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The Schatten current sheet

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Space weather models endeavoring to connect remote observations to *in-situ* measurements at various locations in the heliosphere invariably require a coronal model to connect the photosphere magnetically to the inner heliosphere. The most famous and popular implementation of this connection is a potential field source surface (PFSS) model out to the source surface, typically located at 2.5 solar radii, combined with a Schatten current sheet (SCS) model. While the PFSS model is mostly understood, the SCS has been utilized in heliospheric physics for nearly 50 years with little understanding of it's physical and mathematical underpinnings. In this overview article, I lay out the mathematical formalism of the SCS, describe how it differs from the PFSS, and summarize several techniques used to combine the PFSS and SCS to create a global coronal model from the photosphere to the inner heliosphere.

KEYWORDS

heliosphere, magnetic fields, solar corona, electric current, Magnetohydrodynamics

1 Background

The ultimate goal of heliospheric modeling is rapid and reliable space weather forecasting at 1 AU for operational capabilities. Such models require a number of inputs from the surface as initial or boundary conditions which are then propagated from the Sun to Earth via a combination of empirical or magnetohydrodynamic (MHD) models. A major challenge for space weather forecasting models is the lack of remote measurements of the magnetic field in the solar atmosphere, above the photosphere. Although there are a number of obstacles to such measurements, recent advances in radio astronomy have started to enable magnetic field measurements in the solar corona (Alissandrakis and Gary, 2021). Nevertheless, there are still no global measurements of the magnetic field in the solar atmosphere, necessitating the development of approximate models for the three dimensional structure of the Sun's magnetic field.

The three most common models of the magnetic field in the solar atmosphere are the potential field source surface (PFSS; Altschuler and Newkirk, 1969) model, the linear force free magnetic field (LFF; Nakagawa, 1973; Levine and Altschuler, 1974), the nonlinear force free magnetic field (NLFF; Wiegelmann, 2007) model, which are summarized nicely in Mackay and Yeates (2012). Another approach that avoids many of the flawed assumptions in the first three techniques is the non force-free field (NFFF; Hu and Dasgupta, 2006) model. By themselves, all of these models suffer from a lack of sufficient constraints: while the photospheric magnetic field is well observed, and can be used as a boundary condition to calculate the solution to each model, global models still require an outer boundary condition to fully constrain the problem numerically. In PFSS models, the outer boundary condition is assumed to be a perfectly radial magnetic field at a location called the "source surface." Although there are good theoretical reasons to assume the existence of such a surface, early eclipse observations (Schatten, 1971) and MHD (Pneuman and Kopp, 1971) models indicated that polar plumes tended to bend more equator-ward than was predicted by the PFSS model, while the bending of streamers should, contrary to the results of the

PFSS extrapolation, depend on the amplitude of the solar cycle (Mackay and Yeates, 2012). Furthermore, Ulysses observations initially supported the view that the magnetic field was essentially uniform in latitude (Wang and Sheeley, 1990). These considerations prompted the development of the Schatten et al. (1969) model, which we will call the SCS, in which the magnetic field outside the source surface produced by the PFSS was replaced with a similar current free magnetic field into which currents are introduced, essentially, by hand. A subsequent addition to this model attempted to minimize the magnitude of extraneous currents introduced by this process (Schatten, 1971).

Although the SCS is used ubiquitously in the literature, and indeed forms the basis of present-day operational space weather modeling, there is a paucity in the literature of its mathematical formalism. Much of the formalism has been derived previously (Altschuler and Newkirk, 1969; Schatten et al., 1969; Schatten, 1971; Wang and Sheeley, 1992; Zhao and Hoeksema, 1994; Nikolić, 2017; Reiss et al., 2019; Narechania et al., 2021; Song, 2023; Knizhnik et al., 2024a), but different parts of the mathematical basis for the SCS are scattered among these various sources. Curiously, many authors use the SCS model but do not describe its implementation, making those references insufficiently descriptive for scientists to implement the technique from scratch, and the actual equations for the SCS are given, to my knowledge, only in Nikolić (2017) and Knizhnik et al. (2024a). The literature is replete with papers that state that the SCS has been implemented, but do not fully describe it, or do not fully describe whether, how, or if any techniques have been employed to minimize extraneous currents in the region outside the source surface. This review article, therefore, endeavors to derive the mathematical formalism of the SCS by showing how it is an extension of the PFSS model, and lay out the current-minimizing methods described by Schatten (1971) as well as the interface region approach introduced by McGregor et al. (2008). Finally, we describe the process for obtaining the heliospheric current sheet from the SCS model, and we comment on the preferred approach to implementing the SCS.

2 Mathematical formalism

2.1 Potential magnetic field inside a spherical shell

The mathematical formalism of the SCS is closely linked with that of the PFSS. As a result, we will start the derivation of the SCS by deriving the expressions for the magnetic field inside a spherical shell, since at the heart of both the PFSS and SCS magnetic field models is that they are solutions of a Laplace equation in such a shell, bounded from below by a spherical surface magnetic field distribution. In the case of the PFSS, the magnetic field is prescribed between the spherical surface at $r = R_\odot$ and $r = R_{ss}$. In the case of the SCS, the magnetic field is prescribed between $r = R_{ss}$ and ∞. In both cases, the magnetic field in that volume is solenoidal

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.1}$$

and potential

$$\nabla \times \boldsymbol{B} = 0. \tag{2.2}$$

The solution of Equation 2.2 is identically

$$\boldsymbol{B} = -\nabla\Psi, \tag{2.3}$$

and using Equation 2.1, Ψ satisfies

$$\nabla^2\Psi = 0$$

whose general solution in spherical coordinates is given by (Jackson, 1998):

$$\Psi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^l + B_{lm}r^{-(l+1)}\right]Y_{lm}(\theta,\phi). \tag{2.4}$$

The $Y_{lm}(\theta,\phi)$ are orthogonal spherical harmonics

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi},$$

where $P_l^m(\cos\theta)$ are the associated Legendre polynomials (Arfken, 1985). Spherical harmonics satisfy the orthogonality relation

$$\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\ \sin\theta\ Y^*_{l'm'}(\theta,\phi)Y_{lm}(\theta,\phi) = \delta_{l'l}\delta_{m'm}, \tag{2.5}$$

where $\delta_{i,j}$ is the Kroenecker delta. Equation 2.3 then leads to:

$$B_r = -\frac{\partial\Psi}{\partial r} = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[lA_{lm}r^{l-1} - (l+1)B_{lm}r^{-(l+2)}\right]Y_{lm}(\theta,\phi), \tag{2.6}$$

$$B_\theta = -\frac{1}{r}\frac{\partial\Psi}{\partial\theta} = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^{l-1} + B_{lm}r^{-(l+2)}\right]\frac{\partial Y_{lm}(\theta,\phi)}{\partial\theta}, \tag{2.7}$$

$$B_\phi = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\phi} = -\frac{1}{\sin\theta}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^{l-1} + B_{lm}r^{-(l+2)}\right]imY_{lm}(\theta,\phi). \tag{2.8}$$

This is the general form of a potential, solenoidal field **B** in spherical coordinates. The key task of the PFSS and SCS models is determining the forms of the expansion coefficients A_{lm} and B_{lm} in terms of spherical harmonics from Equation 2.6, Equation 2.7, Equation 2.8 by using the appropriate boundary conditions.

