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Pseudospectra and eigenvalue asymptotics for disordered non-selfadjoint operators in the semiclassical limit

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The purpose of this note is to review certain recent results concerning the pseudospectra and the eigenvalues asymptotics of non-selfadjoint semiclassical pseudo-differential operators subject to small random perturbations.

KEYWORDS

semiclassical analysis, non-selfadjoint operators, random matrix, spectral theory, partial differential equation (PDE)

1 Introduction

The spectral theory of non-selfadjoint operators acting on a Hilbert space is an established and highly developed subject. Non-selfadjoint operators are prevalent naturally in a wide range of modern problems. For instance, in the field of quantum mechanics, the study of scattering systems naturally leads to the notion of quantum resonances. These can be described as the complex values of the meromorphic. The continuation of the scattering matrix or of the cut-off resolvent of the Hamiltonian to the non-physical sheet of the complex plane. Alternatively, through a complex deformation of the initial Hamiltonian, these resonances can be characterized as the genuine complex valued eigenvalues of a non-selfadjoint operator [1, 3, 60]. We recommend the reader to reference [18] for an in-depth discussion of the mathematics of scattering poles. Another aspect of quantum mechanics is the examination of a *small* system that is linked to a *larger* environment. The effective dynamics of the small systems are governed by a non-selfadjoint operator: the Lindbladian [39].

A major obstacle to the spectral analysis of non-selfadjoint operators is the possible strong *spectral instability* of their spectrum with respect to small perturbations. This phenomenon, sometimes referred to as the *pseudospectral effect*, was initially considered to be a drawback, as it could lead to the origin of immense numerical errors, see Embree and Trefethen [19] and the references therein. However, a recent line of research has also demonstrated that the pseudospectral effect can provide novel insights into the spectral distribution of non-selfadjoint operators that are subjected to small generic perturbations.

2 Spectral instability of non-selfadjoint operators

We commence by recalling the definition of the *pseudospectrum* of a linear operator, a crucial concept that which quantifies its spectral instability. This notion appears to have originated in the second half of the 20^{th} century in various contexts, see reference [65] for a historic overview. It quickly became an important notion in numerical analysis, as it allows us to quantify how much eigenvalues can *spread out* under the influence of small perturbations, see references [64, 65] and the book [19]. We follow here the latter reference.

Let $\mathcal{H}$ be a complex Hilbert space (assumed separable for simplicity) with norm $\|\cdot\|$ and scalar product $(\cdot|\cdot)$. Let $P:\mathcal{H}\to\mathcal{H}$ be a closed densely defined linear operator, with resolvent set $\rho(P)$ and spectrum $\mathrm{Spec}(P)=\mathbb{C}\backslash\rho(P)$.

Definition 1. For any $\varepsilon>0$, we define the ε-pseudospectrum of P by

$$\mathrm{Spec}_\varepsilon(P) := \mathrm{Spec}(P)\cup\{z\in\rho(P);\ \|(P-z)^{-1}\|>\varepsilon^{-1}\}. \tag{1}$$

We note that some authors define the ε-pseudospectrum with a $\geqslant$ rather than a $>$. We, however, follow here reference [19]. It is noteworthy that with this choice of non-strict inequality results in the $\mathrm{Spec}_\varepsilon(P)$ being an open set in $\mathbb{C}$.

For P selfadjoint (or even normal), the spectral theorem implies that

$$\mathrm{Spec}_\varepsilon(P)\subset\mathrm{Spec}(P)+D(0,\varepsilon), \tag{2}$$

where $D(0,\varepsilon)\subset\mathbb{C}$ denotes the open disk with radius ε centered at 0. For P non-selfadjoint, the pseudospectrum of P can be much larger, as illustrated by the following example.

Example 1. For $N\gg1$, consider the Jordan block matrix

$$P_N=\begin{pmatrix}0&1&0&\dots&0\\ 0&0&1&\ddots&\vdots\\ \vdots&\ddots&\ddots&\ddots&0\\ \vdots&\dots&\ddots&\ddots&1\\ 0&\dots&\dots&\dots&0\end{pmatrix}:\mathbb{C}^N\to\mathbb{C}^N. \tag{3}$$

The spectrum of P_N is given by $\{0\}$. Consider the vector $e_+=(1,z,\dots,z^{N-1})$, $|z|\leqslant r<1$. Then,

$$\|(P_N-z)e_+\|=|z|^N=\mathcal{O}\left(\mathrm{e}^{-N|\log r|}\right)\|e_+\|.$$

So, Theorem 2 shows that for any $\varepsilon>0$ and any $r\in]0,1[$ we have that for $N>1$ sufficiently large

$$D(0,r)\subset\mathrm{Spec}_\varepsilon(P_N).$$

An immediate consequence of Equation 1 is the property that pseudospectra are nested. More precisely,

$$\mathrm{Spec}_{\varepsilon_2}(P)\subset\mathrm{Spec}_{\varepsilon_1}(P),\quad \varepsilon_1>\varepsilon_2>0. \tag{4}$$

The set (Equation 1) describes a region of spectral instability of the operator P, since any point in the ε-pseudospectrum of P lies within the spectrum of a certain ε-perturbation of P [19].

Theorem 1. Let $\varepsilon>0$. Then

$$\mathrm{Spec}_\varepsilon(P)=\bigcup_{\substack{Q\in\mathcal{B}(\mathcal{H})\\ \|Q\|<1}}\mathrm{Spec}(P+\varepsilon Q). \tag{5}$$

Proof. See reference [19, p. 31].

A third, equivalent definition of the ε-pseudospectrum of P is provided by the existence of approximate solutions to the eigenvalue problem $(P-z)u=0$.

Theorem 2. Let $\varepsilon>0$ and $z\in\mathbb{C}$. Then the following statements are equivalent:

1. $z\in\mathrm{Spec}_\varepsilon(P)$;
2. $z\in\mathrm{Spec}(P)$ or there exists a $u_z\in\mathcal{D}(P)$ such that $\|(P-z)u_z\|<\varepsilon\|u_z\|$, where $\mathcal{D}(P)$ denotes the domain of P.

Proof. See reference [19, p. 31].

Such a state u_z is referred to as an ε-quasimode, or simply a *quasimode* of $P-z$.

