frontiers | Frontiers in Applied Mathematics and Statistics

TYPE Original Research
PUBLISHED 24 September 2024
DOI 10.3389/fams.2024.1467441

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OPEN ACCESS

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RECEIVED 19 July 2024
ACCEPTED 27 August 2024
PUBLISHED 24 September 2024

CITATION
Protsakh N (2024) Inverse problem for semilinear wave equation with strong damping. *Front. Appl. Math. Stat.* 10:1467441.
doi: 10.3389/fams.2024.1467441

Inverse problem for semilinear wave equation with strong damping

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The initial-boundary and the inverse coefficient problems for the semilinear hyperbolic equation with strong damping are considered in this study. The conditions for the existence and uniqueness of solutions in Sobolev spaces to these problems have been established. The inverse problem involves determining the unknown time-dependent parameter in the right-hand side function of the equation using an additional integral type overdetermination condition.

KEYWORDS

semilinear wave equation, inverse coefficient problem, existence of solutions, initial-boundary value problem, strong damping

1 Introduction

Propagation of sound in a viscous gas and other similar processes of the same nature can be described by the model hyperbolic equation of the third order, which includes a mixed derivative with respect to spatial and time variables

$$u_{tt} = \eta \Delta_x u_t + \Delta_x u, \tag{1}$$

where η is a positive constant, and $\eta \Delta_x u_t$ represents low viscosity.

Many important physical phenomena can be modeled with the use of Equation 1 and its generalizations. These are, in particular, processes that occur in viscous media (propagation of disturbances in viscoelastic and viscous-plastic rods, movement of a viscous compressible fluid, sound propagation in a viscous gas), wave processes in different media, acoustic waves in environments where wave propagation disrupts the state of thermodynamic and mechanical equilibrium, liquid filtration processes in porous media, heat transfer in a heterogeneous environment, moisture transfer in soils, and longitudinal vibrations in a homogenous bar with viscosity. The term $\Delta_x u_t$ indicates that the level of stress is proportional to the level of strains and to the strain rate [1–5].

Due to its wide range of applications, different problems for Equation 1 were investigated by many authors. For example, the unique solvability of the direct initial-boundary value problems for Equation 1 and its nonlinear generalizations with power nonlinearities have been studied in other research [1, 2, 4–11].

The inverse problems, with the integral overdetermination conditions, of identifying of the coefficients in the right-hand side function of hyperbolic equations without damping or for other types of equations have been investigated in many studies [12–18]. Their unique solvability has been solved with the use of the methods such as integral equations, the Green function, regularization, and the Shauder principle [14] and successive approximations [18]. The unique solvability of a two-dimensional inverse problem for the linear third-order hyperbolic equation with constant coefficients and with the unknown time-dependent lower coefficient has been proved in Mehraliyev et al. [19].

The main objective of this study is to determine the sufficient conditions for the existence and uniqueness of the solution to the inverse problem for the third-order semilinear hyperbolic equation with an unknown time- dependent function on its right-hand side. The unknown function is determined from the equation, subject to initial, boundary, and integral type overdetermination conditions. To prove the main results of the study, we use the properties of the solution for the corresponding initial-boundary value problem and the method of successive approximations. These results are new for semilinear n-dimensional third-order hyperbolic equations with non-constant coefficients and an unknown function on their right-hand side. The unique solvability of the initial-boundary value problem has been proved using of the method of Galerkin approximations and the methods of monotonicity and compactness.

2 Problem setting

Let $\Omega \subset \mathbb{R}^n, n \in \mathbb{N}$, be a bounded domain with the smooth boundary $\partial\Omega \in C^1$ and $0 < T < \infty$. Denote $Q_\tau = \Omega \times (0,\tau), \tau \in (0,T]$; $Q_{t_1,t_2} = \Omega \times (t_1,t_2), t_1,t_2 \in (0,T]$. In this study, we consider the following inverse problem: find the sufficient conditions for the existence of a pair of functions $(u(x,t), g(t))$ that satisfies the equation with strong damping (in the sense of Definition 3.1).

$$u_{tt} - \sum_{i,j=1}^{n}(a_{ij}(x,t)u_{x_i})_{x_j} - \sum_{i,j=1}^{n}(b_{ij}(x,t)u_{x_it})_{x_j} + \varphi_1(x,u) + \varphi_2(x,u_t) = f_1(x)g(t) + f_2(x,t), \quad x \in \Omega,\ t \in [0,T], \tag{2}$$

and the initial, boundary, and overdetermination conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega, \tag{3}$$

$$u|_{\partial\Omega\times(0,T)} = 0, \tag{4}$$

$$\int_\Omega K(x)u(x,t)dx = E(t), \qquad t \in [0,T]. \tag{5}$$

We shall use Lebesgue and Sobolev spaces $L^\infty(\cdot)$, $L^2(\cdot)$, $H^1(\cdot) := W^{1,2}(\cdot)$, $C^k(\cdot)$, $C([0,T]; L^2(G))$, $H_0^1(\cdot) := W_0^{1,2}(\cdot)$ (see, e.g., Gajewski et al. [20]).

Suppose that the data of the problem (2–5) satisfy the following conditions.

(H1) : a_{ij}, b_{ij}, a_{ijt}, b_{ijt}, $b_{ij,x_i} \in C([0,T]; L^\infty(\Omega))$, $a_{ij}(x,t) = a_{ji}(x,t), b_{ij}(x,t) = b_{ji}(x,t)$, and

$$\alpha_0\|\xi\|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \le \alpha_1\|\xi\|^2,$$

$$\beta_0\|\xi\|^2 \le \sum_{i,j=1}^{n} b_{ij}(x,t)\xi_i\xi_j \le \beta_1\|\xi\|^2,$$

for all $\xi \in \mathbb{R}^n$, almost all $x \in \Omega$, all $t \in [0,T]$, and $i,j = 1,...,n$, where α_0, α_1 and β_0, β_1 are positive constants.

(H2) : functions $\varphi_1(x,\xi), \varphi_2(x,\xi)$ are measurable with respect to $x \in \Omega$ for all $\xi \in \mathbb{R}^1$ and continuously differentiable concerning $\xi \in \mathbb{R}$. Moreover,

$$|\varphi_i(x,\xi)| \le L_{i,1}|\xi|, \quad |\varphi_i(x,\xi) - \varphi_i(x,\eta)| \le L_{i,0}|\xi - \eta|,\ i = 1,2,$$

$$(\varphi_2(x,\xi) - \varphi_2(x,\eta))(\xi - \eta) \ge 0$$

for almost all $x \in \Omega$ and $\xi, \eta \in \mathbb{R}$, where $L_{i,0}, L_{i,1}$ are positive constants.

(H3) : $f_1 \in L^2(\Omega)$, $f_2 \in C([0,T]; L^2(\Omega))$, $u_0 \in H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$.

(H4) : $E \in C^2([0,T])$, $\int_\Omega K(x)u_0(x)dx = E(0)$, $\int_\Omega K(x)u_1(x)dx = E'(0)$.

(H5) : $K \in H^2(\Omega) \cap H_0^1(\Omega)$.

Denote $\tilde{f}(x,t) := f_1(x)g(t) + f_2(x,t)$.

