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A Boolean sum interpolation for multivariate functions of bounded variation

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This paper deals with the approximation error of trigonometric interpolation for multivariate functions of bounded variation in the sense of Hardy-Krause. We propose interpolation operators related to both the tensor product and sparse grids on the multivariate torus. For these interpolation processes, we investigate the corresponding error estimates in the L_p norm for the class of functions under consideration. In addition, we compare the accuracy with the cardinality of these grids in both approaches.

KEYWORDS

Boolean sum operator, multivariate function of bounded variation, interpolation problem, sparse grid, tensor product grid, hyperbolic cross

1 Introduction

The interpolation of periodic functions at equidistant nodes by trigonometric polynomials is a basic task of approximation theory with far-reaching applications (see, e.g., Chapter 3 in Plonka et al. [2]). The possibility of using FFT algorithms with huge amounts of data has contributed greatly to the popularity of this approximation method. Accordingly, error estimates for such interpolation methods have been intensively studied in the literature. The decisive difference between approximation methods which are based on integral evaluations of the given function f, for example, the Fourier coefficients, and an interpolation method is that information about f must really be available pointwise. This difference becomes particularly important in the case of interpolation of discontinuous functions, where one will focus on the error in L_p norms in particular. As is well-known, the Riemann integrability of a periodic function f is a condition for the L_p error to tend to 0 as the number of nodes $n \to \infty$ (cf. [3]). For a little more smoothness, the approximation order in L_p can be bounded by the best one-sided approximation in L_p using trigonometric polynomials (cf. [4]).

A particularly important class of functions, generally discontinuous functions, for which one would like to obtain error estimates are functions of bounded variation. A first result in this area comes from Zacharias, who proved in [5] with Hilbert space methods that the L_2 error behaves like $1/\sqrt{n}$. This result was generalized to $1 \leq p < \infty$ in Prestin [6].

To generalize these error estimates to multivariate periodic functions, a suitable concept for multivariate bounded variation is required. The Hardy-Krause definition is appropriate here (see Clarkson and Adams [7] and for more information on these spaces [8], [9] and others). For the dimension $d = 2$ and interpolation on the tensor product, such results can be found in Prestin and Tasche [10], (see also Kolomoitsev et al. [11]). An essential tool for the proof of the error estimates is the consideration of blending operators, which have been extensively analyzed in the study of Delvos et al. (cf., e.g., [12–15]).

In this study, the results for the approximation error of functions of bounded variation are to be transferred to interpolation methods on sparse grids. Such grids were first introduced in Smolyak [16] and since then have been widely used in interpolation problems, quadrature schemes, and other fields. For more details, see Dũng et al. [17]. These sparse grids are very efficient, especially for large spatial dimensions d, that is, the approximation order is only reduced by a logarithmic factor compared to the tensor product interpolation, although the number of interpolation nodes is only by a log factor bigger than in the univariate case. At this point, it should be noted that error estimates for such interpolation methods of continuous functions are known (see Dũng et al. [17, Chap. 5.3]). Such statements are proved for functions belonging to the spaces $\mathbf{H}^r_p$, where $r > 1/p$ is assumed, which implies the continuity of the function to be interpolated. Our larger class of functions of bounded variation then provides an order of convergence as in the case $r = 1/p$. Our approach requires a notation for the definition of bounded variation that is well-suited for large dimensions d. Here, we follow the approach in Aistleitner et al. [1].

Finally, we note that these approximation results for functions of bounded variation are also valid for Fourier sums and the corresponding multivariate hyperbolic cross-variants, where the results can also be obtained using other methods.