3 Spectral instability of semiclassical pseudo-differential operators

Although the notion of ε-pseudospectrum defined in Definition 1 is valid in the context of semiclassical pseudo-differential operators, we present here a somewhat different, but still related notion, which is more suited to the semiclassical setting. Here, the term "semiclassical" implies that our operators are dependent on a parameter $h\in]0,1]$ (often referred to as "Planck's parameter"), and that our focus we will be on the asymptotic (*semiclassical*) regime $h\searrow0$. This small parameter will provide us with a natural threshold for defining the pseudospectrum, and thereby measuring the spectral instability. The following discussion is based on the studies of Davies [13] and Dencker et al. [16].

Let $d\geqslant1$ and $h\in]0,1]$. An *order function* $m\in C^\infty(\mathbb{R}^{2d};[1,\infty[)$, is a function satisfying the following growth condition:

$$\exists C_0\geqslant1,\ \exists N_0>0:\quad m(\rho)\leqslant C_0\langle\rho-\mu\rangle^{N_0}m(\mu),\quad \forall\rho,\mu\in\mathbb{R}^{2d}, \tag{6}$$

where $\langle\rho-\mu\rangle:=\sqrt{1+|\rho-\mu|^2}$ denotes the "Japanese brackets." We will also sometimes write $(x,\xi)=\rho\in\mathbb{R}^{2d}$, so that $\xi\in\mathbb{R}^d$. To such an order function m, we may associate a semiclassical symbol class [17, 71]. We assert that a smooth function $p\in C^\infty(\mathbb{R}^{2d}_\rho,]0,1]_h)$ belongs to the symbol class $S(m)$ if for any multiindex $\alpha\in\mathbb{N}^{2d}$ when there exists a constant $C_\alpha>0$ such that

$$|\partial^\alpha_\rho p(\rho;h)|\leqslant C_\alpha m(\rho),\quad \forall\rho\in\mathbb{R}^{2d},\ \forall h\in]0,1]. \tag{7}$$

We recommend the reader for further reading on semiclassical analysis to [17, 41, 71].

Let the symbol $p\in S(m)$, $m\geqslant1$, be a "classical" symbol, which satisfies an asymptotic expansion in the limit $h\to0$:

$$p(\rho;h)\sim p_0(\rho)+hp_1(\rho)+\dots\quad \text{in } S(m), \tag{8}$$

where each $p_j\in S(m)$ is independent of h. We assume that there exists a $z_0\in\mathbb{C}$ and a $C_0>0$ such that

$$|p_0(\rho)-z_0|\geqslant m(\rho)/C_0,\quad \rho\in T^*\mathbb{R}^d. \tag{9}$$

Here, $T^*\mathbb{R}^d\simeq\mathbb{R}^{2d}$ denotes the cotangent space of $\mathbb{R}^d$. In this case, we call p_0 the (semiclassical) principal symbol of p. We then define two subsets of $\mathbb{C}$ associated with p_0:

$$\Sigma:=\Sigma(p_0):=\overline{p_0(T^*\mathbb{R}^d)},$$
$$\Sigma_\infty:=\{z\in\Sigma;\ \exists(\rho_j)_{j\geqslant1}\text{ s.t. }|\rho_j|\to\infty,\ p_0(\rho_j)\to z\}. \tag{10}$$

Here, the $\overline{p_0(T^*\mathbb{R}^d)}$ denotes the closure of the set $p_0(T^*\mathbb{R}^d)$, and we will use this notation in the sequel. The set Σ is the *classical spectrum*, and Σ_∞ can be called the *classical spectrum at infinity* of the h-Weyl quantization of p was defined by

$$\begin{aligned} P_h u(x) &\coloneqq p^w(x, hD_x)u(x) \\ &= \frac{1}{(2\pi h)^d} \iint e^{\frac{i}{h}(x-y)\cdot\xi} p\left(\frac{x+y}{2}, \xi; h\right) u(y)dyd\xi, \\ &\quad u \in S(\mathbb{R}^d), \end{aligned} \tag{11}$$

seen as an oscillatory integral in ξ. The operator P_h maps $S \to S$, and by duality $S' \to S'$, continuously.

3.1 Semiclassical pseudospectrum

Similar to Dencker et al. [16], we define for a symbol $p \in S(m)$ as in Equation 8 the sets

$$\Lambda_\pm(p) \coloneqq \left\{p(\rho);\ \pm\frac{1}{2i}\{\overline{p}, p\}(\rho) < 0\right\} \subset \Sigma \subset \mathbb{C}, \tag{12}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket. It should be noted that the condition $\frac{1}{2i}\{\overline{p}, p\} \neq 0$ is the classical analog of the $[P_h^*, P_h] \neq 0$. As in Dencker et al. [16], we call the set

$$\Lambda(p) := \overline{\Lambda_- \cup \Lambda_+} \tag{13}$$

the *semiclassical pseudospectrum*.

Theorem 3 ([16]). Suppose that $n \geqslant 2$, $C_b^\infty(T^*\mathbb{R}^d) \ni p \sim p_0 + hp_1 + \dots$, and $p_0^{-1}(z)$ is compact for a dense set of values $z \in \mathbb{C}$. If $P_h = p^w(x, hD_x)$, then

$$\Lambda(p_0)\backslash\Sigma_\infty \subset \overline{\Lambda_+(p_0)}$$

and for every $z \in \Lambda_+(p_0)$ and every $\rho_0 \in T^*\mathbb{R}^d$ with

$$p_0(\rho_0) = z, \qquad \frac{1}{2i}\{\overline{p}_0, p_0\}(\rho_0) < 0,$$

there exists $0 \neq e_+ \in L^2(\mathbb{R}^d)$ such that

$$\|(P_h - z)e_+\| = \mathcal{O}(h^\infty)\|e_+\|, \quad \mathrm{WF}_h(e_+)^1 = \{\rho_0\}. \tag{14}$$

If, in addition, p has a bounded holomorphic continuation to to $\{\rho \in \mathbb{C}^{2d},\ |\mathrm{Im}\,\rho| \leqslant 1/C\}$, then Equation 14 holds with the h^∞ replaced by $\exp(-1/(Ch))$.