Let $\gamma_0 = \gamma_0(\Omega)$ be a coefficient in Friedrich's inequality.

$$\int_\Omega |v(x)|^2\,dx \le \gamma_0 \int_\Omega \sum_{i=1}^{n} |v_{x_i}(x)|^2\,dx, \quad v \in H_0^1(\Omega). \tag{6}$$

3 Initial-boundary value problem

Definition 3.1. A function $u(x,t)$ is considered to be a solution of problem 2–4 if $u \in C([0,T]; H_0^1(\Omega))$, $u_t \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega))$, $u_{tt} \in L^2(Q_T)$, u satisfies (3), and

$$\int_{Q_\tau}\Big(u_{tt}v + \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i}v_{x_j} + \sum_{i,j=1}^{n} b_{ij}(x,t)u_{x_it}v_{x_j} + \varphi_1(x,u)v + \varphi_2(x,u_t)v - \tilde{f}(x,t)v\Big)\,dxdt = 0 \tag{7}$$

for all functions $v \in L^2(0,T; H_0^1(\Omega))$ and $\tau \in (0,T]$.

Theorem 3.2. Under the assumptions (H1)–(H3) and $g \in L^2(0,T)$, $a_{ijt} \le 0$ for all $i,j = 1,2,\ldots,n$, the problem (2–4) has a unique solution.

Proof. First, using Galerkin method, we prove the existence of a solution for the problem. Let $\{w^k\}_{k=1}^\infty, k = 1,2,...$, be a basis in $H_0^1(\Omega)$, orthonormal in $L^2(\Omega)$. We will consider the sequence of functions

$$u^N(x,t) = \sum_{k=1}^{N} c_k^N(t)w^k(x),\ N = 1,2,...,$$

where the set $(c_1^N(t), ..., c_N^N(t))$ is a solution of the initial value problem

$$\begin{aligned}&\int_\Omega\Big(u_{tt}^N w^k + \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i}^N w_{x_j}^k + \sum_{i,j=1}^{n} b_{ij}(x,t)u_{x_it}^N w_{x_j}^k \\ &+\varphi_1(x,u^N)w^k + \varphi_2(x,u_t^N)w^k\Big)dx = \int_\Omega \tilde{f}(x,t)w^k dx, \\ &c_k^N(0) = u_{0,k}^N, \quad c_{kt}^N(0) = u_{1,k}^N, \quad k = 1,...,N.\end{aligned} \tag{8}$$

Here $u_0^N(x) = \sum_{k=1}^{N} u_{0,k}^N w^k(x),\ u_1^N(x) = \sum_{k=1}^{N} u_{1,k}^N w^k(x)$ and

$$\lim_{N\to\infty} \|u_0 - u_0^N\|_{H_0^1(\Omega)} = 0,\ \lim_{N\to\infty} \|u_1 - u_1^N\|_{H_0^1(\Omega)} = 0.$$

The solution of system (8) exists on some interval $[0, \tau_0]$ (Carathéodory's Theorem [21, p. 43]). The estimation (13) from below implies that this solution could be extended on $[0, T]$. Multiplying each equation of (8) on function $(c_k^N(t))'$ respectively, summing up for k from 1 to N and integrating to t on interval $[0, \tau],\ \tau \le \tau_0$, we obtain

$$\int_{Q_\tau} \Big(u_{tt}^N u_t^N + \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i}^N u_{x_j t}^N + \sum_{i,j=1}^{n} b_{ij}(x,t) u_{x_i t}^N u_{x_j t}^N + \varphi_1(x, u^N) u_t^N + \varphi_2(x, u_t^N) u_t^N\Big)\, dx\, dt = \int_{Q_\tau} \tilde{f}(x,t) u_t^N\, dx\, dt. \tag{9}$$

After transformations of terms from (9), we get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u_t^N(x,\tau)|^2 dx + \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i}^N(x,\tau) u_{x_j}^N(x,\tau) dx \\
&-\frac{1}{2}\int_{Q_\tau} \sum_{i,j=1}^{n} a_{ijt}(x,t) u_{x_i}^N u_{x_j}^N dxdt + \int_{Q_\tau} \sum_{i,j=1}^{n} b_{ij}(x,t) u_{x_i t}^N u_{x_j t}^N dxdt \\
&+\int_{Q_\tau} \varphi_1(x, u^N) u_t^N dxdt + \int_{Q_\tau} \varphi_2(x, u_t^N) u_t^N dxdt \\
&= \frac{1}{2}\int_\Omega |u_1^N(x)|^2 dx + \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,t) u_{0x_i}^N(x) u_{0x_j}^N(x) dx \\
&+ \int_{Q_\tau} \tilde{f}(x,t) u_t^N\, dx\, dt.
\end{aligned} \tag{10}$$

Note that

$$\begin{aligned}
&\int_{Q_\tau} (\varphi_1(x,u^N) + \varphi_2(x,u_t^N)) u_t^N dxdt \\
&\le \int_{Q_\tau} \left(L_{1,1}|u^N||u_t^N| + L_{2,1}|u_t^N|^2\right) dx\, dt \\
&\le \frac{1}{2}\int_{Q_\tau} \left(L_{1,1}^2 |u^N|^2 + (2L_{2,1}+1)|u_t^N|^2\right) dx\, dt \\
&\le \frac{1}{2}\int_{Q_\tau} \Big(L_{1,1}^2 \gamma_0 \sum_{i=1}^{n} |u_{x_i}^N|^2 + (2L_{2,1}+1)|u_t^N|^2\Big) dx\, dt,
\end{aligned}$$

then from (10) we obtain

$$\begin{aligned}
&\int_\Omega |u_t^N(x,\tau)|^2 dx + \alpha_0 \int_\Omega \sum_{i=1}^{n} |u_{x_i}^N(x,\tau)|^2 dx + 2\beta_0 \int_{Q_\tau} \sum_{i=1}^{n} |u_{x_i t}^N|^2 dx\, dt \\
&\le \int_\Omega |u_1(x)|^2 dx + \alpha_1 \int_\Omega \sum_{i=1}^{n} |u_{0x_i}(x)|^2 dx + \int_{Q_\tau} (\tilde{f}(x,t))^2 dx\, dt \\
&+ 2(L_{2,1}+1)\int_{Q_\tau} |u_t^N|^2 dx\, dt + (L_{1,1}^2\gamma_0 + \alpha_2)\int_{Q_\tau} \sum_{i=1}^{n} |u_{x_i}^N|^2 dx\, dt.
\end{aligned} \tag{11}$$

We rewrite the last inequality in the form

$$\begin{aligned}
&\int_\Omega \Big(|u_t^N(x,\tau)|^2 + \sum_{i=1}^{n} |u_{x_i}^N(x,\tau)|^2\Big) dx \le A_1 \\
&+ A_2 \int_{Q_\tau} \Big(|u_t^N|^2 + \sum_{i=1}^{n} |u_{x_i}^N|^2\Big) dx\, dt,
\end{aligned} \tag{12}$$

where

$$A_1 := \frac{1}{\min\{1,\alpha_0\}} \Big(\int_\Omega |u_1(x)|^2 dx + \alpha_1 \int_\Omega \sum_{i=1}^{n} |u_{0x_i}(x)|^2 dx + \int_{Q_T} \tilde{f}^2(x,t) dxdt\Big)$$

$$A_2 := \frac{\max\{2(L_{2,1}+1); (L_{1,1}^2\gamma_0 + \alpha_2)\}}{\min\{1,\alpha_0\}}.$$