2 Function of bounded variation

Let $p \in [1,\infty)$, $d \in \mathbb{N}$. For 2π-periodic functions f of d variables on the torus $\mathbb{T}^d$, we consider the space $L_p(\mathbb{T}^d)$, $1 \le p < \infty$, supplied by the following norm:

$$\|f\|_p := \left(\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} |f(\mathbf{z})|^p \mathrm{d}\mathbf{z}\right)^{\frac{1}{p}} < \infty.$$

We denote by $D = \{1,\dots,d\}$ the set of coordinates with cardinality $|D| = d$ and split it into two domains $B \subset D$ and $\overline{B} = D\backslash B$, $|B|+|\overline{B}| = d$. Following Aistleitner et al. [1] by $\mathbf{z} = \mathbf{y}^B:\mathbf{x}$, where $\mathbf{y},\mathbf{x} \in \mathbb{T}^d$, we describe the vector $\mathbf{z} \in \mathbb{T}^d$ consisting of the components $z^j = y^j$ if $j \in B$ and $z^j = x^j$ otherwise. Such a partition will also be used to represent the vector $\mathbf{z} \in \mathbb{T}^d$ as a combination of arguments from B and fixed values along coordinates from $\bar{B}$.

For each coordinate $j = 1,...,d$ we introduce some arbitrary decomposition $\mathcal{Z}_j$, namely

$$\mathcal{Z}_j : 0 = \xi^j_1 < \ldots < \xi^j_{u_j} = 2\pi.$$

Let $\xi = (\xi^1_{k_1},\xi^2_{k_2},\dots,\xi^d_{k_d}) \in \mathbb{T}^d$ be a vector with components $\xi^j_{k_j} \in \mathcal{Z}_j$, $k_j = 1,\dots,u_j$ and $\xi_+ = ((\xi^1_{k_1})_+,(\xi^2_{k_2})_+,\dots,(\xi^d_{k_d})_+) \in \mathbb{T}^d$, where

$$(\xi^j_{k_j})_+ = \begin{cases} \xi^j_{k_j+1}, & k_j < u_j, \\ 2\pi, & \text{otherwise.}\end{cases}$$

Using this notation for a function $f:\mathbb{T}^d \to \mathbb{C}$, we introduce a d-dimensional difference operator in the following way:

$$\Delta_D(f) = \sum_{\boldsymbol{\xi}\in\prod\limits_{j\in D}\mathcal{Z}_j} \left|\sum_{\emptyset\subseteq U\subseteq D} (-1)^{|U|} f(\boldsymbol{\xi}^U:\boldsymbol{\xi}_+)\right|.$$

Furthermore, we consider the difference operator and corresponding variation for $f:\mathbb{T}^d \to \mathbb{C}$ with respect to coordinates $j \in B$ and fixed values z^j for $j \in \overline{B}$:

$$\Delta_B(f,\mathbf{z}^{\overline{B}}) = \sum_{\boldsymbol{\xi}\in\prod\limits_{j\in B}\mathcal{Z}_j} \left|\sum_{\emptyset\subseteq U\subseteq B} (-1)^{|U|} f((\boldsymbol{\xi}^U:\boldsymbol{\xi}_+)^B:\mathbf{z})\right|.$$

Then, we define for all $B \subseteq D$:

$$V^B f(\mathbf{z}^{\overline{B}}) = \sup_{\mathcal{Z}_j, j\in B} \Delta_B(f,\mathbf{z}^{\overline{B}}).$$

In particular, $V^{\emptyset}f(\mathbf{z}) = f(\mathbf{z})$.

For a function $V^B f(\mathbf{z}^{\overline{B}}) \in L_p(\mathbb{T}^{d-|B|})$, we have

$$\|V^B f\|_p = \left(\frac{1}{(2\pi)^{d-|B|}}\int\limits_{\mathbb{T}^{d-|B|}} |V^B f(\mathbf{z}^{\overline{B}})|^p \mathrm{d}\mathbf{z}^{\overline{B}}\right)^{\frac{1}{p}}$$

for $1 \le p < \infty$ and $\|V^B f\|_\infty = \sup\limits_{\mathbf{z}^{\overline{B}}\in\mathbb{T}^{d-|B|}} V^B f(\mathbf{z}^{\overline{B}})$ for $p = \infty$.

Let us mention that for $B = D$, the variation $V^D f(z^{\overline{B}})$ is a constant, which we simply denote as $V^D f$.