1 This implies that the semiclassical wavefront set of e_+ is defined by ρ_0. In other words, the state e_+ is concentrated in position and frequency near the point ρ_0. See, for instance, Zworski [71] for a definition. For $u = (u(h))_{h\in(0,1)}$ a bounded family in $L^2(\mathbb{R}^d)$, its semiclassical wavefront set $\mathrm{WF}_h(u)$ denotes the phase space region where u is h-microlocalized:

$$\begin{aligned}\mathrm{WF}_h(u) \stackrel{\text{def}}{=} \complement \Big\{&(x,\xi) \in T^*\mathbb{R}^d;\ \exists a \in C_c^\infty(T^*\mathbb{R}^d),\ a(x,\xi) = 1, \\ &\|a^w(x, hD_x)u(h)\|_{L^2} = \mathcal{O}(h^\infty), \Big\}\end{aligned}$$

where a^w denotes the Weyl quantization of a, and $\complement U$ denotes the complement of a given set U.

If $n = 1$, then the same conclusion holds, provided that in addition to the general assumptions, each component of $\mathbb{C}\backslash\Sigma_\infty$ has a nonempty intersection with $\complement\Lambda(p)$.[2]

This result can be extended to unbounded symbols $p \in S(T^*\mathbb{R}^d, m)$, as shown in Equation 8, and the corresponding operators P_h with principal symbol p_0, by applying Theorem 3 to $\widetilde{P}_h = (P_h - z_0)^{-1}(P_h - z)$, with principal symbol $\widetilde{p}_0 \in C_b^\infty(T^*\mathbb{R}^d)$ and z_0 as in Equation 9 and $z_0 \neq z$. Indeed, note that $z \in \Sigma(p_0)$ if and only if $0 \in \Sigma(\widetilde{p}_0)$, and that $\rho \in p_0^{-1}(z)$ with $\pm\{\mathrm{Re}\,p_0, \mathrm{Im}\,p_0\}(\rho) < 0$ is equivalent to $\rho \in \widetilde{p}_0^{-1}(0)$ with $\pm\{\mathrm{Re}\,\widetilde{p}_0, \mathrm{Im}\,\widetilde{p}_0\}(\rho) < 0$. Furthermore, a quasimode u as in Theorem 3 for $\widetilde{P}_h$ then provides, after a possible truncation, a quasimode for $P_h - z$ in the same sense.

By replacing P_h with its formal adjoint, P_h^*, and thus p with $\overline{p}$, Theorem 3 yields that for every $z \in \Lambda_-(p)$ and every $\rho_0 \in T^*\mathbb{R}^d$ with

$$p_0(\rho_0) = z, \qquad \frac{1}{2i}\{\overline{p}_0, p_0\}(\rho_0) > 0,$$

there exists $0 \neq e_- \in L^2(\mathbb{R}^d)$ such that

$$\|(P_h - z)^* e_-\| = \mathcal{O}(h^\infty)\|e_-\|, \quad \mathrm{WF}_h(e_-) = \{\rho_0\}.$$

The additional statements of Theorem 3 regarding symbols that permit a holomorphic extension to a complex neighborhood of $\mathbb{R}^{2d}$, and the case where $n = 1$ hold as well.

Example 2. The case study to be considered is the case of the non-selfadjoint Harmonic oscillator

$$P_h = (hD_x)^2 + ix^2$$

is seen as an unbounded operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$. The principal symbol for P_h is given by $p(x,\xi) = \xi^2 + ix^2 \in S(T^*\mathbb{R}, m)$, with a weight function $m(x,\xi) = 1+\xi^2+x^2$. We equip P_h with the domain $H(m) := (P_h + 1)^{-1}L^2(\mathbb{R})$, where the operator on the right is the pseudo-differential inverse of $P_h + 1$. This choice of domain renders P_h a closed and densely defined operator. Using, for instance, the method of complex scaling, it can be observed that the spectrum of P_h is determined by

$$\mathrm{Spec}(P_h) = \{e^{i\pi/4}(2n+1)h;\ n \in \mathbb{N}\}. \tag{15}$$

Furthermore, Σ is the closed first quadrant in the complex plane, whereas $\Sigma_\infty = \emptyset$. For $\rho = (x,\xi) \in T^*\mathbb{R}$, we find that

$$\frac{1}{2i}\{\overline{p}, p\}(x,\xi) = 2\xi \cdot x. \tag{16}$$

Thus, for every $z \in \mathring{\Sigma}^3$ there exist points

$$\begin{aligned}\rho_+^j(z) &= (-1)^j(-\sqrt{|\mathrm{Re}\,z|}, \sqrt{|\mathrm{Im}\,z|}), \\ \rho_-^j(z) &= (-1)^j(-\sqrt{|\mathrm{Re}\,z|}, -\sqrt{|\mathrm{Im}\,z|}), \quad j = 1, 2,\end{aligned}$$

such that

$$\pm\frac{1}{2i}\{\overline{p}, p\}(\rho_\pm^j(z)) < 0, \quad j = 1, 2.$$

2 $\complement\Lambda(p)$ denotes the complement of the set $\Lambda(p)$.

3 $\mathring{\Sigma}$ denotes the interior of the set Σ.

Using the WKB method, it is possible to construct quasimodes of the form $e^j_+(x;h) = a^j_+(x;h)e^{i\phi^j_+(x)/h}$ with $a^j_+(x;h) \in C^\infty_c(\mathbb{R})$ admitting an asymptotic expansion $a^j_+(x;h) \sim a^j_{+,0}(x)+ha^j_{+,1}(x)+\dots$ with $\mathrm{WF}_h(e^j_+) = \{\rho^j_+(z)\}$ and

$$\|(P_h - z)e^j_+\| = \mathcal{O}(e^{-1/Ch}), \tag{17}$$

see Davies [13, 14] for an explicit computation, and Dencker et al. [16] for a more general construction.

In fact, the works of Davies [13, 14] provide an explicit WKB construction for a quasimode u for one-dimensional non-selfadjoint Schrödinger operators $P_h - z = (hD_x)^2 + V(x) - z$ on $L^2(\mathbb{R})$ with $V \in C^\infty(\mathbb{R})$ complex-valued and $z = V(a) + \eta^2$, for some $a \in \mathbb{R}$, $\eta > 0$. Furthermore, one assumes that $\operatorname{Im} V'(a) \neq 0$. These studies served as the foundation for the quasimode construction of non-selfadjoint (pseudo-)differential operators. Zworski [69] compared Davies' quasimode construction under the condition on the gradient of $\operatorname{Im} V$ to a quasimode construction under a non-vanishing condition of the Poisson bracket $\frac{1}{2i}\{\overline{p}, p\}$. Furthermore, Zworski [69] established the link to the famous commutator condition of Hörmander [32, 33]. A full generalization of the quasimode construction under a non-vanishing condition of the poisson bracket, see Theorem 3, was then achieved by Dencker et al. [16]. Finally, Pravda-Starov [46–48] improved these results by modifying a quasimode construction by Moyer and Hörmander, see reference [34, Lemma 26.4.14], for adjoints of operators that do not satisfy the Nirenberg-Tréves condition (Ψ) for local solvability.

