed model suggests that only two major mining pools would have a role in the network. A discrepancy between model predictions and hashrate distribution over Bitcoin may occur if the market is not stabilized, or agents are not fully rational. In addition, note that the proposed model only accounts for the competition among miners, and does not take into consideration the positive externalities produced by the miners (see **Appendix A**). Such positive externalities may motivate a longlasting equilibrium wherein four mining pools take place, as positive externalities naturally serve as incentives for multiple pools to coexist. Our work serves as a plausible model to justify the relatively small number of mining pools, which we posit as being due to the competition among those (Gervais et al., 2014; Arnosti and Weinberg, 2018; Gencer et al., 2018; Bai et al., 2019; Cong et al., 2019; Leshno and Strack, 2019).

Multi-cryptocurrency ecosystem. In the cryptocurrency ecosystem, large mining pools typically decide, dynamically, which blockchain to mine. Such decisions are made based on different thresholds related to the value of the cryptocurrencies and the costs for mining (mining complexity). The churn of computational power across blockchains is a well-known source of price volatility, and different mechanisms have been developed to counteract migrations of miners across platforms (Ulrich, 2017). One of those mechanisms is referred to as emergency difficulty adjustment (EDA), which reduces the difficulty of the puzzle when there are not many miners in the system, preventing the blockchain from dying.

Puzzle complexity. In Bitcoin, puzzle difficulty (complexity) is dynamically adjusted so that the time to mine a block varies between certain pre-established time bounds. Bitcoin target block generation rate is of 10 min. In theory, due to the dynamic adjustment of puzzle complexity, Bitcoin throughput (number of

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[3]For instance, https://miningpoolhub.com/.

[4]https://www.blockchain.com/pools

blocks generated per time unit) does not depend on the number of miners. An increase in the number of miners increases the time between generation of blocks per miner (Meshkov et al., 2017; Wisdom, 2018). In Huberman et al. (2017), the authors argue in favor of adjusting the frequency at which blocks are generated as a function of the congestion in the network.

Users fees. Users pay fees to have their blocks mined. Such fees impact the competition among miners, as they serve as incentives for mining. The higher the fees offered by users, the larger the expected number of miners. In this paper, we have not accounted for the role of blockchain users in the competition among miners. We envision that the interplay between users and miners leads to complex dynamics, which should be studied in light of the tension between positive and negative externalities discussed in **Appendix A**.

9. RELATED WORK

There is a vast literature investigating game theoretical aspects of blockchain systems (Kiayias et al., 2016; Huberman et al., 2017; Abdellatif and Abdelmouttalib, 2018; Ma et al., 2018; Papadis et al., 2018; Spiegelman et al., 2018; Azouvi and Hicks, 2019; Goren and Spiegelman, 2019; Liu et al., 2019a,b; Wang et al., 2019). Nonetheless, the literature on congestion games applied to such systems is scarce. In particular, to the best of our knowledge, there is no prior work investigating the competition at the network edge among miners as a congestion game, and its connection to multi-cryptocurrency markets.

Congestion games have been applied in the field of networking to account for security aspects (Maillé et al., 2011), link congestion (Johari and Tsitsiklis, 2003) and pricing of infrastructures and users (Hassin and Haviv, 1997). In Huberman et al. (2017), the authors study Bitcoin as a congestion game, where the congestion occurs due to an increase in the number of transaction requests from users. In particular, the authors abstract away from several aspects of the competition between miners. In this paper, in contrast, we focus on the competition between miners.

Spiegelman et al. (2018) adopted the framework of congestion games to model competition between miners of multiples cryptocurrencies who try to maximize utilities by choosing which puzzle (cryptocurrency) to mine (the work was then extended at Goren and Spiegelman, 2019). The authors prove that there is no standard potential function for the game they propose, but that an ordinal potential always exists, implying that best response converges to a pure Nash equilibrium. Our work captures different aspects of the problem, and is complementary to Spiegelman et al. (2018). An important similarity between the two works consists of establishing conditions under which pure Nash equilibria exist even when the game does not admit a standard potential function. The major differences between our work and (Spiegelman et al., 2018) are: (1) in the modeling of the probability to succeed in solving a puzzle (see section 8); (2) in the ESP decision, which is out of the scope of Spiegelman et al. (2018); (3) in the action space (mining power), which is continuous in Spiegelman et al. (2018),

precluding the use of crowding game results, and discrete in this paper (except in section 7), allowing us to rely on Milchtaich (1996) to prove existence of pure Nash equilibria. We refer the reader to Spiegelman et al. (2018) and Ulrich (2017) for additional references on the multi-cryptocurrency ecosystem and its security challenges.

Sánchez (2019) initiates a preliminary study on the so-called price of crypto-anarchy based on the models introduced here. In this paper, in contrast, we focused on the distributed competition among miners, and have not assessed the loss of efficiency due to the absence of a central controller to perturb the competition. We envision that a more in-depth study of the loss of efficiency due to the lack of controllers, and a study of the role of authorities in regulating the crypto-market, e.g., as indicated in section 7.2 (see also BBC News, 2019 and Fernández-Villaverde and Sanches, 2019), is an important open aspect, and leave that topic as subject for future work.

10. CONCLUSION

Competition among miners is at the core of public blockchain systems. Competition is one of the most fundamental elements ensuring that miners will strive to reach a consensus about the current state of the blockchain. We modeled the competition over several ESPs and over several blockchains characterizing multiple cryptocurrencies as a non-cooperative game. Then, we specialized our game to two cases: the ESP connection game and the cryptocurrency selection game. For each game, we showed properties of the Nash equilibrium. In particular, leveraging results about congestion games, we establish the existence of pure Nash equilibria and characterize such equilibria through problems that admit efficient algorithmic solutions.

We believe that this work opens up a number of interesting directions for future work. In particular, we did not account for strategic decisions related to punishment and cooperation between miners over repeated games. Those games naturally emerge in the sequential solution of multiple puzzles. The study of those is left as subject for future work.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/**Supplementary Material**.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

ACKNOWLEDGMENTS

by CAPES and CNPq. This paper extends our conference version of the paper which appeared in Altman et al. (2018). In particular, section 7 is novel as well as the background discussion on mining competition, accounting for dynamic puzzle complexity (section 2).

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