2.2 The potential field source surface model

2.2.1 Boundary conditions

In the PFSS model (Altschuler and Newkirk, 1969), the boundary conditions are specified at some inner radius $r = R_1$ and some outer radius $r = R_2$. The boundary condition at $r = R_1$ is the Neumann condition

$$\frac{\partial\Psi}{\partial r}(r,\theta,\phi)\bigg|_{R_1} = -B_r(R_1,\theta,\phi)\hat{\mathbf{r}}. \tag{2.9}$$

This boundary condition applies to $B_r(R_1,\theta,\phi)$ only, leaving $B_\theta(R_1,\theta,\phi)$ and $B_\phi(R_1,\theta,\phi)$ unconstrained.

The boundary condition at R_2 is the Dirichlet condition

$$\Psi(R_2,\theta,\phi) = 0. \tag{2.10}$$

2.2.2 The PFSS solution

From Equation 2.4, Equation 2.10 means that

$$B_{lm} = -A_{lm} R_2^{2l+1}.$$

Thus, the general solution for the radial magnetic field component in the region $R_1 \le r \le R_2$ is:

$$\frac{\partial \Psi}{\partial r} = -B_r(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[lA_{lm}r^{l-1} + (l+1)A_{lm}R_2^{2l+1}r^{-(l+2)}\right]Y_{lm}(\theta,\phi)$$
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}Y_{lm}(\theta,\phi)\left(lr^{l-1} + lR_2^{2l+1}r^{-(l+2)} + R_2^{2l+1}r^{-(l+2)}\right). \quad (2.11)$$

The coefficients A_{lm} are obtained via the orthogonality condition in Equation 2.5. Multiplying both sides of Equation 2.11 by $\sin\theta Y^*_{l'm'}(\theta,\phi)$, evaluating at $r = R_1$ using boundary condition Equation 2.9, and integrating over θ and ϕ:

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta Y^*_{l'm'}(\theta,\phi) B_r(R_1,\theta,\phi)$$
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(lR_1^{l-1} + lR_2^{2l+1}R_1^{-(l+2)} + R_2^{2l+1}R_1^{-(l+2)}\right)$$
$$\times A_{lm}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta Y^*_{l'm'}(\theta,\phi)Y_{lm}(\theta,\phi),$$

so that

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta Y^*_{l'm'}(\theta,\phi) B_r(R_1,\theta,\phi)$$
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(lR_1^{l-1} + lR_2^{2l+1}R_1^{-(l+2)} + R_2^{2l+1}R_1^{-(l+2)}\right)A_{lm}\delta_{l'l}\delta_{m'm}.$$

The Kroenecker δ's set all terms to 0 except $l' = l$ and $m' = m$, so that the double sums each only have a single term, yielding:

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta Y^*_{lm}(\theta,\phi) B_r(R_1,\theta,\phi)$$
$$= A_{lm}\left(lR_1^{l-1} + lR_2^{2l+1}R_1^{-(l+2)} + R_2^{2l+1}R_1^{-(l+2)}\right).$$

Therefore:

$$A_{lm} = \frac{a_{lm}}{lR_1^{l-1} + lR_2^{2l+1}R_1^{-(l+2)} + R_2^{2l+1}R_1^{-(l+2)}}, \quad (2.12)$$

where

$$a_{lm} \equiv \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta Y^*_{lm}(\theta,\phi) B_r(R_1,\theta,\phi). \quad (2.13)$$

Combining Equation 2.11, Equation 2.12, and Equation 2.13, the radial magnetic field in the volume is therefore:

$$B_{r,pfss}(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\theta,\phi)\frac{lr^{l-1} + lR_2^{2l+1}r^{-(l+2)} + R_2^{2l+1}r^{-(l+2)}}{lR_1^{l-1} + lR_2^{2l+1}R_1^{-(l+2)} + R_2^{2l+1}R_1^{-(l+2)}}$$
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\theta,\phi)\left(\frac{R_1}{r}\right)^{(l+2)}\left(\frac{l(r/R_2)^{2l+1} + l + 1}{l(R_1/R_2)^{2l+1} + l + 1}\right)$$
$$\equiv \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}c_l(r)Y_{lm}(\theta,\phi). \quad (2.14)$$

where

$$c_l(r) \equiv \left(\frac{R_1}{r}\right)^{(l+2)}\left(\frac{l(r/R_2)^{2l+1} + l + 1}{l(R_1/R_2)^{2l+1} + l + 1}\right).$$

Similarly, following Wang and Sheeley (1992) and defining

$$d_l(r) \equiv -\left(\frac{R_1}{r}\right)^{l+2}\left(\frac{(r/R_2)^{2l+1} - 1}{l(R_1/R_2)^{2l+1} + l + 1}\right),$$

The other two components of the magnetic field become, from Equation 2.7 and Equation 2.8:

$$B_{\theta,pfss}(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}d_l(r)\frac{\partial Y_{lm}(\theta,\phi)}{\partial\theta}, \quad (2.15)$$

$$B_{\phi,pfss}(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} i\ m\ a_{lm}d_l(r)\frac{Y_{lm}(\theta,\phi)}{\sin\theta}. \quad (2.16)$$

A key feature of the PFSS solution is that at $r = R_2$, $d_l(r) = 0$ for all l, meaning that

$$B_{r,pfss}(R_2,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}\left(\frac{R_1}{R_2}\right)^{(l+2)}\left(\frac{2l+1}{l(R_1/R_2)^{2l+1} + l + 1}\right)Y_{lm}(\theta,\phi) \quad (2.17)$$

$$B_{\theta,pfss}(R_2,\theta,\phi) = 0, \quad (2.18)$$

and

$$B_{\phi,pfss}(R_2,\theta,\phi) = 0. \quad (2.19)$$

In other words, the magnetic field determined from the PFSS solution is purely radial at $r = R_2$.