For a quasimode construction for non-selfadjoint boundary value problems, we recommend the reader refer to the study of Galkowski [20].

It is noteworthy, that Equation 14 (or Equation 17 in the aforementioned example) implies that if the resolvent $(P_h - z)^{-1}$ exists then its norm is larger than any power of h when $h \to 0$, or even larger than $e^{1/Ch}$ in the analytical case. Each family $(e^j_+(z,h))$ is an h^∞-quasimode of $P_h - z$, or for short a quasimode of $P_h - z$.

From the quasimode Equation 14, it is easy to observe an operator Q of unity norm and a parameter $\delta = \mathcal{O}(h^\infty)$, such that the perturbed operator $P_h + \delta Q$ has an eigenvalue at z. For instance, if we call the error $r_+ = (P_h - z)e_+$, we may take the rank 1 operator $\delta Q = -r_+ \otimes (e_+)^*$. According to Theorem 3, it can be observed that the interior of the set $\Lambda(p)$, situated away from the set Σ_∞, is a zone of strong spectral instability for P_h. For this reason, we may refer to the semiclassical pseudospectrum $\Lambda(p)$ also as the (h^∞-) pseudospectrum of P_h. Finally, we recommend the reader also to the refer studies of Pravda-Starov [46–48] for further refinement of the notion of semiclassical pseudospectrum.

3.2 Outside the semiclassical pseudospectrum

When

$$z \in \mathbb{C}\backslash\Sigma(p),$$

then by condition (Equation 9), we have $(p_0(\rho) - z) \geqslant m(\rho)/C$ for some sufficiently large $C > 0$ and so we know that the inverse $(P_h - z)^{-1}$ is a pseudo-differential operator with principal symbol $(p_0 - z)^{-1} \in S(1/m) \subset S(1)$. Hence, $(P_h - z)^{-1}$ maps $L^2 \to L^2$ and

$$\|(P_h - z)^{-1}\| = \mathcal{O}(1) \tag{18}$$

uniformly in $h > 0$. Therefore, from the semiclassical point of view, we may consider $\mathbb{C}\backslash\Sigma$ as a *zone of spectral stability*.

3.3 At the boundary of the semiclassical pseudospectrum

At the boundary of the semiclassical pseudospectrum, a transition occurs between the zone of strong spectral instability and stability. Indeed, at the boundary we find an improvement over the resolvent bounds, assuming some additional non-degeneracy:

Splitting a symbol $p \in C^\infty_b(T^*\mathbb{R}^d)$ into real and imaginary part, $p = p_1 + ip_2$, we consider the iterated Poisson bracket

$$p_I := \{p_{i_1}, \{p_{i_2}, \{\dots, \{p_{i_{k-1}}, p_{i_k}\}\}\dots\}\}$$

where $I \in \{1,2\}^k$, and $|I| = k$ is called the *order* of the Poisson bracket. The *order* of p at $\rho \in T^*\mathbb{R}^d$ is given by

$$k(\rho) := \max\{j \in \mathbb{N};\ p_I(\rho) = 0,\ 1 < |I| \leqslant j\}.$$

The *order* of $z_0 \in \Sigma\backslash\Sigma_\infty$ is the maximum of $k(\rho)$ for $\rho \in p^{-1}(z_0)$.

Theorem 4. See Dencker et al. [16, 56] Assume that $C^\infty_b(T^*\mathbb{R}^d) \ni p \sim p_0 + hp_1 + \dots$. Let $P_h = p^w(x, hD_x)$ and let $z_0 \in \partial\Sigma(p_0)\backslash\Sigma_\infty(p_0)$. Assume that $dp_0 \neq 0$ at every point in $p_0^{-1}(z_0)$, and that z_0 has a finite order $k \geqslant 1$ for p. Then, k is equal and $h > 0$ is small enough for

$$\|(P_h - z)^{-1}\| \leqslant Ch^{-\frac{k}{k+1}}.$$

In particular, there exists a $c_0 > 0$, such that $h > 0$ is small enough for

$$\{z \in \mathbb{C};\ |z - z_0| \leqslant c_0 h^{\frac{k}{k+1}}\} \cap \operatorname{Spec}(P_h) = \emptyset.$$

This result was proven in dimension 1 by Zworski [70], and in certain cases by Boulton [8]. Further refinements have been obtained from Sjöstrand [56]. Similar to the discussion after Theorem 3, we can extend Theorem 4 to unbounded symbols $p \in S(T^*\mathbb{R}^d, m)$ and their corresponding quantizations.

Example 3. Recall the non-selfadjoint Harmonic oscillator $P_h = (hD_x)^2 + ix^2$ from Example 2. Here $\partial\Sigma = \mathbb{R}_+ \cup i\mathbb{R}_+$, so we see by Equation 16 that for $0 \neq z_0 \in \Sigma$

$$\frac{1}{2i}\{\overline{p}, p\}(\rho) = \{\operatorname{Re} p, \operatorname{Im} p\}(\rho) = 0, \quad \rho \in p^{-1}(z_0).$$

However,

$$\text{either } \{\operatorname{Re} p, \{\operatorname{Re} p, \operatorname{Im} p\}\}(\rho) = 4\xi^2 \neq 0,$$
$$\text{or } \{\operatorname{Im} p, \{\operatorname{Re} p, \operatorname{Im} p\}\}(\rho) = -4x^2 \neq 0,$$

indicating that z_0 is of order 2 for $p = \xi^2 + ix^2$, and Theorem 4 reveals that

$$\|(P_h - z_0)^{-1}\| \leqslant Ch^{-\frac{2}{3}}.$$

In order for a the ε-pseudospectrum of P_h to reach the boundary of Σ, we require $\varepsilon > h^{2/3}/C$.