Then, the total variation of a function $f:\mathbb{T}^d \to \mathbb{C}$ is determined by the quantity

$$HV(f) = \sum_{\emptyset\subset B\subseteq D} \|V^B f\|_\infty.$$

A function $f:\mathbb{T}^d \to \mathbb{C}$ for which $HV(f)$ is finite we call function of bounded variation on $\mathbb{T}^d$ in the sense of Hardy-Krause and write $f \in HV(\mathbb{T}^d)$.

Remark 2.1. An alternative definition of this kind of bounded variation is discussed in Bakhvalov [18, Lemma 4]. So, $f \in HV(\mathbb{T}^d)$ if $V^D f < \infty$ and for any $j \in D$ there are z^j_0 such that $f(z^j_0:\mathbf{z}) \in HV(\mathbb{T}^{d-1})$, that is, f has bounded variation up to coordinates $i \in D\setminus\{j\}$.

Remark 2.2. Let $d > 1$. By definition $f \in HV(\mathbb{T}^d)$ iff $\|V^B f\|_\infty$ is finite for all $B \subseteq D$. All these 2^d conditions are pairwise independent of each other as can be seen by the following examples [for the case $d = 2$ cf. ([7], p. 827)].

Let $B_1 \neq B_2$ be arbitrary subsets of D. W.l.o.g. we assume $1 \in B_1$, $1 \notin B_2$ and we distinguish the 4 possible cases:

a) $2 \in B_1 \cap B_2$, b) $2 \in B_1, 2 \notin B_2$, c) $2 \notin B_1, 2 \in B_2$, d) $2 \notin B_1 \cup B_2$.

Now, we consider functions $F:\mathbb{T}^d \to \mathbb{C}$ of the form

$$F(\mathbf{x}) = f(\alpha,\beta)\prod_{k=3}^{d} g_k(x^k)$$

with $g_k \in HV(\mathbb{T}^1)$ and $0 < V_0^{2\pi}(g_k) < \infty$ for all $k = 3,\dots,d$, where $V_0^{2\pi}$ denotes the one-dimensional total variation on $[0,2\pi]$. If $D \supseteq A = B\cup C$ with $B \subseteq \{1,2\}$ and $C \subseteq \{3,\dots,d\}$, then

$$\|V^A F\|_\infty = \|V^B f\|_\infty \prod_{k\in C} V_0^{2\pi}(g_k) \prod_{k>2,k\notin C} \sup_{z\in\mathbb{T}} |g_k(z)|.$$

Hence, $\|V^A F\|_\infty$ is finite, if $\|V^B f\|_\infty$ is finite.

As examples $f_j : \mathbb{T}^2 \to \mathbb{C}, j = 1,2,3,4$ we choose

$$f_1(\alpha,\beta) = \begin{cases} 1, \text{ if } 0 < \alpha < \beta < 2\pi, \\ 0, \text{ otherwise in } [0,2\pi)^2, \end{cases}$$

$$f_2(\alpha,\beta) = \begin{cases} \frac{1}{\beta}, \text{ if } 0 < \beta < 2\pi, \\ 0, \text{ otherwise in } [0,2\pi)^2, \end{cases}$$

$$f_3(\alpha,\beta) = f_4(\beta,\alpha) = \begin{cases} \sin\frac{1}{\alpha}, \text{ if } 0 < \alpha < 2\pi, \\ 0, \quad \text{otherwise in } [0,2\pi)^2. \end{cases}$$

On the one hand, we conclude for

$$\text{a), b)} \begin{array}{l} \|V^{B_1} f_1\|_\infty = \infty \\ \|V^{B_2} f_1\|_\infty = 1 \end{array}, \quad \text{for c)} \begin{array}{l} \|V^{B_1} f_3\|_\infty = \infty \\ \|V^{B_2} f_3\|_\infty = 0 \end{array},$$

$$\text{for d)} \begin{array}{l} \|V^{B_1} f_3\|_\infty = \infty \\ \|V^{B_2} f_3\|_\infty = 1 \end{array}.$$

On the other hand, we conclude for

$$\text{a), b), d)} \begin{array}{l} \|V^{B_1} f_2\|_\infty = 0 \\ \|V^{B_2} f_2\|_\infty = \infty \end{array}, \quad \text{for c)} \begin{array}{l} \|V^{B_1} f_4\|_\infty = 0 \\ \|V^{B_2} f_4\|_\infty = \infty \end{array}.$$

The main aim of our investigation is to study the approximation order of trigonometric interpolation processes on tensor product and sparse grids for multivariable functions $f \in HV(\mathbb{T}^d)$.