In terms of the Legendre Polynomials, the solution of the PFSS can be written as (Nikolić, 2017; 2019):

$$B_{r,pffs}(r,\theta,\phi) = \sum_{l=1}^{\infty}\sum_{m=0}^{l}\left[(l+1)\left(\frac{R_1}{r}\right)^{l+2} + m\left(\frac{R_1}{R_2}\right)^{l+2}\left(\frac{r}{R_2}\right)^{l-1}\right]$$
$$\times P_l^m(\theta)\left(g_{lm}\cos(m\phi) + h_{lm}l\sin(m\phi)\right),$$

$$B_{\theta,pffs}(r,\theta,\phi) = -\sum_{l=1}^{\infty}\sum_{m=0}^{l}\left[\left(\frac{R_1}{r}\right)^{l+2} - \left(\frac{R_1}{R_2}\right)^{l+2}\left(\frac{r}{R_2}\right)^{l-1}\right]$$
$$\times \frac{dP_l^m(\theta)}{d\theta}\left(g_{lm}\cos(m\phi) + h_{lm}\sin(m\phi)\right),$$

$$B_{\phi,pffs}(r,\theta,\phi) = \sum_{l=1}^{\infty}\sum_{m=0}^{l}\left[\left(\frac{R_1}{r}\right)^{l+2} - \left(\frac{R_1}{R_2}\right)^{l+2}\left(\frac{r}{R_2}\right)^{l-1}\right]$$
$$\times P_l^m(\theta)\frac{m}{\sin\theta}\left(g_{lm}\sin(m\phi) - h_{lm}\cos(m\phi)\right),$$

where

$$g^{lm} = \frac{2l+1}{4\pi(l+1+l(\frac{R_1}{R_2})^{2l+1})}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta P_l^m(\theta)B_r(R_1,\theta,\phi)\cos(m\phi),$$

and

$$h_{lm} = \frac{2l+1}{4\pi(l+1+l(\frac{R_1}{R_2})^{2l+1})}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta P_l^m(\theta)B_r(R_1,\theta,\phi)\sin(m\phi),$$

and $P^l_m(\theta)$ are the Schmidt functions, related to the Legendre polynomials $P^l_m \cos(\theta)$ via

$$P^l_m(\theta) = \sqrt{\frac{(2-\delta_{m,0})(l-m)!}{(l+m)!}}P^l_m\cos(\theta).$$

2.3 The Schatten model

2.3.1 Boundary conditions

In the SCS model (Schatten, 1971), the boundary conditions are specified at the inner radius R_2 (which coincides with the outer boundary R_2 of the PFSS model[1]), and the outer radius at ∞. This time, the boundary conditions on Ψ are the Neumann boundary condition at R_2:

$$\frac{\partial \Psi}{\partial r}(r,\theta,\phi)\Big|_{R_2} = |B_{r,pfss}(R_2,\theta,\phi)|\hat{\mathrm{r}}, \tag{2.20}$$

and the Dirichlet condition

$$\Psi(r\to\infty,\theta,\phi) = 0. \tag{2.21}$$

There are two crucial features of Equation 2.20. First, the absolute value sign ensures that all of the field is pointing outward from R_2, enabling the solution to represent a current free magnetic field, and in the process ensuring that the potential field solution will be dominated by a monopole term. Although this violates Equation 2.1, the solenoidality of the magnetic field will be enforced in a later step, described in Section 3.

Second, the Neumann boundary condition at R_2 in the SCS is a direct contrast to the Dirichlet boundary condition at R_2 in the PFSS. As will be shown below, this creates a tangential discontinuity and a current sheet at R_2 that requires various minimization techniques.

2.3.2 The SCS solution

Combining Equation 2.21 and the general expression for the three magnetic field components given in Equation 2.6, Equation 2.7, Equation 2.8 we obtain that

$$A_{lm} = \delta_{l0},$$

since keeping terms that go like r^{l-1} will cause the magnetic field to blow up as $r\to\infty$ unless $l<1$. Thus, Equation 2.6, Equation 2.7, Equation 2.8 simplify to:

$$B_r = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[(l+1)B_{lm}r^{-(l+2)}\right]Y_{lm}(\theta,\phi), \tag{2.22}$$

$$B_\theta = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[B_{lm}r^{-(l+2)}\right]\frac{\partial Y_{lm}(\theta,\phi)}{\partial\theta}, \tag{2.23}$$

$$B_\phi = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[B_{lm}r^{-(l+2)}\right]imY_{lm}(\theta,\phi). \tag{2.24}$$

Orthogonality of the spherical harmonics allows us to determine the expansion coefficients B_{lm} by multiplying Equation 2.22 by $\sin\theta Y^*_{l'm'}(\theta,\phi)$, evaluating at R_2 using Equation 2.20, and integrating over θ and ϕ:

$$\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta Y^*_{l'm'}(\theta,\phi)|B_{r,pfss}(R_2,\theta,\phi)| = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[(l+1)B_{lm}R_2^{-(l+2)}\right]\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta Y^*_{l'm'}(\theta,\phi)Y_{lm}(\theta,\phi).$$

The orthonormality of the spherical harmonics in Equation 2.5 yields:

$$\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta Y^*_{l'm'}(\theta,\phi)|B_{r,pfss}(R_2,\theta,\phi)| = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(l+1)B_{lm}R_2^{-(l+2)}\delta_{l'l}\delta_{m'm}.$$

All contributions to the double sums vanish due to the Kroenecker δ's, with the exception of $l'=l$ and $m'=m$, so that

$$\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta Y^*_{lm}(\theta,\phi)|B_{r,pfss}(R_2,\theta,\phi)| = (l+1)B_{lm}R_2^{-(l+2)}.$$

If we define

$$b_{lm}\equiv\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta Y^*_{lm}(\theta,\phi)|B_{r,pfss}(R_2,\theta,\phi)|,$$