Then by Grönwall's lemma, from (12), we get

$$\int_\Omega \Big(|u_t^N(x,\tau)|^2 + \sum_{i=1}^{n} |u_{x_i}^N(x,\tau)|^2\Big) dx \le A_1 e^{A_2 T}. \tag{13}$$

Therefore, from (11) we also get

$$\int_{Q_\tau} \sum_{i=1}^{n} |u_{x_i t}^N|^2 dx\, dt \le \frac{A_1(1 + A_2 T e^{A_2 T}) \min\{1,\alpha_0\}}{2\beta_0}. \tag{14}$$

Multiplying each equation of (8) on function $(c_k^N(t))''$ respectively, summing up with respect to k from 1 to N and integrating on interval $[0,\tau],\ \tau \le T$, we obtain

$$\int_{Q_\tau} \Big((u_{tt}^N)^2 + \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i}^N u_{x_j tt}^N + \sum_{i,j=1}^{n} b_{ij}(x,t) u_{x_i t}^N u_{x_j tt}^N + \varphi_1(x,u^N) u_{tt}^N + \varphi_2(x,u_t^N) u_{tt}^N\Big) dx\, dt = \int_{Q_\tau} \tilde{f}(x,t) u_{tt}^N\, dx\, dt. \tag{15}$$

After transformations in all terms from (15), we get

$$\begin{aligned}
&\frac{1}{2}\int_{Q_\tau} |u_{tt}^N|^2 dxdt + \frac{\beta_0}{4}\int_\Omega \sum_{i=1}^{n} |u_{x_i t}^N(x,\tau)|^2 dx \\
&\le \frac{n\alpha_1^2}{\beta_0}\int_\Omega \sum_{i=1}^{n} |u_{x_i}^N(x,\tau)|^2 dx \\
&+ \int_\Omega \sum_{i=1}^{n} \Big(\frac{n\alpha_1^2}{2}|u_{0x_i}^N(x)|^2 + \frac{\beta_1+1}{2}|u_{1x_i}^N(x)|^2\Big) dx \\
&+ \frac{2\beta_2 + \beta_0 + 4\alpha_1}{4}\int_{Q_\tau} \sum_{i=1}^{n} |u_{x_i t}^N|^2 dx\, dt \\
&+ \Big(\frac{\alpha_2^2}{\beta_0} + \frac{3L_{1,1}^2\gamma_0}{2}\Big)\int_{Q_\tau} \sum_{i=1}^{n} |u_{x_i}^N|^2 dx\, dt \\
&+ \frac{3L_{2,1}}{2}\int_{Q_\tau} |u_t|^2 dx\, dt + \frac{3}{2}\int_{Q_\tau} |\tilde{f}(x,t)|^2 dx\, dt,
\end{aligned} \tag{16}$$

where $\sum_{i,j=1}^{n} b_{ijt}(x,t)\xi_i\xi_j \leq \beta_2\|\xi\|^2$, $\beta_2 > 0$, $\alpha_2 = \max_{i,j}\sup_{t\in[0,T]} |a_{ijt}(x,t)|$. Taking into account (13), (14), from (16) we obtain

$$\int_{Q_\tau} |u_{tt}^N|^2\,dx\,dt + \frac{\beta_0}{2}\int_\Omega \sum_{i=1}^{n} |u_{x_it}^N(x,\tau)|^2\,dx \leq A_3, \qquad (17)$$

where

$$A_3 := \frac{A_1}{4\beta_0}\Big(8n\alpha_1^2 e^{A_2T} + \max\{8\alpha_2^2 + 12L_{1,1}^2\beta_0\gamma_0;\, 12\beta_0 L_{2,1}\}\, Te^{A_2T} + (2\beta_2+\beta_0+4\alpha_1)\times \times(1+A_2Te^{A_2T})\min\{1,\alpha_0\}\Big) + \int_\Omega \sum_{i=1}^{n}\left(n\alpha_1^2|u_{0x_i}(x)|^2 + (\beta_1+1)|u_{1x_i}(x)|^2\right)dx + 3\int_{Q_T} |\tilde f(x,t)|^2 dx\,dt.$$

The right-hand sides of the estimates (13), (14), and (17) are positive constants, independent of N. Therefore, there exists a subsequence of $\{u^N\}_{N=1}^\infty$ (which will be denoted by the same notation), such that as $N\to\infty$

$$\begin{aligned} &u^N \to u \ *-\text{weakly in } L^\infty(0,T,H_0^1(\Omega)),\\ &u_t^N \to u_t \ *-\text{weakly in } L^\infty(0,T,H_0^1(\Omega)),\\ &u^N \to u \text{ weakly in } L^2(0,T,H_0^1(\Omega)),\\ &u_t^N \to u_t \text{ weakly in } L^2(0,T,H_0^1(\Omega)),\\ &u_{tt}^N \to u_{tt} \text{ weakly in } L^2(Q_T). \end{aligned} \qquad (18)$$

It follows from (18) that $u^N \to u$ in $L^2(Q_T)$, and therefore, $\varphi_1(x,u^N) \to \varphi_1(x,u)$ weakly in $L^2(Q_T)$ as $N\to\infty$. Besides, $u\in C([0,T];H_0^1(\Omega))$, $u_t\in C([0,T];L^2(\Omega))$ and $\varphi_2(x,u_t^N)\to\chi$ weakly in $L^2(Q_T)$.

Equations 8 and 18 imply the equality

$$\int_{Q_\tau}\Big(u_{tt}v + \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i}v_{x_j} + \sum_{i,j=1}^{n} b_{ij}(x,t)u_{x_it}v_{x_j} + \varphi_1(x,u)v + \chi v - \tilde f(x,t)v\Big)dxdt = 0 \qquad (19)$$

for all functions $v\in L^2(0,T;H_0^1(\Omega))$ and $\tau\in(0,T]$.

Let us prove that $\chi = \varphi_2(x,u_t)$.

Note that $\|\varphi_1(x,u^{N+k}) - \varphi_1(x,u^N)\|_{L^2(Q_T)} \leq L_{1,0}\|u^{N+k} - u^N\|_{L^2(Q_T)}$ for all $k\in\mathbb{N}$. Due to (18), $\{u\}_{k=1}^\infty$ is fundamental in $L^2(Q_T)$. So, for any $\varepsilon>0$, there exists such a number N_0 that for all N, $k\in\mathbb{N}$, $N>N_0$ the inequality $\|u^{N+k}-u^N\|_{L^2(Q_T)}\leq\varepsilon$ holds; thus, $\{\varphi_1(x,u)\}_{k=1}^\infty$ is also fundamental in $L^2(Q_T)$ and, therefore,

$$\varphi_1(x,u^N)\to\varphi_1(x,u) \text{ in } L^2(Q_T) \text{ as } N\to\infty. \qquad (20)$$