3 Interpolation on the tensor product grid

In this section, we study an interpolation operator for multivariable functions on tensor product grids. Our approach continues the investigations in Prestin [6] and Prestin and Tasche [10], where the trigonometric interpolation for univariate and bivariate functions and the corresponding approximation bounds were established.

Let T_n^d be the space of trigonometric polynomials such that

$$T_n^d := \operatorname{span}\{\, e^{\mathbf{ikx}}, |\mathbf{k}|_\infty \leq 2^n\}.$$

We define a set of an odd number of equidistant nodes in direction x^j by

$$X_n^j := \{x_k^j = \frac{2k\pi}{2^{n+1}+1}, \quad k = 0, ..., 2^{n+1}\}. \tag{1}$$

Then, the tensor product $\otimes_{j=1}^d X_n^j$ is called a full interpolation grid on $\mathbb{T}^d$.

For an univariate bounded function $f : \mathbb{T} \to \mathbb{C}$, the interpolation operator L_n is of the form

$$L_n f(x) = \frac{2}{2^{n+1}+1} \sum_{k=0}^{2^{n+1}} f(x_k) K_n(x - x_k),$$

where

$$K_n(x) = \frac{1}{2} + \sum_{j=1}^{2^n} \cos jx = \frac{1}{2} \sum_{j=-2^n}^{2^n} e^{ijx} \tag{2}$$

is the 2^n-th Dirichlet kernel. For a multivariate function $f : \mathbb{T}^d \to \mathbb{C}$, the corresponding interpolation operator with respect to the coordinate j takes the form

$$\begin{aligned} L_{r_j}^j f(\mathbf{x}) &:= I \otimes \ldots \otimes L_{r_j} \otimes \ldots \otimes I f(\mathbf{x}) \\ &= \frac{2}{2^{r_j+1}+1} \sum_{i=0}^{2^{r_j+1}} f(x_i^j : \mathbf{x}) K_{r_j}(x^j - x_i^j), \end{aligned}$$

where I is the identity operator and $A \otimes B$ is the algebraic tensor product of A and B.

It is obvious that the operator $L_{r_j}^j$ satisfies the interpolation conditions

$$L_{r_j}^j f(x_i^j : \mathbf{x}) = f(x_i^j : \mathbf{x}), \quad i = 0, \ldots, 2^{r_j+1} \tag{3}$$

for each $j = 1, \ldots, d$.

Let us consider the tensor product of interpolation operators with respect to arguments belonging to the set $B \subseteq D$, that is, we define the corresponding interpolation operator for the grid $\otimes_{j\in B} X_{r_j}^j$ as

$$L^B = \bigotimes_{j\in B} L_{r_j}^j.$$

Moreover, the interpolation property

$$L^B f(\mathbf{x}_0^B : \mathbf{x}) = f(\mathbf{x}_0^B : \mathbf{x})$$

holds for any $\mathbf{x}_0^B \in \otimes_{j\in B} X_{r_j}^j$.

Furthermore, we give the representation for the operator L^B by its Fourier series. Let $\mathbf{k} = \{k^j\}_{j\in B}$ and $|k_j|_\infty \leq 2^{r_j}$. So, using Equation 2 we immediately get that

$$L^B f(\mathbf{x}) = \sum_{j\in B} \sum_{\mathbf{k}=-2^{\mathbf{r_j}}}^{2^{\mathbf{r_j}}} c_{\mathbf{k}}^B e^{\mathbf{ikx}^B}$$

with

$$c_{\mathbf{k}}^B f(\mathbf{x}) = \prod_{j\in B} \frac{1}{(2^{r_j+1}+1)} \sum_{\mathbf{x}_0^B \in \otimes_{j\in B} X_{r_j}^j} f(\mathbf{x}_0^B : \mathbf{x}) e^{\mathbf{ik}(\mathbf{x}_0^B)}.$$