Then B_{lm} is given by

$$B_{lm} = \frac{b_{lm}}{(l+1)R_2^{-(l+2)}}.$$

Plugging this into Equation 2.22, Equation 2.23, Equation 2.24, the magnetic field components outside R_2 are, then[2],

$$B_{r,sch}(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}g_l(r)b_{lm}Y_{lm}(\theta,\phi), \tag{2.25}$$

$$B_{\theta,sch}(r,\theta,\phi) = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}h_l(r)\;b_{lm}\frac{\partial Y_{lm}(\theta,\phi)}{\partial\theta}, \tag{2.26}$$

$$B_{\phi,sch}(r,\theta,\phi) = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}h_l(r)\;i\;m\;b_{lm}\frac{Y_{lm}}{\sin\theta}(\theta,\phi), \tag{2.27}$$

where

$$g_l(r)\equiv\left(\frac{R_2}{r}\right)^{(l+2)}, \tag{2.28}$$

and

$$h_l(r)\equiv\left(\frac{R_2}{r}\right)^{(l+2)}\frac{1}{l+1}. \tag{2.29}$$

In terms of the Schmidt-Legendre functions, the SCS can be written as (Nikolić, 2019):

$$B_{r,sch}(r,\theta,\phi) = \sum_{l=1}^{\infty}\sum_{m=0}^{l}(l+1)\left(\frac{R_2}{r}\right)^{l+2}P_l^m(\theta)\left(f_{lm}\cos(m\phi)+q_{lm}\sin(m\phi)\right) \tag{2.30}$$

$$B_{\theta,sch}(r,\theta,\phi) = -\sum_{l=1}^{\infty}\sum_{m=0}^{l}\left(\frac{R_2}{r}\right)^{l+2}\frac{dP_l^m(\theta)}{d\theta}\left(f_{lm}\cos(m\phi)+q_{lm}\sin(m\phi)\right) \tag{2.31}$$

$$B_{\phi,sch}(r,\theta,\phi) = \sum_{l=1}^{\infty}\sum_{m=0}^{l}\left(\frac{R_2}{r}\right)^{l+2}\frac{m}{\sin(\theta)}P_l^m(\theta)\left(f_{lm}\sin(m\phi)-q_{lm}\cos(m\phi)\right) \tag{2.32}$$

1 As will be described below, this is often not how the SCS is implemented (McGregor et al., 2008).

2 There is a typo in Equations C13-C14 of Knizhnik et al. (2024a) in the expression for $B_{\theta,sch}(r,\theta,\phi)$ and $B_{\phi,sch}(r,\theta,\phi)$.

where

$$f_{lm} = \frac{2l+1}{4\pi(l+1)} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta P_l^m(\theta) |B_r(R_2,\theta,\phi)| \cos(m\phi),$$

and

$$q_{lm} = \frac{2l+1}{4\pi(l+1)} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta P_l^m(\theta) |B_r(R_2,\theta,\phi)| \sin(m\phi),$$

There are three important features of Equation 2.25, Equation 2.26, Equation 2.27. First, since the SCS is effectively just another calculation of the PFSS solution where the outer boundary is at ∞, it makes sense that the transformation $R_2 \to \infty$ and $R_1 \to R_2$ makes $c_l(r) = g_l(r)$ and $d_l(r) = h_l(r)$ (and similarly makes $g_l^m = f_l^m$ and $h_l^m = q_l^m$), thus allowing the PFSS solution Equation 2.14, Equation 2.15 and Equation 2.16 to reduce to the SCS solution Equation 2.25, Equation 2.26, Equation 2.27 when the outer boundary is at ∞. Second, at $r = R_2$ it can readily be shown that $B_{r,sch}(R_2,\theta,\phi) = |B_{r,pfss}(R_2,\theta,\phi)|$. Third, it can also be readily shown that

$$B_{\theta,sch}(R_2,\theta,\phi) \neq 0,$$

and

$$B_{\phi,sch}(R_2,\theta,\phi) \neq 0,$$

in stark contrast to Equation 2.18, Equation 2.19. Mathematically, this discontinuity results from using a Dirichlet boundary condition at $r = R_2$ when computing the PFSS, but a Neumann condition at $r = R_2$ when computing the SCS. Physically, this occurs because the PFSS solution places all of the currents which generate the potential field in the volume $R_1 < r < R_2$ inside or at R_1 and outside R_2. In contrast, the SCS puts all of these currents at R_2. However, these sets of currents are fundamentally inconsistent with the idea that there is no current anywhere in the entire volume (Schuck et al., 2022). In models that splice together the PFSS and SCS solutions, such as WSA (Wang and Sheeley, 1990; Arge and Pizzo, 2000), this discontinuity in the tangential magnetic field components creates a sharp current sheet at $r = R_2$. This feature of the SCS also might create the impression that the magnetic field is singular at R_2, since Ψ is discontinuous there. However, the presence of this current sheet implies that $\boldsymbol{B} \neq -\nabla\Psi$, so there is no singularity in the magnetic field. However, this discontinuity does produce an unphysical 'kink' in magnetic field lines traced from outside R_2 to R_1. This creates significant issues for identifying source regions of plasma measured *in-situ*, as well as for predicting *in-situ* plasma parameters McGregor et al. (2008).

3 Dealing with the pfss-scs discontinuity

One approach to deal with the discontinuity is simply to assume that its effects are small or negligible, which is the assumption underlying the heliospheric modeling in Knizhnik et al. (2024a). However, Zhao and Hoeksema (1994) showed that this unphysical kinking of the field at R_2 produces noticeable deviations from observations.

To mitigate this issue, two approaches have been proposed.

3.1 Schatten's minimization

In the approach originally proposed by Schatten (1971) and implemented by, e.g., Zhao and Hoeksema (1994) and Reiss et al. (2019), the sum of squared residuals between the PFSS solution at R_2 and the SCS solution is minimized to obtain expansion coefficients from the SCS solution which are not, in general, the same as those derived in Section 2.3. This approach does not require solving the Laplace equation in the SCS region $r > R_2$. Instead, a matrix equation is solved, which produces the expansion coefficients in this region as functions of the PFSS solution at R_2.