Consider the sequence

$$0\leq X_N = \int_{Q_T}\left(\varphi_2(x,u_t^N) - \varphi_2(x,\eta_t)\right)(u_t^N - \eta_t)\,dx\,dt = \int_{Q_T}\left(\varphi_2(x,u_t^N)u_t^N - \varphi_2(x,\eta_t)(u_t^N-\eta_t) - \varphi_2(x,u_t^N)\eta_t\right)dx\,dt, \qquad (21)$$

where $\eta\in C([0,T];H_0^1(\Omega))$, $\eta_t\in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$, $\eta_{tt}\in L^2(Q_T)$. From (9), it follows that

$$\begin{aligned}\int_{Q_T}\varphi_2(x,u_t^N)u_t^N\,dxdt &= \int_{Q_T}\Big(\tilde f(x,t)u_t^N - u_{tt}^N u_t^N - \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i}^N u_{x_jt}^N - \sum_{i,j=1}^{n} b_{ij}(x,t)u_{x_it}^N u_{x_jt}^N - \varphi_1(x,u^N)u_t^N\Big)\,dx\,dt\\ &= -\frac12\int_\Omega\Big((u_t^N(x,T))^2 + \sum_{i,j=1}^{n} a_{ij}(x,T)u_{x_i}^N(x,T)u_{x_j}^N(x,T)\Big)\,dx\\ &+ \frac12\int_\Omega\Big((u_1^N(x))^2 + \sum_{i,j=1}^{n} a_{ij}(x,0)u_{0,x_i}^N(x)u_{0,x_j}^N(x)\Big)\,dx\\ &+ \int_{Q_T}\Big(\tilde f(x,t)u_t^N + \frac12\sum_{i,j=1}^{n} a_{ijt}(x,t)u_{x_i}^N u_{x_j}^N - \sum_{i,j=1}^{n} b_{ij}(x,t)u_{x_it}^N u_{x_jt}^N - \varphi_1(x,u^N)u_t^N\Big)\,dx\,dt.\end{aligned} \qquad (22)$$

After substitution (22) in (21), passing to the limit as $N\to\infty$, taking into account (18), (20), and the assumptions of Theorem 3.2, we obtain

$$\begin{aligned}0\leq \liminf_{N\to\infty} X_N &\leq -\frac12\int_\Omega\Big((u_t(x,T))^2 + \sum_{i,j=1}^{n} a_{ij}(x,T)u_{x_i}(x,T)u_{x_j}(x,T)\Big)\,dx\\ &+ \frac12\int_\Omega\Big((u_1(x))^2 + \sum_{i,j=1}^{n} a_{ij}(x,0)u_{0,x_i}(x)u_{0,x_j}(x)\Big)\,dx\\ &+ \int_{Q_T}\Big(\tilde f(x,t)u_t + \frac12\sum_{i,j=1}^{n} a_{ijt}(x,t)u_{x_i}u_{x_j} - \sum_{i,j=1}^{n} b_{ij}(x,t)u_{x_it}u_{x_jt}\\ &-\varphi_1(x,u)u_t - \varphi_2(x,\eta_t)(u_t-\eta_t) - \chi\eta_t\Big)\,dx\,dt\\ &\leq \int_{Q_T}(\chi - \varphi_2(x,\eta_t))(u_t-\eta_t)\,dxdt.\end{aligned}$$

Choosing here $\eta = u - \varkappa w$, $\varkappa>0$, $w\in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$, $w_t\in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$, $w_{tt}\in L^2(Q_T)$, dividing the result on $\varkappa$ and then tending $\varkappa\to0$ we obtain $\chi=\varphi_2(x,u_t)$. Hence, from (19), it follows (7).

Now we prove the uniqueness of the solution for the problems (2–4). On the contrary, suppose that there exist two solutions $u_{(1)}(x,t)$ and $u_{(2)}(x,t)$ of problems (2–4). Then $\tilde u := \tilde u(x,t) = u_{(1)}(x,t) - u_{(2)}(x,t)$ satisfies the conditions $\tilde u(x,0)\equiv0$, $\tilde u_t(x,0)\equiv0$, and the equality

$$\int_{Q_\tau}\Big(\tilde u_{tt}v + \sum_{i,j=1}^{n} a_{ij}(x,t)\tilde u_{x_i}v_{x_j} + \sum_{i,j=1}^{n} b_{ij}(x,t)\tilde u_{x_it}v_{x_j} + (\varphi_1(x,u_{(1)}) - \varphi_1(x,u_{(2)}))v + (\varphi_2(x,(u_{(1)})_t) - \varphi_2(x,(u_{(2)})_t))v\Big)\,dx\,dt = 0 \qquad (23)$$

holds for all $v \in L^2(0,T;H_0^1(\Omega))$, $\tau \in (0,T]$.

After choosing $v=\tilde{u}_t$ in (23) we get

$$\int_{Q_\tau}\Big(\tilde{u}_{tt}\tilde{u}_t+\sum_{i,j=1}^{n}a_{ij}(x,t)\tilde{u}_{x_i}\tilde{u}_{x_jt}+\sum_{i,j=1}^{n}b_{ij}(x,t)\tilde{u}_{x_it}\tilde{u}_{x_jt} \\ +(\varphi_1(x,u_{(1)})-\varphi_1(x,u_{(2)}))\tilde{u} \\ +(\varphi_2(x,(u_{(1)})_t)-\varphi_2(x,(u_{(2)})_t))\tilde{u}_t\Big)\,dx\,dt=0. \quad (24)$$

From (24) by the same way as from (11) we got (12), we find the following estimate

$$\int_\Omega\Big(|u_t^N(x,\tau)|^2+\sum_{i=1}^{n}|u_{x_i}^N(x,\tau)|^2\Big)\,dx \\ \le A_2\int_{Q_\tau}\left(|u_t^N|^2+\sum_{i=1}^{n}|u_{x_i}^N|^2\right)dxdt. \quad (25)$$

Then from Grönwall's lemma and (25) we obtain $\int_\Omega\big(|\tilde{u}_t(x,\tau)|^2+\sum_{i=1}^{n}|\tilde{u}_{x_i}(x,\tau)|^2\big)\,dx \le 0$ and $\int_{Q_\tau}\sum_{i=1}^{n}|\tilde{u}_{x_it}(x,t)|^2\,dx\,dt \le 0$, hence, $\tilde{u}\equiv 0$, and, therefore, $u_{(1)}=u_{(2)}$ in Q_T.

4 Inverse problem

Definition 4.1. A pair of functions $(u(x,t),g(t))$ is a solution to the problem (2–5), if $u\in C([0,T];H_0^1(\Omega))$, $u_t\in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$, $u_{tt}\in L^2(Q_T)$, and $g\in C([0,T])$, and it satisfies (5) and

$$\int_{Q_\tau}\left(u_{tt}v+\sum_{i,j=1}^{n}a_{ij}(x,t)u_{x_i}v_{x_j} \\ +\sum_{i,j=1}^{n}b_{ij}(x,t)u_{x_it}v_{x_j}+\varphi_1(x,u)v+\varphi_2(x,u_t)v\right)dx\,dt \\ =\int_{Q_\tau}\big(f_1(x)g(t)+f_2(x,t)\big)v\,dx\,dt \quad (26)$$

holds for all functions $v\in L^2(0,T;H_0^1(\Omega))$ and $\tau\in(0,T]$.

4.1 The equivalent problem

In this section, we shall find the equivalent problem for the problem (2–5).