We also introduce the intermediate interpolation operator often called blending operator, namely

$$M^B = \bigoplus_{j\in B} L_{r_j}^j,$$

where $A \oplus C = A + C - AC$ is the boolean sum operation. As is known (cf. [14], p. 141), the sum representation for M^B is

$$M^B = \sum_{k=1}^{|B|} (-1)^{k-1} \left(\sum_{U=\{j_1,j_2,\ldots j_k\}, U\subseteq B, |U|=k} L_{r_{j_1}}^{j_1} L_{r_{j_2}}^{j_2} \cdots L_{r_{j_k}}^{j_k} \right)$$

and for the remainder operator, we have the product representation

$$I - M^B = \prod_{j\in B}(I - L^j_{r_j}).$$

In the next theorem, we establish the approximation property of the blending interpolation operator on a $|B|$-variate tensor product grid.

Theorem 3.1. Let $f \in \mathbb{T}^d \to \mathbb{C}$, $1 < p < \infty$ and $B \subseteq D$ be some index set. If $\|V^U f\|_p$ for all $U \subseteq B$ exists and is a finite number, then it holds true that

$$\|f - M^B f\|_p \le c\|V^B f\|_p \prod_{j\in B}(2^{r_j+1}+1)^{-1/p}, \tag{4}$$

where c is some constant depending only on p and $|B|$.

Proof. For a univariate function $f : \mathbb{T} \to \mathbb{C}$ in Prestin [6], it was proved that for $1 < p < \infty$ the inequality

$$\|f - L^j_{r_j} f\|_p \le c(2^{r_j+1}+1)^{-1/p} V_0^{2\pi}(f) \tag{5}$$

holds with some constant c depending only on p.

Let $B = \{j_1, j_2, \ldots j_q\}$. Thus, using Lemma 2 in Prestin and Tasche [10] and Equation 5 by $|B|$ times, we immediately get that

$$\left\|\prod_{j\in B}(I - L^j_{r_j})f\right\|_p \le c(2^{r_{j_1}+1}+1)^{-1/p}$$
$$\left\|V^{\{j_1\}}\left(\prod_{j\in B\setminus\{j_1\}}(I - L^j_{r_j})f(\mathbf{x}^{D\setminus\{j_1\}})\right)\right\|_p$$

$$\le \ldots \le c\prod_{j\in B}(2^{r_j+1}+1)^{-1/p}\|V^B f(\mathbf{x}^{\overline{B}})\|_p \tag{6}$$

what has to be proved.

Corollary 3.2. In the case of $B = D$, **Theorem 3.1** states that

$$\|f - M^D f\|_p \le cV^D f \prod_{j\in D}(2^{r_j+1}+1)^{-1/p}$$

and for $r_j = n$ for all $j \in D$ we immediately have

$$\|f - M^D f\|_p \le c(2^{n+1}+1)^{-d/p} V^D f.$$

Theorem 3.3. Let $f \in HV(\mathbb{T}^d)$ and $1 < p < \infty$. Then,

$$\|(I - L^B)f\|_p \le \sum_{\emptyset\subset U\subseteq B}\prod_{j\in U}(2^{r_j+1}+1)^{-1/p}\|V^U f\|_p. \tag{7}$$

Proof. According to Delvos [14, Proposition 4.1], we can express the remainder as a combination of the remainders of blending operators with lower dimensions:

$$I - L^B = \sum_{k=1}^{|B|}\sum_{U\subseteq B, |U|=k}(-1)^{k-1}(I - L^{j_1}_{r_{j_1}})(I - L^{j_2}_{r_{j_2}})\cdots(I - L^{j_k}_{r_{j_k}}).$$

Then, the proof follows the same estimate as Equation 6.