3.4 Pseudospectra and random matrices

In this section, we present a brief discussion on pseudospectra for large $N \times N$ random matrices. One may interpret the $1/N$, where $N \gg 1$, as an analog to the semiclassical parameter. By recalling the example of the non-selfadjoint harmonic oscillator, as illustrated in Example 2, we see that pseudospectra can be very large in general. However, in a generic setting, they are typically much smaller.

Let $M \in \mathbb{C}^{N\times N}$ be a complex $N \times N$ matrix and let $s_1(M) \geqslant \ldots \geqslant s_N(M) \geqslant 0$ denotes its singular values, which are the eigenvalues of $\sqrt{M^*M}$ ordered in a decreasing manner and counting multiplicities. It should be noted that if $M - z$ is bijective for some $z \in \mathbb{C}$, then

$$\|(M - z)^{-1}\| = s_N(M - z)^{-1}.$$

In view of Equation 1, the ε-pseudospectrum of M is then characterized by the condition that $z \in \mathrm{Spec}_\varepsilon(M)$

$$z \in \mathrm{Spec}_\varepsilon(M) \iff s_N(M - z) < \varepsilon.$$

A classical result from Sankar et al. [51, Lemma 3.2] (stated there for real Gaussian random matrices) indicates that with a high probability, the smallest singular value of a deformed random matrix is not too small.

Theorem 5 ([51]). There exists a constant $C > 0$ such that the following holds true. Let $N \geqslant 2$, let X_0 be an arbitrary complex $N \times N$ matrix, and let Q be an $N \times N$ complex Gaussian random matrix, whose entries are all independent copies of a complex Gaussian random variable $q \sim \mathcal{N}_{\mathbb{C}}(0, 1)$. Subsequently, for any $\delta > 0$

$$\mathbf{P}\left(s_N(X_0 + \delta Q) < \delta t\right) \leqslant CNt^2.$$

Proof. For real matrices the proof can be found in Sankar et al. [51, Lemma 3.2], see also reference [63, Theorem 2.2]. For complex matrices a proof is presented for instance in Vogel [66, Appendix A].

Theorem 5 states us that any fixed $z \in \mathbb{C}$ is not included in the ε-pseudospectrum of $X + \delta Q$ with a probability $\geqslant 1 - CN\varepsilon^2\delta^{-2}$. This result suggests that the pseudospectrum of random matrices is typically *not too large*. Theorem 5 has received many extensions. For instance Rudelson and Vershynin [50] consider the case of random matrices with iid (independent and identically distributed) sub-Gaussian entries. Tao and Vu [62] consider iid entries with a nonzero variance. Cook [12] considers the case of random matrices whose of entries have an inhomogeneous variance profile under appropriate assumptions. We conclude this section by noting the following, quantitative outcome obtained by Tao and Vu.

Theorem 6 ([63]). Let q be a random variable with a mean zero and a bounded second moment, and let $\gamma \geqslant 1/2$, $A \geqslant 0$ be constants. Then, there exists a constant $C > 0$, depending on q, γ, and A such that the following holds true. Let Q be the random matrix of size N, whose entries are independent and identically distributed copies of q, and let X_0 be a deterministic matrix satisfying $\|X_0\| \leqslant N^\gamma$. Then,

$$\mathbf{P}\left(s_n(X_0 + Q) \leqslant n^{-\gamma(2A+2)+1/2}\right) \leqslant C\left(n^{-A+o(1)} + \mathbf{P}(\|Q\| \geqslant n^\gamma)\right). \tag{19}$$

Example 4. Consider the case where q is a random variable satisfying the moment conditions

$$\mathbb{E}[q] = 0, \quad \mathbb{E}[|q|^2] = 1, \quad \mathbb{E}[|q|^4] < +\infty. \tag{20}$$

Form [37] reveals that Equation 20 implies that $\mathbb{E}[\|Q\|] \leqslant CN^{1/2}$, which, using Markov's inequality, yields that for any $\varepsilon > 0$

$$\mathbf{P}\left[\|Q\| \geqslant CN^{1/2+\varepsilon}\right] \leqslant C^{-1}N^{-1/2-\varepsilon}\mathbb{E}[\|Q\|] \leqslant N^{-\varepsilon}. \tag{21}$$

In this case (Equation 19) becomes

$$\mathbf{P}\left(s_n(X_0 + Q) \leqslant n^{-(\varepsilon+1/2)(2A+2)+1/2}\right) \leqslant C\left(n^{-A+o(1)} + N^{-\varepsilon}\right). \tag{22}$$

4 Eigenvalue asymptotics for non-selfadjoint (random) operators

Consider the operator $P_h = p^w(x, hD_x)$ depicted in Equations 8, 11, which is viewed as an unbounded operator $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. We equip P_h with the domain $H(m) := (P_h - z_0)^{-1}L^2(\mathbb{R}^d)$. It should be noted that $(P_h - z_0)^{-1}$ exists for $h > 0$ that is sufficiently small by the elipticity condition (Equation 9). We will denote by $\|u\|_m := \|(P_h - z_0)u\|$ the associated norm on $H(m)$. Although this norm depends on the selection of the symbol $p_0 - z_0$, it is equivalent to the norm defined by any operator with an elliptic principal symbol $q \in S(m)$, so that the space $H(m)$ solely depends on the order function m. Since $H(m)$ contains the Schwartz functions $\mathcal{S}(\mathbb{R}^d)$, it is dense in $L^2(\mathbb{R}^d)$.

Let us verify that P_h equipped with domain $H(m)$ is closed. Let $(P_h - z_0)u_j \to v$ and $u_j \to u$ in L^2. Since $(P_h - z_0) : H(m) \to L^2$ is bijective, it follows that $u_j \to (P_h - z_0)^{-1}v$ in $H(m)$ and also in L^2. So $u = (P_h - z_0)^{-1}v$. In summary, P_h equipped with the domain $H(m)$ is a densely defined closed linear operator.

Recall Equation 10, and let

$$\Omega \Subset \mathbb{C}\backslash\Sigma_\infty \tag{23}$$

be open, relatively compact, not entirely contained in Σ and so that $\overline{\Omega} \subset \mathbb{C}\backslash\Sigma_\infty$. Using the ellipticity assumption (Equation 9), it was proven in reference [25, Section 3] that

- $\mathrm{Spec}(P_h) \cap \Omega$ is discrete for $h > 0$ small enough,
- For all $\varepsilon > 0$ there exists an $h(\varepsilon) > 0$ such that

  $$\mathrm{Spec}(P_h) \cap \Omega \subset \Sigma + D(0, \varepsilon), \quad 0 < h \leqslant h(\varepsilon),$$

  where $D(0, \varepsilon)$ denotes the disc in $\mathbb{C}$ of radius ε and centered at 0.