Lemma 4.2. *Let $\int_\Omega K(x)f_1(x)\,dx\neq 0$, the assumptions of Theorem 3.2, (H4), and (H5) hold. A pair of functions $(u(x,t),g(t))$, where $u\in C([0,T]],H_0^1(\Omega))$, $u_t\in L^2(0,T;H_0^1(\Omega))\cap C(0,T,L^2(\Omega))$, $u_{tt}\in L^2(Q_T)$, $g\in C([0,T])$, is a solution to the problem (2–5) if and only if it satisfies (26) for all functions $v\in L^2(0,T;H_0^1(\Omega))$, and for $\tau\in(0,T]$, the equality*

$$g(t)\int_\Omega K(x)f_1(x)\,dx=E''(t)+\int_\Omega\left(\sum_{i,j=1}^{n}K_{x_j}(x)a_{ij}(x,t)u_{x_i} \\ -\sum_{i,j=1}^{n}(K_{x_j}(x)b_{ij}(x,t))_{x_i}u_t+K(x)\varphi_1(x,u)+K(x)\varphi_2(x,u_t) \\ -K(x)f_2(x,t)\right)dx \quad (27)$$

holds for $t\in[0,T]$.

Proof. Necessity: Let $(u(x,t),g(t))$ be a solution to the problem (2–5). From (26) and (5), it follows that

$$\int_\Omega\big(f_1(x)g(t)K(x)+f_2(x,t)K(x)-\varphi_1(x,u)K(x)-\varphi_2(x,u_t)K(x) \\ -\sum_{i,j=1}^{n}a_{ij}(x,t)u_{x_i}K_{x_j}(x)-\sum_{i,j=1}^{n}b_{ij}(x,t)u_{x_it}K_{x_j}(x)\big)\,dx \\ =E''(t),\ t\in(0,T]. \quad (28)$$

By integrating by parts in (28) and using the condition (H4), we get the equality

$$g(t)\int_\Omega K(x)f_1(x)\,dx+\int_\Omega\Big(K(x)f_2(x)-K(x)\varphi(x,u) \\ -\sum_{i,j=1}^{n}a_{ij}(x,t)K_{x_j}(x)u_{x_i} \\ +\sum_{i,j=1}^{n}(b_{ij}(x,t)K_{x_j}(x))_{x_i}u_t\Big)\,dx=E''(t),\ t\in(0,T]. \quad (29)$$

From (29), we can obtain (27).

Sufficiency: Let $g^*\in C([0,T])$, $u^*\in C([0,T];H_0^1(\Omega))$, $u_t^*\in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$, $u_{tt}^*\in L^2(Q_T)$, and they satisfy (4), (26), and (27). Then u^* is a solution to the problem (2–4) with g^* instead of g in (2).

We set $E^*(t)=\int_\Omega K(x)u^*(x,t)\,dx$, $t\ge 0$. In exactly the same way as in the proof of necessity, we obtain

$$g^*(t)\int_\Omega K(x)f_1(x)\,dx=(E^*(t))''+\int_\Omega\left(\sum_{i,j=1}^{n}K_{x_j}(x)a_{ij}(x,t)u_{x_i}^*\right. \\ -\sum_{i,j=1}^{n}(K_{x_j}(x)b_{ij}(x,t))_{x_i}u_t^*+K(x)\varphi_1(x,u^*) \\ \left.+K(x)\varphi_2(x,u_t^*)-K(x)f_2(x,t)\right)dx,\quad t\in(0,T]. \quad (30)$$

On the other hand $g^*(t)$ and $u^*(x,t)$ satisfy (27)

$$g^*(t)\int_\Omega K(x)f_1(x)\,dx=\Big(E''(t) \\ +\int_\Omega\big(\sum_{i,j=1}^{n}K_{x_j}(x)a_{ij}(x,t)u_{x_i}^*-\sum_{i,j=1}^{n}(K_{x_j}(x)b_{ij}(x,t))_{x_i}u_t^* \\ +K(x)\varphi_1(x,u^*)+K(x)\varphi_2(x,u_t^*)-K(x)f_2(x,t)\big)\,dx\Big),\quad t\in(0,T]. \quad (31)$$

It follows from (30), (31) that

$$(E^*(t))'' = E''(t), \qquad t \in (0, T]. \tag{32}$$

Integrating (32) with the use of the equalities $E^*(0) = E(0) = \int\limits_\Omega K(x)u_0(x)\,dx$, $(E^*)'(0) = E'(0) = \int\limits_\Omega K(x)u_1^*(x)\,dx$, implies $E^*(t) = E(t)$, $t \geq 0$. Hence, $u^*(x,t)$ satisfies the overdetermination condition (5).

4.2 Main results

Let $f_1 := \int\limits_\Omega (f_1(x))^2\,dx$, $\alpha_2 := \max\limits_{i,j} \sup\limits_{Q_T} |a_{ijt}|$. Denote

$$M_1 := \max\left\{ n \max_i \sup_{[0,T]} \int\limits_\Omega \left(\sum_{j=1}^n K_{x_j}(x)a_{ij}(x,t) \right)^2 dx \right.$$
$$+ L_{1,0}^2\gamma_0 \int\limits_\Omega (K(x))^2\,dx;$$
$$\left. L_{2,0}^2 \int\limits_\Omega (K(x))^2\,dx + \sup_{[0,T]} \int\limits_\Omega \left(\sum_{i,j=1}^n (K_{x_j}(x)b_{ij}(x,t))_{x_i} \right)^2 dx \right\},$$

$$M_2 := \frac{4M_1}{\left(\int\limits_\Omega K(x)f_1(x)\,dx \right)^2};$$

$$M_3 := \frac{2f_1\gamma_0}{\beta_0} \exp\left(\frac{\max\left\{2L_{2,0};\, \frac{2\gamma_0^2 L_{1,0}^2}{\beta_0} + \alpha_2\right\} T_0}{\min\{1, \alpha_0\}} \right);$$

$$M_4 := M_2 M_3,$$

T_0 is such a number that $M_4 T_0 < 1$.

Theorem 4.3. Let $\int\limits_\Omega K(x)f_1(x)\,dx \neq 0$, $a_{ijt} \leq 0$ for all $i,j = 1, 2, \dots, n$, and the assumptions (H1) – (H5) hold. Then there exists a unique solution to the problem (2–5).

Proof. I. In the first step, we shall prove the theorem for $T \leq T_0$.