Corollary 3.4. In the case of $B = D$ for a function $f \in HV(\mathbb{T}^d)$, the inequality (Equation 7) takes the form

$$\|(I - L^D)f\|_p \le c\sum_{\emptyset\subset B\subseteq D}\prod_{j\in B}(2^{r_j+1}+1)^{-1/p}\|V^B f\|_p.$$

Furthermore, if $r_j = n$ for all $j \in D$, then **Theorem 3.3** implies that

$$\begin{aligned}\|(I - L^D)f\|_p &\le c\sum_{\emptyset\subset B\subseteq D}(2^{n+1}+1)^{-|B|/p}\|V^B f\|_p\\ &\le c2^{-n/p}HV(f).\end{aligned}$$

Remark 3.5. In the case $p = 1$, the inequality Equation 5 has the form

$$\|f - L^j_{r_j} f\|_1 \le cr_j(2^{r_j+1}+1)^{-1}V_0^{2\pi}(f)$$

and Equations 4, 7 read as follows:

$$\|f - M^B f\|_1 \le c\prod_{j\in B} r_j(2^{r_j+1}+1)^{-1}\|V^B f\|_1$$

and

$$\|f - L^B f\|_1 \le \sum_{\emptyset\subset U\subseteq B}\prod_{j\in U} r_j(2^{r_j+1}+1)^{-1}\|V^U f\|_1,$$

respectively.

Remark 3.6. For $f \in L_1(\mathbb{T}^d)$, we consider the $\mathbf{m}$-th Fourier coefficients

$$c_{\mathbf{m}}(f) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(\mathbf{z})e^{-i\mathbf{m}\mathbf{z}}d\mathbf{z}, \qquad \mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d.$$

With $B(\mathbf{m}) \subseteq D$, we denote the set of indices j such that $m_j \ne 0$. Then, according to Fülöp and Móricz [19] for all $\mathbf{m} \in \mathbb{Z}^d$, the trigonometric Fourier coefficients $c_{\mathbf{m}}(f)$ of $f \in HV(\mathbb{T}^d)$ can be estimated by

$$|c_{\mathbf{m}}(f)| \le \frac{\|V^{B(\mathbf{m})}f\|_1}{(2\pi)^{|B(\mathbf{m})|}\prod_{j\in B(\mathbf{m})}|m_j|}. \tag{8}$$

This estimate is best possible, as demonstrated by the example

$$f(\mathbf{z}) = \prod_{j\in B}\chi_{[0,\pi/m_j]}(z_j), \tag{9}$$

where we have equality in Equation 8.

For $p = 2$, we want to compare the tensor product interpolation with the best approximation. The best approximation in the Hilbert space $L_2(\mathbb{T})^d$ is given by the Fourier partial sum

$$S_n f(x) = \sum_{\mathbf{m}\in T_n^d} c_{\mathbf{m}}(f)e^{i\mathbf{m}\mathbf{x}}.$$

By Parseval equation, we estimate

$$\begin{aligned}\|f - S_n f\|_2^2 &= \sum_{|\mathbf{m}|_\infty > n} |c_{\mathbf{m}}(f)|^2 \\ &\le \sum_{r=1}^{d} \sum_{\substack{|\mathbf{m}|_\infty > 2^n \\ |B(\mathbf{m})| = r}} \frac{\| V^{B(\mathbf{m})} f\|_1^2}{(2\pi)^{2r} \prod_{j \in B(\mathbf{m})} |m_j|^2} \\ &\le \sum_{r=1}^{d} \sum_{|B| = r} \frac{\| V^B f\|_1^2}{(2\pi)^{2r}} \left(\frac{1}{2^{n-1}} \right)^r .\end{aligned}$$

Hence,

$$\|f - S_n f\|_2 \le \sum_{B \ne \emptyset} \frac{\| V^B f\|_1}{(\sqrt{2}\pi)^{|B|} 2^{n|B|/2}} \le \frac{HV(f)}{\pi \sqrt{2^{n+1}}}.$$

Based on the examples provided in Equation 9, it is evident that the order of this estimate cannot be improved.