This is derived by Schatten (1971), Zhao and Hoeksema (1994) and Reiss et al. (2019) as follows. The quantity to be minimized can be written as

$$F = \sum_{i=1}^{N_i}\sum_{j=1}^{N_j}\sum_{k=1}^{3} \left[B_{k,pfss}(R_2,\theta_i,\phi_j) - B_{k,sch}(R_2,\theta_i,\phi_j)\right]^2, \qquad (3.1)$$

where i, j index the angular directions at a given radius, in this case taken to be $r = R_2$, and $k = 1,2,3$ corresponds to the r,θ,ϕ directions. Taking the PFSS solution as the "ground truth" at $r = R_2$, the minimization of the functional F attempts to bring the three SCS components of the magnetic field as close as possible to the corresponding PFSS magnetic field components while maintaining the current-free assumption in the region $r > R_2$. Using Equation 2.25, Equation 2.26, Equation 2.27, Equation 2.28, Equation 2.29, (evaluated at $r = R_2$, such that $g_l(R_2) = 1$ and $h_l(R_2) = (l+1)^{-1}$), and having l only go up to N_l for practical purposes, rather than ∞, Equation 3.1 can be written as

$$\begin{aligned} F &= \sum_{i=1}^{N_i}\sum_{j=1}^{N_j}\sum_{k=1}^{3}\left[B_{ijk} - \sum_{l=0}^{N_l}\sum_{m=-l}^{l}\Upsilon_{lmk}b_{lm}\right]^2 \\ &= \sum_{i=1}^{N_i}\sum_{j=1}^{N_j}\sum_{k=1}^{3}\left[B_{ijk}^2 + \sum_{l=0}^{N_l}\sum_{m=-l}^{l}\Upsilon_{lmk}b_{lm}\sum_{n=0}^{N_l}\sum_{p=-n}^{n}\Upsilon_{npk}b_{np}\right. \\ &\qquad \left. -2B_{ijk}\sum_{l=0}^{N_l}\sum_{m=-l}^{l}\Upsilon_{lmk}b_{lm}\right], \end{aligned}$$

where $B_{ijk} \equiv B_{k,pfss}(R_2,\theta_i,\phi_j)$. From Equation 2.25, Equation 2.26, Equation 2.27, Equation 2.28, Equation 2.29, we have, respectively,

$$\Upsilon_{lm1} = Y_{lm}(\theta,\phi),$$

$$\Upsilon_{lm2} = \frac{1}{l+1}\frac{\partial Y_{lm}(\theta,\phi)}{\partial\theta},$$

and,

$$\Upsilon_{lm3} = \frac{im}{l+1}Y_{lm}(\theta,\phi).$$

The minimization can then be performed by differentiating F with respect to the coefficients b_{lm}:

$$\frac{\partial F}{\partial b_{lm}} = 0.$$

This can be written down as

$$\sum_{i=1}^{N_i}\sum_{j=1}^{N_j}\sum_{k=1}^{3}\Upsilon_{lmk}\sum_{n=0}^{N_l}\sum_{p=-n}^{n}\left[\Upsilon_{npk}b_{np} - 2B_{ijk}\right] = 0$$

The sum over l, m disappeared since we differentiated only with respect to a specific l, m. For each l and m, this can be written in matrix form as

$$\overleftrightarrow{\mathbf{U}}\overleftrightarrow{\mathbf{U}}^{\top} \cdot \mathbf{b} = 2\overleftrightarrow{\mathbf{U}} \cdot \overleftrightarrow{\mathbf{B}},$$

where $\overleftrightarrow{\mathbf{B}}$ is a matrix of length $N_i \times N_j \times 3$:

$$\overleftrightarrow{\mathbf{B}} = \begin{bmatrix} B_{r,pfss}(R_2,\theta_1,\phi_1) \\ B_{r,pfss}(R_2,\theta_1,\phi_2) \\ \vdots \\ B_{r,pfss}(R_2,\theta_2,\phi_1) \\ B_{r,pfss}(R_2,\theta_2,\phi_2) \\ \vdots \\ B_{\theta,pfss}(R_2,\theta_1,\phi_1) \\ B_{\theta,pfss}(R_2,\theta_1,\phi_2) \\ \vdots \\ B_{\theta,pfss}(R_2,\theta_2,\phi_1) \\ B_{\theta,pfss}(R_2,\theta_2,\phi_2) \\ \vdots \\ B_{\phi,pfss}(R_2,\theta_1,\phi_1) \\ B_{\phi,pfss}(R_2,\theta_1,\phi_2) \\ \vdots \\ B_{\phi,pfss}(R_2,\theta_2,\phi_1) \\ B_{\phi,pfss}(R_2,\theta_2,\phi_2) \\ \vdots \end{bmatrix}$$

and similarly $\overleftrightarrow{\mathrm{U}}$ is a matrix of dimensions $N_l^2 \times 3N_iN_j$ such that