We construct an approximation $(u^m(x,t), g^m(t))$ of the solution of problem (2–5), where $g^1(t) := 0$, the functions $g^m(t)$, $m \geq 2$, satisfy the equality

$$g^m(t) = \left(\int\limits_\Omega K(x)f_1(x)\,dx \right)^{-1} \left(E''(t) + \int\limits_\Omega \left(\sum_{i,j=1}^n K_{x_j}(x)a_{ij}(x,t)u_{x_i}^{m-1} \right.\right.$$
$$- \sum_{i,j=1}^n (K_{x_j}(x)b_{ij}(x,t))_{x_i} u_t^{m-1} + K(x)\varphi_1(x, u^{m-1})$$
$$\left.\left. + K(x)\varphi_2(x, u_t^{m-1}) - K(x)f_2(x,t) \right) dx \right), \; t \in [0, T_0], \tag{33}$$

and u^m satisfies the equality

$$\int\limits_{Q_\tau} \Big(u_{tt}^m v + \sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}^m v_{x_j} + \sum_{i,j=1}^n b_{ij}(x,t)u_{x_i t}^m v_{x_j} + \varphi_1(x, u^m)v$$
$$+ \varphi_2(x, u_t^m)v \Big)\,dxdt = \int\limits_{Q_\tau} (f_1(x)g^m(t) + f_2(x,t))v\,dx\,dt,$$
$$\tau \in [0, T_0], \qquad m \geq 1, \tag{34}$$

for all $v \in L^2(0, T_0; H_0^1(\Omega))$, and the conditions

$$u^m(x,0) = u_0(x), \qquad u_t^m(x,0) = u_1(x), \;\; x \in \Omega. \tag{35}$$

It follows from Theorem 3.2 that for each $m \in \mathbb{N}$ there exists a unique function $u^m \in C([0,T_0]; H_0^1(\Omega))$, $u_t^m \in L^2(0,T_0; H_0^1(\Omega)) \cap C([0,T_0]; L^2(\Omega))$, $u_{tt}^m \in L^2(Q_{T_0})$, that satisfies (34), (35). Now we show that $\{(u^m(x,t), g^m(t))\}_{m=1}^\infty$ converges to the solution of the problem (2–5). Denote

$$z^m := z^m(x,t) = u^m(x,t) - u^{m-1}(x,t),$$
$$r^m(t) := g^m(t) - g^{m-1}(t), \qquad m \geq 2.$$

Equation 33 for $t \in (0, T_0]$ and $m \geq 3$, implies the equality

$$r^m(t) = \left(\int\limits_\Omega K(x)f_1(x)\,dx \right)^{-1} \int\limits_\Omega \Big(\sum_{i,j=1}^n K_{x_j}(x)a_{ij}(x,t)z_{x_i}^{m-1}$$
$$- \sum_{i,j=1}^n (K_{x_j}(x)b_{ij}(x,t))_{x_i} z_t^{m-1}$$
$$+ K(x)(\varphi_1(x, u^{m-1}) - \varphi_1(x, u^{m-2}))$$
$$+ K(x)(\varphi_2(x, u_t^{m-1}) - \varphi_2(x, u_t^{m-2})) \Big)\,dx. \tag{36}$$

We square both sides of equality (36) and integrate the result with respect to t, taking into account the hypotheses (H5), then we obtain

$$\int\limits_0^\tau (r^m(t))^2\,dt \leq \frac{4}{\left(\int\limits_\Omega K(x)f_1(x)\,dx \right)^2}$$
$$\int\limits_0^\tau \Bigg(\Big(\int\limits_\Omega \sum_{i,j=1}^n K_{x_j}(x)a_{ij}(x,t)z_{x_i}^{m-1}\,dx \Big)^2$$
$$+ \Big(\int\limits_\Omega \sum_{i,j=1}^n (K_{x_j}(x)b_{ij}(x,t))_{x_i} z_t^{m-1}\,dx \Big)^2$$
$$+ \Big(\int\limits_\Omega K(x)(\varphi_1(x, u^{m-1}) - \varphi_1(x, u^{m-2}))\,dx \Big)^2$$
$$+ \Big(\int\limits_\Omega K(x)(\varphi_2(x, u_t^{m-1}) - \varphi_2(x, u_t^{m-2}))\,dx \Big)^2 \Bigg)\,dt, \; m \geq 3. \tag{37}$$

Then (37) implies the estimate

$$\int\limits_0^\tau (r^m(t))^2\,dt \leq M_2 \int\limits_{Q_\tau} \left((z_t^{m-1})^2 + \sum_{i=1}^n (z_{x_i}^{m-1})^2 \right) dx\,dt,$$
$$\tau \in (0, T_0], \qquad m \geq 3. \tag{38}$$

It follows from (35) that $z^m(x,0) = 0,\ z_t^m(x,0) = 0, x \in \Omega,\ m \geq 2$. Hence, from (34) with $v = z_t^m$, $\tau \in (0, T_0]$, we get

$$\int_{Q_\tau} \Big(z_{tt}^m z_t^m + \sum_{i,j=1}^{n} a_{ij}(x,t) z_{x_i}^m z_{x_j t}^m + \sum_{i,j=1}^{n} b_{ij}(x,t) z_{x_i t}^m z_{x_j t}^m \\ +(\varphi_1(x,u^m) - \varphi_1(x,u^{m-1}))z_t^m \\ +(\varphi_2(x,u_t^m) - \varphi_2(x,u_t^{m-1}))z_t^m\Big)\, dx\, dt \\ = \int_{Q_\tau} f_1(x) r^m(t) z_t^m \, dx\, dt,\ m \geq 1. \tag{39}$$

The last term in (39)

$$\int_{Q_\tau} f_1(x) r^m(t) z_t^m \, dx\, dt \leq \frac{\beta_0}{4\gamma_0} \int_{Q_\tau} (z_t^m)^2 \, dx\, dt + \frac{f_1\gamma_0}{\beta_0} \int_0^\tau (r^m(t))^2\, dt \\ \leq \frac{\beta_0}{4} \int_{Q_\tau} \sum_{i=1}^{n} (z_{x_i t}^m)^2 \, dx\, dt + \frac{f_1\gamma_0}{\beta_0} \int_0^\tau (r^m(t))^2\, dt.$$

Besides,

$$\int_{Q_\tau} (\varphi_1(x,u^m) - \varphi_1(x,u^{m-1}))z_t^m \, dx\, dt \\ \leq \int_{Q_\tau} L_{1,0} |z^m||z_t^m| \, dx\, dt \\ \leq \int_{Q_\tau} \left(\frac{L_{1,0}^2\gamma_0}{\beta_0}(z^m)^2 + \frac{\beta_0}{4\gamma_0}(z_t^m)^2 \right) dx\, dt \\ \leq \int_{Q_\tau} \left(\frac{L_{1,0}^2\gamma_0^2}{\beta_0} \sum_{i=1}^{n} (z_{x_i}^m)^2 + \frac{\beta_0}{4} \sum_{i=1}^{n} (z_{x_i t}^m)^2 \right) dx\, dt$$

and

$$\int_{Q_\tau} (\varphi_2(x,u_t^m) - \varphi_2(x,u_t^{m-1}))z_t^m \, dx\, dt \leq L_{2,0} \int_{Q_\tau} |z_t^m|^2 \, dx\, dt.$$

Then, taking into account (H1) – (H5), from (39) we get inequality

$$\min\{1,\alpha_0\} \int_\Omega \Big((z_t^m(x,\tau))^2 + \sum_{i=1}^{n} (z_{x_i}^m(x,\tau))^2\Big)\, dx \\ + \beta_0 \int_{Q_\tau} \sum_{i=1}^{n} (z_{x_i t}^m)^2 \, dx\, dt \\ \leq \max\left\{2L_{2,0};\ \frac{2\gamma_0^2 L_{1,0}^2}{\beta_0} + \alpha_2\right\} \int_{Q_\tau} \left((z_t^m)^2 + \sum_{i=1}^{n} (z_{x_i}^m)^2\right) dx\, dt \\ + \frac{2f_1\gamma_0}{\beta_0} \int_0^\tau (r^m(t))^2\, dt,\ m \geq 2. \tag{40}$$