4 Interpolation on the sparse grid

In the following section, we study an interpolation operator on a sparse grid related to a corresponding Boolean sum operator for the d-dimensional case. Our error estimates for functions of bounded variation complement the results proved in Baszenski and Delvos [12, 13].

To construct a chain of interpolation operators, we consider for each coordinate $j \in D$ the following set of an even number of equidistant nodes:

$$\tilde{X}_n^j := \{x_k^j = \frac{2k\pi}{2^{n+1}}, \quad k = 0, ..., 2^{n+1} - 1\}. \tag{10}$$

It is known that for a univariate bounded function $f : \mathbb{T} \to \mathbb{C}$, the interpolation operator $\tilde{L}_n$ on the grid (Equation 1) has the form

$$\tilde{L}_n f(x) = \frac{1}{2^n} \sum_{k=0}^{2^{n+1}-1} f(x_k) K_n^\star(x - x_k),$$

where

$$K_n^\star(x) = \frac{1}{2} + \sum_{k=1}^{2^n - 1} \cos kx + \frac{1}{2} \cos nx$$

is the 2^n-th modified Dirichlet kernel. In the same way as it was done in Section 3, we will introduce the operators $\tilde{L}^B$ and $\tilde{M}^B$. Then, the same error estimates are obtained for these approximation methods as in Section 3. The only change is the error estimate for the one-dimensional interpolation. Here, one can refer to Corollary 3.6 in Prestin and Xu [4], where the exact error bound is derived although no explicit constants are given.

Remark 4.1. It is well-known that $K_n^\star$ is a Lagrange basis function for system of nodes (Equation 1). It is easy to check that for any $m \ge 1$ the relation $\mathrm{Im}\tilde{L}_n \subset \mathrm{Im}\tilde{L}_{n+m}$ as well as $\tilde{X}_n \subset \tilde{X}_{n+m}$ are satisfied. Then taking into account Remark 2.2 [13] we have that for operators $\tilde{L}_n$ and $\tilde{L}_{n+m}$ the ordering $\tilde{L}_n < \tilde{L}_{n+m}$ and the relation

$$\tilde{L}_{n+m} \tilde{L}_n = \tilde{L}_n \tilde{L}_{n+m} = \tilde{L}_n \tag{11}$$

hold for all n such that $0 \le n < n + m$.

Now, we introduce a d-dimensional Boolean sum interpolation operator of n-th order in the following way

$$G_n^d = \bigoplus_{r_1 + r_2 + ... + r_d = n} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \dots \tilde{L}_{r_d}^d .$$

In an analogous manner as in Section 3, a partial variant G_n^B with $B \subset D$ can be introduced here and error estimates can be proven. The approach remains the same. To simplify the notation, we therefore restrict ourselves to the case $B = D$.

To determine the set of interpolation points of the operator G_n^d, we note (cf. [15]) that the grid for the operator $\tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \dots \tilde{L}_{r_d}^d$ is $\tilde{X}_{r_1}^1 \times \tilde{X}_{r_2}^2 \times \dots \times \tilde{X}_{r_d}^d$ and for $\tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \dots \tilde{L}_{r_d}^d \oplus \tilde{L}_{l_1}^1 \tilde{L}_{l_2}^2 \dots \tilde{L}_{l_d}^d$ is

$$\tilde{X}_{r_1}^1 \times \tilde{X}_{r_2}^2 \times \dots \times \tilde{X}_{r_d}^d \cup \tilde{X}_{l_1}^1 \times \tilde{X}_{l_2}^2 \times \dots \times \tilde{X}_{l_d}^d .$$

Thus, for the operator G_n^d, we have the sparse grid of n-th order in the following form

$$\tilde{X}_{\text{sparse}}^n := \bigcup_{r_1 + r_2 + ... + r_d = n} \bigotimes_{j = 1, ..., d} \tilde{X}_{r_i}^j .$$

Due to Equation 3, it follows that G_n^d interpolates f on each point such that $\mathbf{x}_0 \in \tilde{X}_{\text{sparse}}^n$, that is,

$$G_n^d f(\mathbf{x}_0) = f(\mathbf{x}_0)$$

for all $x_0 \in \tilde{X}_{\text{sparse}}^n$.