$$\overleftrightarrow{\mathbf{U}}^{\top} = \begin{bmatrix} \Upsilon_{001}(\theta_1,\phi_1) & \cdots & \Upsilon_{0N_l1}(\theta_1,\phi_1) & \cdots & \Upsilon_{N_l01}(\theta_1,\phi_1) & \cdots & \Upsilon_{N_lN_l1}(\theta_1,\phi_1) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \Upsilon_{001}(\theta_1,\phi_j) & \cdots & \Upsilon_{0N_l1}(\theta_1,\phi_j) & \cdots & \Upsilon_{N_l01}(\theta_1,\phi_j) & \cdots & \Upsilon_{N_lN_l1}(\theta_1,\phi_j) \\ \Upsilon_{001}(\theta_2,\phi_1) & \cdots & \Upsilon_{0N_l1}(\theta_2,\phi_1) & \cdots & \Upsilon_{N_l01}(\theta_2,\phi_1) & \cdots & \Upsilon_{N_lN_l1}(\theta_2,\phi_1) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \Upsilon_{001}(\theta_i,\phi_j) & \cdots & \Upsilon_{0N_l1}(\theta_i,\phi_j) & \cdots & \Upsilon_{N_l01}(\theta_i,\phi_j) & \cdots & \Upsilon_{N_lN_l1}(\theta_i,\phi_j) \\ \Upsilon_{002}(\theta_1,\phi_1) & \cdots & \Upsilon_{0N_l2}(\theta_1,\phi_1) & \cdots & \Upsilon_{N_l02}(\theta_1,\phi_1) & \cdots & \Upsilon_{N_lN_l2}(\theta_1,\phi_1) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \Upsilon_{002}(\theta_1,\phi_j) & \cdots & \Upsilon_{0N_l2}(\theta_1,\phi_j) & \cdots & \Upsilon_{N_l02}(\theta_1,\phi_j) & \cdots & \Upsilon_{N_lN_l2}(\theta_1,\phi_j) \\ \Upsilon_{002}(\theta_2,\phi_1) & \cdots & \Upsilon_{0N_l2}(\theta_2,\phi_1) & \cdots & \Upsilon_{N_l02}(\theta_2,\phi_1) & \cdots & \Upsilon_{N_lN_l2}(\theta_2,\phi_1) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \Upsilon_{002}(\theta_i,\phi_j) & \cdots & \Upsilon_{0N_l2}(\theta_i,\phi_j) & \cdots & \Upsilon_{N_l02}(\theta_i,\phi_j) & \cdots & \Upsilon_{N_lN_l2}(\theta_i,\phi_j) \\ \Upsilon_{003}(\theta_1,\phi_1) & \cdots & \Upsilon_{0N_l3}(\theta_1,\phi_1) & \cdots & \Upsilon_{N_l03}(\theta_1,\phi_1) & \cdots & \Upsilon_{N_lN_l3}(\theta_1,\phi_1) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \Upsilon_{003}(\theta_1,\phi_j) & \cdots & \Upsilon_{0N_l3}(\theta_1,\phi_j) & \cdots & \Upsilon_{N_l03}(\theta_1,\phi_j) & \cdots & \Upsilon_{N_lN_l3}(\theta_1,\phi_j) \\ \Upsilon_{003}(\theta_2,\phi_1) & \cdots & \Upsilon_{0N_l3}(\theta_2,\phi_1) & \cdots & \Upsilon_{N_l03}(\theta_2,\phi_1) & \cdots & \Upsilon_{N_lN_l3}(\theta_2,\phi_1) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \Upsilon_{003}(\theta_i,\phi_j) & \cdots & \Upsilon_{0N_l3}(\theta_i,\phi_j) & \cdots & \Upsilon_{N_l03}(\theta_i,\phi_j) & \cdots & \Upsilon_{N_lN_l3}(\theta_i,\phi_j) \end{bmatrix},$$

Such that $\mathbf{b}$ is a vector of length N_l^2, which contains the expansion coefficients:

$$\mathbf{b} = \begin{bmatrix} b_{00} \\ b_{10} \\ \vdots \\ b_{lm} \end{bmatrix}.$$

If working with the more common Legendre polynomial formulation, Equation 2.30, Equation 2.31, Equation 2.32, the minimization has been derived by several authors: (Schatten, 1971; Zhao and Hoeksema, 1994; Reiss et al., 2019; Nikolić, 2017). The most thorough derivation is by Nikolić (2017), who writes down the vector of expansion coefficients

$$\overleftrightarrow{\mathbf{FQ}} = \begin{bmatrix} f_{00} \\ f_{10} \\ \vdots \\ f_{N_lN_l} \\ \vdots \\ q_{00} \\ q_{10} \\ \vdots \\ q_{N_lN_l} \end{bmatrix},$$

In terms of the vector $\overleftrightarrow{\mathbf{B}}$ above (Equation 2.65) and the matrix $\overleftrightarrow{\mathbf{D}}$, where

$$\overleftrightarrow{\mathbf{D}}^{\top} = \begin{bmatrix} \alpha_{001}(\theta_1,\phi_1) & \cdots & \alpha_{0N_l1}(\theta_1,\phi_1) & \cdots & \alpha_{N_l01}(\theta_1,\phi_1) & \cdots & \alpha_{N_lN_l1}(\theta_1,\phi_1) & \cdots & \alpha \Leftrightarrow \beta \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \alpha_{001}(\theta_1,\phi_j) & \cdots & \alpha_{0N_l1}(\theta_1,\phi_j) & \cdots & \alpha_{N_l01}(\theta_1,\phi_j) & \cdots & \alpha_{N_lN_l1}(\theta_1,\phi_j) & \cdots & \alpha \Leftrightarrow \beta \\ \alpha_{001}(\theta_2,\phi_1) & \cdots & \alpha_{0N_l1}(\theta_2,\phi_1) & \cdots & \alpha_{N_l01}(\theta_2,\phi_1) & \cdots & \alpha_{N_lN_l1}(\theta_2,\phi_1) & \cdots & \alpha \Leftrightarrow \beta \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \alpha_{001}(\theta_i,\phi_j) & \cdots & \alpha_{0N_l1}(\theta_i,\phi_j) & \cdots & \alpha_{N_l01}(\theta_i,\phi_j) & \cdots & \alpha_{N_lN_l1}(\theta_i,\phi_j) & \cdots & \alpha \Leftrightarrow \beta \\ \alpha_{002}(\theta_1,\phi_1) & \cdots & \alpha_{0N_l2}(\theta_1,\phi_1) & \cdots & \alpha_{N_l02}(\theta_1,\phi_1) & \cdots & \alpha_{N_lN_l2}(\theta_1,\phi_1) & \cdots & \alpha \Leftrightarrow \beta \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \alpha_{002}(\theta_1,\phi_j) & \cdots & \alpha_{0N_l2}(\theta_1,\phi_j) & \cdots & \alpha_{N_l02}(\theta_1,\phi_j) & \cdots & \alpha_{N_lN_l2}(\theta_1,\phi_j) & \cdots & \alpha \Leftrightarrow \beta \\ \alpha_{002}(\theta_2,\phi_1) & \cdots & \alpha_{0N_l2}(\theta_2,\phi_1) & \cdots & \alpha_{N_l02}(\theta_2,\phi_1) & \cdots & \alpha_{N_lN_l2}(\theta_2,\phi_1) & \cdots & \alpha \Leftrightarrow \beta \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \alpha_{002}(\theta_i,\phi_j) & \cdots & \alpha_{0N_l2}(\theta_i,\phi_j) & \cdots & \alpha_{N_l02}(\theta_i,\phi_j) & \cdots & \alpha_{N_lN_l2}(\theta_i,\phi_j) & \cdots & \alpha \Leftrightarrow \beta \\ \alpha_{003}(\theta_1,\phi_1) & \cdots & \alpha_{0N_l3}(\theta_1,\phi_1) & \cdots & \alpha_{N_l03}(\theta_1,\phi_1) & \cdots & \alpha_{N_lN_l3}(\theta_1,\phi_1) & \cdots & \alpha \Leftrightarrow \beta \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \alpha_{003}(\theta_1,\phi_j) & \cdots & \alpha_{0N_l3}(\theta_1,\phi_j) & \cdots & \alpha_{N_l03}(\theta_1,\phi_j) & \cdots & \alpha_{N_lN_l3}(\theta_1,\phi_j) & \cdots & \alpha \Leftrightarrow \beta \\ \alpha_{003}(\theta_2,\phi_1) & \cdots & \alpha_{0N_l3}(\theta_2,\phi_1) & \cdots & \alpha_{N_l03}(\theta_2,\phi_1) & \cdots & \alpha_{N_lN_l3}(\theta_2,\phi_1) & \cdots & \alpha \Leftrightarrow \beta \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \alpha_{003}(\theta_i,\phi_j) & \cdots & \alpha_{0N_l3}(\theta_i,\phi_j) & \cdots & \alpha_{N_l03}(\theta_i,\phi_j) & \cdots & \alpha_{N_lN_l3}(\theta_i,\phi_j) & \cdots & \alpha \Leftrightarrow \beta \end{bmatrix}.$$