According to Grönwall's Lemma, we obtain

$$\int_\Omega \Big((z_t^m(x,\tau))^2 + \sum_{i=1}^{n} (z_{x_i}^m(x,\tau))^2\Big)\, dx \\ \leq M_3 \int_0^\tau (r^m(t))^2\, dt,\ \tau \in (0, T_0],\ m \geq 2. \tag{41}$$

Interating (41) with respect to τ, we get the estimate

$$\int_{Q_{T_0}} \left((z_t^m)^2 + \sum_{i=1}^{n} (z_{x_i}^m)^2\right) dx\, dt \leq M_3 T_0 \int_0^{T_0} (r^m(t))^2\, dt,\ m \geq 2. \tag{42}$$

Besides, (40) and (42) for $m \geq 2$ imply the estimates

$$\int_{Q_{T_0}} \sum_{i=1}^{n} (z_{x_i t}^m)^2 \, dx\, dt \leq \frac{M_5}{\beta_0} \int_0^{T_0} (r^m(t))^2\, dt \tag{43}$$

and

$$\int_\Omega \left((z_t^m(x,\tau))^2 + \sum_{i=1}^{n} (z_{x_i}^m(x,\tau))^2\right) dx \leq \frac{M_5}{\min\{1,\alpha_0\}} \int_0^{T_0} (r^m(t))^2\, dt, \tag{44}$$

where $M_5 := \max\left\{2L_{2,0};\ \frac{2\gamma_0^2 L_{1,0}^2 + \alpha_2\beta_0}{\beta_0}\right\} M_3 T_0 + \frac{2f_1\gamma_0}{\beta_0}$.

Note that $r^2(t) = g^2(t)$. Then, taking into account (33), we have

$$\int_0^\tau (r^2(t))^2\, dt = \int_0^\tau (g^2(t))^2\, dt \leq 6\Big(\int_\Omega K(x) f_1(x)\, dx\Big)^{-2} \\ \Big(\int_0^\tau (E''(t))^2\, dt + n^2 \max_i \sup_t \sum_{j=1}^{n} \int_\Omega (K_{x_j}(x) a_{ij}(x,t))^2\, dx \\ \int_{Q_{T_0}} \sum_{i=1}^{n} (u_{x_i}^1)^2\, dx\, dt \\ + n^2 \max_i \sup_t \sum_{j=1}^{n} \int_\Omega (K_{x_j}(x) b_{ij}(x,t))^2\, dx \int_{Q_{T_0}} \sum_{i=1}^{n} (u_{x_i t}^1)^2\, dx\, dt \\ + L_{1,0}^2 \int_\Omega (K(x))^2\, dx \int_{Q_{T_0}} |u^1|^2\, dx\, dt + L_{2,0}^2 \\ \int_\Omega (K(x))^2\, dx \int_{Q_{T_0}} |u_t^1|^2\, dx\, dt + \int_0^\tau \Big(\int_\Omega K(x) f_2(x,t)\, dx\Big)^2 dt\Big) \leq M_6,$$

where M_6 is a positive constant. It follows from (42) and (38) that for $m \geq 3$

$$\int_0^{T_0} (r^m(t))^2\, dt \leq M_4 T_0 \int_0^{T_0} (r^{m-1}(t))^2\, dt \\ \leq (M_4 T_0)^{m-2} \int_0^{T_0} (r^2(t))^2\, dt \leq M_6 (M_4 T_0)^{m-2}. \tag{45}$$

By using (45) and the assumption $M_4T_0 < 1$, we can show the estimate

$$\|g^{m+k} - g^m\|_{L^2(0,T_0)} \le \sum_{i=m+1}^{m+k} \left(\int_0^{T_0} (r^i(t))^2\, dt \right)^{\frac{1}{2}} \le$$
$$\le \sum_{i=m+1}^{m+k} M_6^{\frac{1}{2}} (M_4T_0)^{\frac{i-2}{2}} \le \frac{M_6^{\frac{1}{2}} (M_4T_0)^{\frac{m-1}{2}}}{1-(M_4T_0)^{\frac{1}{2}}}, \qquad k \in \mathbb{N},\ m \ge 3. \tag{46}$$

Due to (42)

$$\int_{Q_{T_0}} \left((z_t^m)^2 + \sum_{i=1}^n (z_{x_i}^m)^2 \right) dx\, dt$$
$$\le M_3 T_0 \int_0^{T_0} (r^m(t))^2\, dt \le M_3M_6T_0(M_4T_0)^{m-2}, \quad m \ge 2. \tag{47}$$

Besides, (43) and (44) for $m \ge 2$ imply the estimates

$$\int_{Q_{T_0}} \sum_{i=1}^n (z_{x_i t}^m)^2\, dx\, dt \le \frac{M_5M_6(M_4T_0)^{m-2}}{\beta_0}, \tag{48}$$

and

$$\int_\Omega \left((z_t^m(x,\tau))^2 + \sum_{i=1}^n (z_{x_i}^m(x,\tau))^2 \right) dx \le \frac{M_5M_6(M_4T_0)^{m-2}}{\min\{1,\alpha_0\}}. \tag{49}$$

And, therefore,

$$\sum_{j=1}^n \|u_{x_j}^{m+k} - u_{x_j}^m\|_{L^2(Q_{T_0})} + \|u_t^{m+k} - u_t^m\|_{L^2(Q_{T_0})}$$
$$\le \sum_{i=m+1}^{m+k} \left(\sum_{j=1}^n \|z_{x_j}^i\|_{L^2(Q_{T_0})} + \|z_t^i\|_{L^2(Q_{T_0})} \right)$$
$$\le (n+1) \sum_{i=m+1}^{m+k} \Big(M_3M_6T_0 \Big)^{\frac{1}{2}} (M_4T_0)^{\frac{i-2}{2}}$$
$$\le (n+1) \Big(M_3M_6T_0 \Big)^{\frac{1}{2}} \frac{(M_4T_0)^{\frac{m-1}{2}}}{1-(M_4T_0)^{\frac{1}{2}}}, \qquad k \in \mathbb{N},\ m \ge 2 \tag{50}$$

and

$$\sum_{j=1}^n \|u_{x_j t}^{m+k} - u_{x_j t}^m\|_{L^2(Q_{T_0})} \le \sum_{i=m+1}^{m+k} \sum_{j=1}^n \|z_{x_j t}^i\|_{L^2(Q_{T_0})}$$
$$\le n \sum_{i=m+1}^{m+k} \left(\frac{M_5M_6}{\beta_0} \right)^{\frac{1}{2}} (M_4T_0)^{\frac{i-2}{2}}$$
$$\le n \left(\frac{M_5M_6}{\beta_0} \right)^{\frac{1}{2}} \frac{(M_4T_0)^{\frac{m-1}{2}}}{1-(M_4T_0)^{\frac{1}{2}}}, \qquad k \in \mathbb{N},\ m \ge 2, \tag{51}$$

and for $k \in \mathbb{N}$, $m \ge 2$

$$\sum_{j=1}^n \|u_{x_j}^{m+k} - u_{x_j}^m\|_{C([0,T_0];L^2(\Omega))} + \|u_t^{m+k} - u_t^m\|_{C([0,T_0];L^2(\Omega))}$$
$$\le \sum_{i=m+1}^{m+k} \left(\sum_{j=1}^n \|z_{x_j}^i\|_{C([0,T_0];L^2(\Omega))} + \|z_t^i\|_{C([0,T_0];L^2(\Omega))} \right)$$
$$\le (n+1) \left(\frac{M_5M_6}{\min\{1,\alpha_0\}} \right)^{\frac{1}{2}} \sum_{i=m+1}^{m+k} (M_4T_0)^{\frac{i-2}{2}}$$
$$\le (n+1) \left(\frac{M_5M_6}{\min\{1,\alpha_0\}} \right)^{\frac{1}{2}} \frac{(M_4T_0)^{\frac{m-1}{2}}}{1-(M_4T_0)^{\frac{1}{2}}}. \tag{52}$$