4.1 The selfadjoint setting

If P_h above is selfadjoint, which implies in particular that p is real-valued, we have the classical Weyl asymptotics. We follow here Dimassi and Sjöstrand [17] for a brief review.

Theorem 7. Let Ω be as in Equation 23. For every h-independent interval $I \subset \Omega \cap \mathbb{R}$ with $\mathrm{Vol}_{\mathbb{R}^{2d}}(\partial I) = 0$,

$$\#(\mathrm{Spec}(P_h) \cap I) = \frac{1}{(2\pi h)^d}\left(\int_{p_0^{-1}(I)} dxd\xi + o(1)\right), \quad h \to 0. \quad (24)$$

This result is, in increasing generality, attributed to Chazarin [10], Helffer and Robert [26, 27], Petkov and Robert [45] and Ivrii [35]. See also Dimassi and Sjöstrand [17] for an overview. We highlight two special cases: when $I = [a, b]$, $a < b$, and a, b are not critical points of p_0, then the error term becomes $\mathcal{O}(h)$, see Chazarin [10], Helffer-Robert [26], and Ivrii [35]. When additionally the unions of periodic H_{p_0} trajectories[4] in the energy shell $p_0^{-1}(a)$ and $p_0^{-1}(b)$ are of the Liouville measure 0, then the error term is of the form

$$h\left(\int_{p_0=a} p_1(\rho)L_a(d\rho) - \int_{p_0=b} p_1(\rho)L_b(d\rho)\right) + o(h), \quad (25)$$

where L_λ denotes the Liouville measure on $p_0^{-1}(\lambda)$. See Petkov and Robert [45] and Ivrii [35] and Dimassi and Sjöstrand [17] for details. Let us also highlight that similar results obtained from Theorem 7 are also valid for compact smooth manifolds, see, for instance, Grigis and Sjöstrand [21, Chapter 12] and the references therein.

The corresponding results in the setting of self-adjoint partial differential operators in the high energy limit go back to the seminal study of Weyl [68] and have a long and very rich history. These are, however, beyond the scope of this review.

Example 5. The guiding example to keep in mind is the self-adjoint Harmonic oscillator

$$P_h = (hD_x)^2 + x^2 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

seen as an unbounded operator. The principal symbol of P_h is represented by $p(x, \xi) = \xi^2 + x^2 \in S(T^*\mathbb{R}, m)$, and the weight function $m(x, \xi) = 1 + \xi^2 + x^2$. P_h is represented by the domain $H(m) := (P_h + 1)^{-1}L^2(\mathbb{R})$, where the operator on the right is the pseudo-differential inverse of $P_h + 1$. This choice of domain makes P_h a densely defined closed operator. It is widely acknowledged (see, for instance, reference [71, Theorem 6.2]) that the spectrum of P_h is determined by

$$\mathrm{Spec}(P_h) = \{(2n + 1)h;\, n \in \mathbb{N}\}.$$

Counting the points $(2n + 1)h$ contained in an interval $[a, b]$, $0 \leqslant a < b < \infty$, gives

$$\#(\mathrm{Spec}(P_h) \cap [a, b]) = \frac{b - a}{2h} + \mathcal{O}(1).$$

Since $\mathrm{Vol}_{\mathbb{R}^2}(\{a \leqslant \xi^2 + x^2 \leqslant b\}) = \pi(b - a)$, we confirm Theorem 7 for the Harmonic oscillator.

4 H_{p_0} denotes the Hamilton vector field induced by p_0.

4.2 The non-self-adjoint setting

The natural counterpart of Theorem 7 for non-self-adjoint operators would be eigenvalue asymptotics in a complex domain $\Omega \Subset \mathbb{C}$ as in Equation 23. Recall the non-self-adjoint Harmonic oscillator P_h from Example 2 with principal symbol $p(x, \xi) = \xi^2 + ix^2$. In this case, $\Sigma = \{z \in \mathbb{C};\, \mathrm{Re}\, z, \mathrm{Im}\, z \geqslant 0\}$ and $\Sigma_\infty = \emptyset$. Any $\emptyset \neq \Omega \Subset \Sigma$ away from the line $e^{i\pi/4}\mathbb{R}_+$, indicates the view of Equation 15 that

$$\#(\mathrm{Spec}(P_h) \cap \Omega) = 0.$$

On the other hand,

$$\frac{1}{2\pi h}\int_{p^{-1}(\Omega)} dxd\xi > 0.$$

This example suggests that a direct generalization of Theorem 7 to non-self-adjoint operators with a complex valued principal symbol cannot hold.

Let us comment on two settings where a form of Weyl asymptotics is known to hold: Upon assuming analyticity, one may recover a sort of Weyl asymptotics. More precisely, as shown in the studies of Melin and Sjöstrand [43], Sjöstrand [53], Hitrik and Sjöstrand [28–30], Hitrik et al. [31], and Rouby [49], the discrete spectrum of certain analytic non-self-adjoint pseudo-differential operators is confined to curves in Σ. Moreover, one can recover eigenvalue asymptotics using Bohr-Sommerfeld quantization conditions.

The second setting occurs when the non-self-adjointness of the operator P_h arises not from the principal symbol p_0 (assumed to be real-valued), but from the subprincipal symbol p_1. For instance, when studying the damped wave equation on a compact Riemannian manifold X, one is led to study the eigenvalues of the corresponding stationary operator

$$P_h(z) = -h^2\Delta + 2ih\sqrt{a(x)}\sqrt{z}, \quad a \in C^\infty(X; \mathbb{R}).$$

Here, Δ denotes the Laplace-Beltrami operator on X, and we call $z \in \mathbb{C}$ an eigenvalue of $P_h(z)$ if there exists a corresponding L^2 function u is present in the kernel of $P_h(z) - z$. In fact, such a u is smooth by elliptic regularity. Using Fredholm theory, one can show that these eigenvalues form a discrete set in $\mathbb{C}$.