Taking into account (Equation 11), we have the sum representation (cf. [13])

$$G_n^d = \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} \sum_{r_1 + r_2 + ... + r_d = n - j} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \dots \tilde{L}_{r_d}^d .$$

Remark 4.2. If we put $d = 2$, then the operator G_n^2 has the form (see for details [13]):

$$G_n^2 f = \sum_{r_1 + r_2 = n} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 f - \sum_{r_1 + r_2 = n - 1} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 f .$$

For $d = 3$, we immediately get the following Boolean sum operator:

$$\begin{aligned}G_n^3 f = \sum_{r_1 + r_2 + r_3 = n} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \tilde{L}_{r_3}^3 f - 2 \sum_{r_1 + r_2 + r_3 = n - 1} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \tilde{L}_{r_3}^3 f \\ + \sum_{r_1 + r_2 + r_3 = n - 2} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \tilde{L}_{r_3}^3 f .\end{aligned}$$

Theorem 4.3. If $f \in HV(\mathbb{T}^d)$ and $1 < p < \infty$, then for all n

$$\|(I - G_n^d) f\|_p \le c n^{d-1} 2^{-\frac{n}{p}} HV(f), \tag{12}$$

where c is some constant depending on d and p.

Proof. Following Baszenski and Delvos [12], we have

$$I - G_n^d = \sum_{j=1}^{d} \sum_{q=j}^{d} (-1)^{j-1} \binom{q-1}{j-1} \sum_{B, |B| = q} \sum_{r_{i_1} + ... + r_{i_q} = n - d + j} (I - \tilde{L}_{r_{i_1}}^{i_1}) \times \dots \times (I - \tilde{L}_{r_{i_q}}^{i_q}).$$

Then using **Theorem 3.1**, we get

$$\begin{aligned}
&\|(I - G_n^d)f\|_p \\
&\leq c\sum_{j=1}^{d}\sum_{q=j}^{d}\binom{q-1}{j-1}\sum_{B,|B|=q}\sum_{r_{i_1}+\ldots+r_{i_q}=n-d+j}\|(I-\tilde{L}_{r_{i_1}}^{i_1}) \\
&\times \ldots \times (I-\tilde{L}_{r_{i_q}}^{i_q})f\|_p \\
&\leq c\sum_{j=1}^{d}\sum_{q=j}^{d}\binom{q-1}{j-1}\prod_{j\in B}(2^{r_j+1}+1)^{-1/p} \\
&\leq c\sum_{j=1}^{d}\sum_{q=j}^{d}\binom{q-1}{j-1}\sum_{B,|B|=q}\|V^B f\|_p\,(2^{n-d+j+q})^{-1/p}n^{d-1} \\
&\leq c2^{-\frac{n}{p}}n^{d-1}HV(f)\sum_{j=1}^{d}\sum_{q=j}^{d}\binom{q-1}{j-1}(2^{2j-d})^{-1/p}.
\end{aligned}$$

Now, the result follows from

$$\sum_{j=1}^{d}\sum_{q=j}^{d}\binom{q-1}{j-1}(2^{2j-d})^{-1/p} < 2^{\frac{d}{p}}\sum_{j=1}^{d}\sum_{q=j}^{d}\binom{q-1}{j-1} = 2^{\frac{d}{p}}(2^d-1).$$

Remark 4.4. Let us compare the cardinality of the tensor product grid $X_{\text{prod}}^n := \otimes_{j\in D}X_n^j$ and the sparse grid $\tilde{X}_{\text{sparse}}^n$. The grid X_{prod}^n has 2^{dn} nodes which is essentially more than $n^{d-1}2^n$ nodes of grid $\tilde{X}_{\text{sparse}}^n$. Nevertheless, the approximation order for $f \in HV(\mathbb{T})$ is only worse by a logarithmic factor n^{d-1}.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

YS: Writing – original draft, Writing – review & editing. JP: Writing – original draft, Writing – review & editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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