Here the notation $\alpha \Leftrightarrow \beta$ is meant to indicate that the preceding columns should be duplicated, but swapping α and β, which are defined as

$$\alpha_{lm1} = (l+1)P_l^m(\theta)\cos(m\phi),$$

$$\alpha_{lm2} = -\frac{\partial P_l^m(\theta)}{\partial\theta}\cos(m\phi),$$

$$\alpha_{lm3} = \frac{m}{\sin\theta}P_l^m(\theta)\sin(m\phi),$$

$$\beta_{lm1} = (l+1)P_l^m(\theta)\sin(m\phi),$$

$$\beta_{lm2} = -\frac{\partial P_l^m(\theta)}{\partial\theta}\sin(m\phi),$$

$$\beta_{lm3} = \frac{m}{\sin\theta}P_l^m(\theta)\cos(m\phi).$$

with these definitions, the solution that minimizes the sum of squared residuals (Equation 2.57) in the region $r > R_2$ is Equation 2.30, Equation 2.31, Equation 2.32, but with f_{lm} and q_{lm} given by

$$\overleftrightarrow{\mathbf{FQ}} = (\overleftrightarrow{\mathbf{D}}^{-1})^{\top}\overleftrightarrow{\mathbf{D}}^{-1} \cdot \overleftrightarrow{\mathbf{D}} \cdot \overleftrightarrow{\mathbf{B}}$$

3.1.1 Weakness of the minimization approach

The problem with the approach described in Section 2.4 is that in attempting to minimize the magnitude of the current sheet created at R_2 by the discontinuity in the tangential components of the magnetic field, the least squares minimization created a discontinuity in the radial component of the magnetic field, in direct violation of Gauss's law Equation 2.1. Such a trade-off, where a physical law of nature is sacrificed, is unlikely to produce more realistic results. Furthermore, the net gains from this minimization are unclear. Zhao and Hoeksema (1994) argues that this approach brings modeled coronal plumes and streamers into better agreement with eclipse observations than not performing this minimization. However, McGregor et al. (2008) find that kinks in the field lines in the non-minimized SCS model do not generally affect *in-situ* measurements at 1 AU, at least when using the WSA model for velocity prediction. Furthermore, the minimization approach seems to decrease the current density at the source surface only by a factor of 5 – 10 (relative to the direct SCS or McGregor et al. (2008) approaches), at the cost of a significant increase in the divergence of **B** (Zaveri et al., 2024 in prep).

3.2 The interface region

A second approach, developed by McGregor et al. (2008) and Meadors et al. (2020), and curently implemented in the EUHFORIA code Pomoell and Poedts (2018), is to compute the PFSS solution as described above, but use the radial magnetic field at a radius $R_2 - \epsilon$ as a boundary condition for the SCS, rather than R_2. Here ϵ is a small radial distance in from the radius R_2, such that at $R_2 - \epsilon$, the magnetic field was not forced to be entirely radial, thus minimizing the discontinuity between the PFSS and SCS solution. To best match the two solutions, McGregor et al. (2008) attempted simultaneously to 1) conserve the amount of open flux calculated by the 'new' SCS model and 2) minimize the total current along what they termed the "interface surface." Using a number of different combinations of choices for R_2 and ϵ, McGregor et al. (2008) found that values of $R_2 = 2.6R_\odot$ and $\epsilon = 0.3$ performed best at conserving total open flux and minimizing total current. They demonstrated that these choices reduces, but does not get rid of, unphysical kinks in magnetic fields traced near R_2, and that this resulted in improved prediction skill scores for the solar wind velocity and interplanetary magnetic field polarity. Meadors et al. (2020) showed that the choices for R_2 and ϵ significantly affected the prediction skill score. In the current implementation of WSA, $R_2 = 2.51$ and $\epsilon = 0.02$ are used (Meadors et al., 2020), and the SCS model seems to be truncated at $5R_\odot$, whereupon the field direction is reversed, as will be discussed in Section 3.

4 Reversing the polarities

Equation 2.20 used the absolute value of the radial magnetic field at R_2 as a boundary condition, creating a magnetic field that is everywhere positive. This not only creates a magnetic monopole, violating the fundamental physical law in Equation 2.1, but also creates a discontinuity in the radial component of the magnetic field, since the boundary condition on B_r in the region $r \geq R_2$ is inconsistent with the PFSS solution of B_r at $r = R_2$, which was not positive definite. Both issues are resolved with the following procedure.

After the expansion coefficients are calculated as described above, either by directly implementing the expansion in spherical harmonics at R_2 (Knizhnik et al., 2024a; b) or at some interface radius $R_2 - \epsilon$ (McGregor et al., 2008; Meadors et al., 2020), or by performing a minimization procedure (§subsection 2.4; Zhao and Hoeksema, 1994; Reiss et al., 2019), the correct polarity of the magnetic field must be recovered. This has typically been done by brute force: a magnetic field line is traced towards the source surface from each target location where the polarity is needed. The sign of the magnetic field at the footpoint of the traced field line at the source surface is identified by interpolating from the sign of nearest grid points. All locations along this field line are assigned to the polarity of the footpoint.

In practice, this brute force technique is unavoidable, but it also becomes computationally intensive when a large number of target points are required. For example, if the sign of the field is needed on some surface to be used as input to an MHD model (e.g., Reiss et al., 2019; Knizhnik et al., 2024a), a field line needs to be traced from each grid location on that surface. Even more computationally intensive is a scenario when the magnitude and polarity of the magnetic field needs to be known in the entire volume, requiring field line tracing from every single grid point in the volume (e.g., Scott et al., 2018).