The principal part of $P_h = P_h(z)$ is given by $-h^2\Delta$, and thus is self-adjoint. The principal symbol is $p_0(x, \xi) = |\xi|_x^2$ (the norm here is with respect to the Riemannian metric on X). However, the subprincipal part is complex valued and non-self-adjoint.

Lebeau [38] has established that there exists $a_\pm \in \mathbb{R}$, wherein for every $\varepsilon > 0$ there exist a finite number of eigenvalues such that

$$\frac{\mathrm{Im}\, z}{h} \notin [a_- - \varepsilon, a_+ + \varepsilon].$$

Remark 1. In fact Lebeau provided precise expressions for $a_\pm$ in terms of the infimum and the supremum over the co-sphere bundle S^*X of the long time average of the damping function a evolved via the geodesic flow. Further refinements have been obtained by Sjöstrand [52], and when X is negatively curved by Anantharaman [2] and Jin [36].

Additionally, Markus and Matsaev [40] and Sjöstrand [52] have demonstrated the following analog of the Weyl law. For $0 < E_1 < E_2 < \infty$ and for $C > 0$ sufficiently large

$$\begin{aligned}&\#\big(\mathrm{Spec}(P_h) \cap ([E_1, E_2] + i[-Ch, Ch])\big)\\&= \frac{1}{(2\pi h)^d}\left(\iint_{p_0^{-1}([E_1,E_2])} dxd\xi + \mathcal{O}(h)\right).\end{aligned} \tag{26}$$

Finer results have been obtained by Anantharaman [2] and Jin [36] when X is negatively curved.

4.3 Probabilistic Weyl asymptotics

In a series of studies by Hager [23–25] and Sjöstrand [54, 55], the authors proved a Weyl law, with overwhelming probability, for the eigenvalues in a compact set $\Omega \Subset \mathbb{C}$ as in Equation 23 for randomly perturbed operators

$$P^\delta = P_h + \delta Q_\omega, \quad 0 < \delta = \delta(h) \ll 1, \tag{27}$$

where P_h is as per in Section 3, and the random perturbation Q_ω is one of the following two types.

4.3.1 Random matrix

Let $N(h) \to \infty$ sufficiently fast as $h \to 0$. Let $q_{j,k}$, $0 \leqslant j, k < N(h)$ be independent copies of a complex Gaussian random variable $\alpha \sim \mathcal{N}_{\mathbb{C}}(0, 1)$. We consider the random matrix

$$Q_\omega = \sum_{0 \leqslant j,k < N(h)} q_{j,k}\, e_j \otimes e_k^*, \tag{28}$$

where $\{e_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R}^d)$ is an orthonormal basis and $e_j \otimes e_k^* u = (u|e_k)e_j$ for $u \in L^2(\mathbb{R})$. The condition on $N(h)$ is determined by the requirement that the microsupport of the vectors in the orthonormal system $\{e_j\}_{j<N(h)}$, "covers" the compact set $p_0^{-1}(\Omega) \subset T^*\mathbb{R}^d$, where p_0 is the principal symbol of P_h. For instance, we could consider the first $N(h)$ eigenfunctions (ordered according to increasing eigenvalues) of the Harmonic oscillator $P_h = -h^2\Delta + x^2$ on $\mathbb{R}^d$. The number $N(h)$ is then determined by the condition that the semiclassical wavefront sets of e_j, $j \geqslant N(h)$, are disjoint from $p_0^{-1}(\Omega)$. Alternatively, as in Hager and Sjöstrand [25], one may also take $N(h) = \infty$; however, then one must conjugate Q_ω by suitable elliptic Hilbert–Schmidt operators. We recommend the reader to Hager and Sjöstrand [25] for further information.

4.3.2 Random potential

We take $N(h)$ and an orthonormal family $(e_k)_{k\in\mathbb{N}}$ as above. Let v be real or complex random vector in $\mathbb{R}^{N(h)}$ or $\mathbb{C}^{N(h)}$, respectively, with joint probability law

$$v_*(d\mathbf{P}) = Z_h^{-1}\, \mathbf{1}_{B(0,R)}(v)\, \mathrm{e}^{\phi(v)} L(dv), \tag{29}$$

where $Z_h > 0$ is a normalization constant, $B(0, R)$ is either the real ball $\Subset \mathbb{R}^{N(h)}$ or the complex ball $\Subset \mathbb{C}^{N(h)}$ of radius $R = R(h) \gg 1$, and centered at 0, $L(dv)$ denotes the Lebesgue measure on either $\mathbb{R}^{N(h)}$ or $\mathbb{C}^{N(h)}$ and $\phi \in C^1$ with

$$\|\nabla_v \phi\| = \mathcal{O}(h^{-\kappa_4}) \tag{30}$$

uniformly, for an arbitrary but fixed value of $\kappa_4 \geqslant 0$. In Hager [24] the case of non-compactly supported probability law was considered. More precisely, the entries of the random vector v were supposed to be independent and identically distributed (iid) complex Gaussian random variables $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$. In Sjöstrand [54, 55], the law Equation 29 was considered. For the sake of simplicity, we will not elaborate here the precise conditions on the e_k, $R(h)$, and $N(h)$, in this case, but refer the reader to Sjöstrand [54, 55]. However, one example of a random vector v with law (Equation 30) is a truncated complex or real Gaussian random variables with expectation 0, and uniformly bounded covariances. In fact, the methods in Sjöstrand [54, 55] can be extended to non-compactly supported probability distributions, provided sufficient decay conditions at infinity are assumed. For instance, iid complex Gaussian random variables, as in the one dimensional case [24], are permissable. Finally, we conclude that the methods in Sjöstrand [54, 55] can probably also be modified to allow for the case of more general independent and identically distributed random variables. We define the random function as

$$V_\omega = \sum_{0 \leqslant j < N(h)} v_j\, e_j. \tag{31}$$

We call this perturbation a "random potential," even though V_ω is complex valued. When we consider this type of perturbation, we will make the additional symmetry assumption:

$$p(x, \xi; h) = p(x, -\xi; h). \tag{32}$$

Let $\Omega \Subset \mathbb{C}$ be an open simply connected set as in Equation 23. For $z \in \Omega$ and $0 \leqslant t \ll 1$ we set

$$V_z(t) = \mathrm{Vol}\{\rho \in T^*\mathbb{R}^d;\ |p_0(\rho) - z|^2 \leqslant t\}. \tag{33}$$

Let $\Gamma \Subset \Omega$ be open with $\mathcal{C}^2$ boundary and make the following non-flatness assumption

$$\begin{aligned}&\exists \kappa \in]0, 1], \text{ such that } V_z(t) = \mathcal{O}(t^\kappa),\\&\text{uniformly for } z \in \mathrm{neigh}(\partial\Gamma),\ 0 \leqslant t \ll 1.\end{aligned} \tag{34}$$

The above mentioned works have yielded the following result.