The end result of the polarity reversal is that the boundary condition on the SCS region implicitly matches the PFSS solution at $r = R_2$, and Equation 2.1 is satisfied. Physically, the polarity reversal creates a large current sheet at the boundary between positive and negative polarities. This is known as the heliospheric current sheet (HCS).

It should be noted that the Lorentz force

$$\mathbf{J} \times \mathbf{B} = \nabla \cdot \overleftrightarrow{\mathbf{M}},$$

and the associated magnetic stresses remain constant before and after the polarity reversal, since the Maxwell stress tensor

$$\overleftrightarrow{\mathbf{M}} = \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right),$$

Is quadratic in magnetic field components, so changing the sign of all components of the magnetic field leaves $\overleftrightarrow{\mathbf{M}}$ unchanged (Schatten, 1971).

5 Discussion

The SCS model is a key component for many coronal and heliospheric models. It connects the PFSS regime to the inner heliosphere, and is used as input for MHD and WSA heliospheric models. Unfortunately, the mathematical formalism, reasoning, and physical motivation behind the use of the SCS model is very difficult to find in the literature. This article has endeavored to describe the SCS completely. The SCS has several important properties.

First, it smooths out the magnetic field in the inner heliosphere from an initially complicated and salt-and-pepper photospheric distribution and a slightly smoother, yet still complex magnetic field on the source surface, to an essentially bipolar magnetic field in the inner heliosphere. At large radii, the SCS magnetic field has essentially constant magnitude as a function of both longitude and latitude, and creates a well defined, smooth heliospheric current sheet separating positive and negative polarities.

Second, it shows better agreement with observations of solar plumes and streamers by bending magnetic field lines toward the equator outside of the source surface, in direct contrast to the PFSS model, which assumes perfectly radial magnetic field outside the source surface (Schatten, 1971; Zhao and Hoeksema, 1994).

Third, it is analytically tractable, conserves the solenoidality of the magnetic field (at least when not employing the minimization techniques of Section 2.4), and is relatively simple to compute. The most computationally intensive step is the field line tracing, which for reasonable model resolutions only takes up to several hours on a single CPU.

Nevertheless, the model has significant downsides. First, there is no theoretical reason to believe that the magnetic field in the solar atmosphere is divided into 1) a current free region below R_2, 2) a region outside R_2 that contains no currents except for the HCS and a current sheet at R_2. The division of the coronal volume into two distinct magnetic domains was done to better match observations, and can physically be justified by arguing that the distance $r = R_2$ is the location where the plasma and magnetic pressures balance, and the plasma is able to open up any remaining closed field lines. This surface, known as the Alfvén surface (Cranmer et al., 2023; Chhiber et al., 2024), where the solar wind speed and Alfvén speed match, divides a plasma dominated region from a magnetic field dominated region. While this explains the assumption of a purely radial field at R_2 (with the dominant solar wind pressure opening up all field lines outside the Alfvén surface), the assumption of a vanishing current above and below R_2 remains unjustified. Recent measurements from Parker Solar Probe entering the magnetically dominated corona revealed significant structure in the magnetic field on both sides of the Alfvén surface, suggesting the presence of non-zero electric currents (Kasper et al., 2021). Furthermore, Schuck et al. (2022) has convincingly argued that there must be significant currents in the corona, likely contributing to 20 – 30% of the photospheric flux. Since currents are the source of free energy needed for solar eruptions (as potential fields are the lowest energy state of any magnetic field configuration), they must exist in the solar atmosphere. The locations of these currents in the corona will affect the magnetic field distribution at and above the photosphere, and influence the boundary condition for the region $r \geq R_2$. Finally, there are likely to be significant currents away from the Alfvén surface generated by turbulence in the solar wind. Additionally, the assumption of spherical symmetry of the source surface is suspect, as the radius of Alfvén surface is likely to vary in longitude and latitude (Levine et al., 1982; Verscharen et al., 2021). While conceptually it is straightforward to extend the PFSS + SCS model to non-spherical surfaces, mathematically defining and using the boundary conditions to calculate an analytic model of the magnetic field everywhere is likely to be a major challenge.

A further issue with the SCS model is the approach often employed to minimize the current sheet at $r = R_2$. The trade-off of sacrificing the solenoidality of $\mathbf{B}$ in favor of a slightly smaller current sheet at $r = R_2$ is suspect, and it is not clear whether this truly produces improved *in-situ* predictions. Although the approach proposed by McGregor et al. (2008) removes any unphysical divergence of $\mathbf{B}$, the extent to which it further minimizes the current sheet at $r = R_2$ has not yet been explored (see, however, Zaveri et al., 2024, in prep).

Finally, the view that the interplanetary magnetic field was essentially uniform with heliospheric latitude, as indicated by early Ulysses observations, and which is meant to be reproduced by the SCS model, has been shown to be on poor observational footing. Later Ulysses measurements show significant variability in the interplanetary magnetic field (e.g., Khabarova and Obridko, 2012; Khabarova, 2013; Erdős and Balogh, 2014), as well as in the solar wind velocity and density (McComas et al., 2008; Khabarova et al., 2018). Although significant variation in the velocity and density is seen with the WSA model due to its reliance on the expansion factor and distance from the nearest coronal hole boundary, the magnetic variations seen by Ulysses are not reproduced by the SCS model, which typically creates a bipolar magnetic field outside R_2. The velocity and density variations around the HCS, meanwhile, form the heliospheric plasma sheet (HSP; e.g., Bavassano et al., 1997), which typically shows slow wind streams (at least at 1 AU; Eselevich and Fainshtein, 1991) and higher plasma densities than higher latitude solar wind streams (Kislov et al., 2015; Maiewski et al., 2018; Lavraud et al., 2020). In *in-situ* measurements, the HPS can lead or follow the HCS signatures (Eselevich and Fainshtein, 1991; Winterhalter et al., 1994). Since near the HPS the expansion factor is large and the distance to the nearest coronal hole is small, the WSA model naturally predicts a slow solar wind speed and, subsequently, an increased plasma density (McGregor, 2011; Merkin et al., 2016). Indeed, all of these features are well reproduced by MHD models that employ the WSA formalism as a boundary condition at R_2 (*cf.* Figures 2, 4–6 of Knizhnik et al., 2024a).

Author contributions

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Conflict of interest

The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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