Theorem 8 (Probabilistic Weyl's law). Let Ω be as in Equation 23. Let $\Gamma \Subset \Omega$ be open with $\mathcal{C}^2$ boundary. Let P_h^δ be a randomly perturbed operators as in Equation 27 with $\mathrm{e}^{-1/Ch} \ll \delta \leqslant h^\theta$ with $\theta > 0$ sufficiently large. Then, in the limit $h \to 0$,

$$\begin{aligned}\#\big(\mathrm{Spec}(P_h^\delta) \cap \Gamma\big) &= \frac{1}{(2\pi h)^d}\left(\iint_{p_0^{-1}(\Gamma)} dxd\xi + o(1)\right)\\&\quad\text{with probability} \geqslant 1 - Ch^\eta,\end{aligned} \tag{35}$$

for some fixed $\eta > 0$.

The studies [23–25, 54, 55] also provide an explicit control over θ, the error term in Weyl's law, and the error term in the probability estimate. Theorem 8 is remarkable because such Weyl laws are typically a feature of self-adjoint operator, whereas in the non-selfadjoint case they generally fail. Indeed, as laid out in Section 4.2, the discrete spectrum of the (unperturbed) non-selfadjoint operator P_h is usually localized to curves in the pseudospectrum

Σ, see Melin and Sjöstrand [43], Hitrik and Sjöstrand [28–31], and Rouby [49]. In contrast, Theorem 8 shows that a "generic" perturbation of size $\mathcal{O}(h^\infty)$ is sufficient for the spectrum to "fill out" Σ.

To illustrate this phenomenon, recall the non-selfadjoint harmonic oscillator $P_h = -h^2\partial_x^2 + ix^2$ on $\mathbb{R}$ from Example 2. Its spectrum is given by $\{e^{i\pi/4}(2n+1)h; n \in \mathbb{N}\}$ [14] on the line $e^{i\pi/4}\mathbb{R}_+ \subset \mathbb{C}$. The Theorem 8 shows that a "generic" perturbation of arbitrarily small size is sufficient to produce spectrum roughly equidistributed in any fixed compact set in its classical spectrum Σ, which is in this case the upper right quadrant of $\mathbb{C}$.

As observed in Christiansen and Zworski [11], the real analytic p condition (Equation 34) consistently holds for some $\kappa > 0$. Similarly, when p is truly analytical and such that $\Sigma \subset \mathbb{C}$ has non-empty interior, then

$$\forall z \in \partial\Omega: \; dp\restriction_{p^{-1}(z)} \neq 0 \quad \Longrightarrow \quad (4.12) \text{ holds with } \kappa > 1/2. \tag{36}$$

For smooth p, we have that when for every $z \in \partial\Omega$

$$dp, d\overline{p} \text{ are linearly independent at every point of } p^{-1}(z), \text{ then } (4.12) \text{ holds with } \kappa = 1. \tag{37}$$

Observe that dp and $d\overline{p}$ are linearly independent at ρ when $\{p,\overline{p}\}(\rho) \neq 0$, where $\{a,b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b$ denotes the Poisson bracket. Moreover, in dimension $d = 1$, the condition $\{p,\overline{p}\} \neq 0$ on $p^{-1}(z)$ is equivalent to dp, with $d\overline{p}$ being linearly independent at every point of $p^{-1}(z)$. However, in dimensions $d > 1$, this cannot in hold general, as the integral of $\{p,\overline{p}\}$ with respect to the Liouville measure on $p^{-1}(z)$ vanishes on every compact connected component of $p^{-1}(z)$, see reference [42, Lemma 8.1]. Furthermore, condition (Equation 37) cannot hold when $z \in \partial\Sigma$. However, some iterated Poisson brackets may not have zero there. For example, it has been observed in [25, Example 12.1] that if

$$\forall \rho \in p^{-1}(\partial\Omega): \; \{p,\overline{p}\}(\rho) \neq 0 \text{ or } \{p,\{p,\overline{p}\}\}(\rho) \neq 0, \quad \text{then } (4.12) \text{ holds with } \kappa = \frac{3}{4}. \tag{38}$$

4.3.3 Related results

Theorem 8 has also been extended to the case of elliptic semiclassical differential operators on compact manifolds by Sjöstrand [55], to the Toeplitz quantization of the torus by Christiansen and Zworski [11] and Vogel [66], and to general Berezin-Toeplitz quantizations on compact Kähler manifolds by Oltman [44] in the context of complex Gaussian noise. A further extension of Theorem 8 has been achieved by Becker, Oltman and the author in Becker et al. [6]. There we prove a probabilistic Weyl law for the non-selfadjoint off-diagonal operators of the Bistritzer-MacDonald Hamiltonian [7] for twisted bilayer graphene, see also Cancés et al. [9] and Watson et al. [67], subject to random tunneling potentials. This probabilistic Weyl has an interesting physical consequence as it demonstrates the instability of the so-called *magic angels* for this model of twisted bilayer graphene. Similar results have been achieved in random matrix theory. The case of Toeplitz matrices is represented by symbols on $\mathbb{T}^2$ of the form $\sum_{n\in\mathbb{Z}} a_n e^{in\xi}$, $(x,\xi) \in \mathbb{T}^2$, has been conducted in a series of recent studies by Śniady [61], Davies and Hager [15], Guionnet et al. [22], Basak et al. [4, 5], Sjöstrand and the author of this text [57–59]. Such symbols amount to the case of symbols which are constant in the x variable. In these studies the non-selfadjointness of the problem, however, does not come from the symbol itself, but from the boundary conditions destroying it. The periodicity of the symbol in x is achieved by allowing for a discontinuity. Nevertheless, these studies demonstrate that by adding a small random matrix, the limit of the empirical eigenvalues counting measure μ_N of the perturbed operator converges in probability (or even almost surely in some cases) to $p_*(d\rho)$.

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