Besides, we square both sides of equality (36), taking into account the hypotheses (H5) and obtain

$$(r^m(t))^2 \le \frac{4}{\left(\int_\Omega K(x) f_1(x)\, dx \right)^2} \Bigg(\Bigg(\int_\Omega \sum_{i,j=1}^n K_{x_j}(x) a_{ij}(x,t) z_{x_i}^{m-1}\, dx \Bigg)^2$$
$$+ \left(\int_\Omega \sum_{i,j=1}^n (K_{x_j}(x) b_{ij}(x,t))_{x_i} z_t^{m-1}\, dx \right)^2$$
$$+ \left(\int_\Omega K(x)(\varphi_1(x,u^{m-1}) - \varphi_1(x,u^{m-2}))\, dx \right)^2$$
$$+ \left(\int_\Omega K(x)(\varphi_2(x,u_t^{m-1}) - \varphi_2(x,u_t^{m-2}))\, dx \right)^2 \Bigg), \qquad m \ge 3. \tag{53}$$

From (53), we can conclude that:

$$(r^m(t))^2 \le M_2 \int_\Omega \left((z_t^{m-1}(x,t))^2 + \sum_{i=1}^n (z_{x_i}^{m-1}(x,t))^2 \right) dx$$
$$\le \frac{M_2M_5M_6}{\min\{1,\alpha_0\}} (M_4T_0)^{m-2},\ m \ge 3. \tag{54}$$

Therefore,

$$\max_{[0,T_0]} \|g^{m+k} - g^m\|_{C([0,T_0])} \le \sum_{i=m+1}^{m+k} \|r^i\|_{C([0,T_0])}$$
$$\le \left(\frac{M_2M_5M_6}{\min\{1,\alpha_0\}} \right)^{\frac{1}{2}} \sum_{i=m+1}^{m+k} (M_4T_0)^{\frac{i-2}{2}}$$
$$\le \left(\frac{M_2M_5M_6}{\min\{1,\alpha_0\}} \right)^{\frac{1}{2}} \frac{(M_4T_0)^{\frac{m-1}{2}}}{1-(M_4T_0)^{\frac{1}{2}}},\ k \in \mathbb{N}, m \ge 2. \tag{55}$$

It follows from (46), (50), (51), (52), and (55) that for any $\varepsilon > 0$, there exists m_0 such that for all $k, m \in \mathbb{N}$, $m > m_0$, the following inequalities hold:

$$\|g^{m+k}-g^m\|_{L^2(0,T_0)}\le\varepsilon,\|g^{m+k}-g^m\|_{C([0,T_0])}\le\varepsilon,$$
$$\sum_{j=1}^n\|u^{m+k}_{x_jt}-u^m_{x_jt}\|_{L^2(Q_{T_0})}\le\varepsilon,$$
$$\sum_{j=1}^n\|u^{m+k}_{x_j}-u^m_{x_j}\|_{L^2(Q_{T_0})}+\|u^{m+k}_t-u^m_t\|_{L^2(Q_{T_0})}\le\varepsilon,$$
$$\sum_{j=1}^n\|u^{m+k}_{x_j}-u^m_{x_j}\|_{C([0,T_0];L^2(\Omega))}+\|u^{m+k}_t-u^m_t\|_{C([0,T_0];L^2(\Omega))}\le\varepsilon$$

are true. Hence, the sequence $\{g^m\}_{m=1}^\infty$ is fundamental in $L^2(0,T_0)$ and in $C([0,T_0])$, $\{u^m\}_{m=1}^\infty$ is fundamental in $L^2(0,T_0;H^1_0(\Omega))$ and in $C(0,T_0;H^1_0(\Omega))$, and $\{u^m_t\}_{m=1}^\infty$ is fundamental in $L^2(0,T_0;H^1_0(\Omega))$ and in $C([0,T_0];L^2(\Omega))$. Therefore, as $m\to\infty$

$$g^m\to g\text{ in }C([0,T_0]),\qquad u^m\to u\text{ in }C([0,T_0];H^1_0(\Omega)),$$
$$u^m_t\to u_t\text{ in }L^2(0,T_0;H^1_0(\Omega))\cap C([0,T_0];L^2(\Omega)). \tag{56}$$

Theorem 3.2 implies the following estimate

$$\begin{aligned}\int_{Q_{T_0}}(u^m_{tt})^2\,dx\,dy\,dt\le\frac{A_1}{4\beta_0}&\Big(8n\alpha_1^2e^{A_2T_0}\\&+\max\left\{8\alpha_2^2+12L^2_{1,1}\beta_0\gamma_0;\,12\beta_0L_{2,1}\right\}T_0e^{A_2T_0}\\&+(2\beta_2+\beta_0+4\alpha_1)(1+A_2T_0e^{A_2T_0})\min\{1,\alpha_0\}\Big)\\&+n\alpha_1^2\int_\Omega\sum_{i=1}^n|u_{0x_i}(x)|^2\,dx\\&+(\beta_1+1)\int_\Omega\sum_{i=1}^n|u_{1x_i}(x)|^2\,dx\\&+3\int_{Q_{T_0}}|f_1(x)g^m(t)+f_2(x,t)|^2dx\,dt,\qquad m\ge2.\end{aligned}\tag{57}$$

and, by virtue of (56) $\|g^m\|_{C([0,T_0])}<M_7$, where M_7 is independent on m, and therefore the right-hand side of (57) is bounded with the constant, independent on m. Hence, we can select a subsequence of sequence $\{u^m\}_{m=1}^\infty$ (we preserve the same notation for this subsequence), such that

$$u^m_{tt}\to u_{tt}\text{ weakly in }L^2(Q_{T_0})\qquad\text{as }m\to\infty. \tag{58}$$

Taking into account (56) and (58), from (33), (34) we get that the pair of functions $(u(x,t),g(t))$ satisfies (27) and (26). By virtue of Lemma 4.2 $(u(x,t),g(t))$ is a solution of the problem (2–5) in Q_{T_0}.

II. Uniqueness of solution of the problem (2–5), with $T\le T_0$.

Assume that $(u_{(1)}(x,t),g_{(1)}(t))$ and $(u_{(2)}(x,t),g_{(2)}(t))$ be two solutions of problem (2–5). Then the pair of functions $(\tilde u(x,t),\tilde g(t))$, where $\tilde u(x,t)=u_{(1)}(x,t)-u_{(2)}(x,t)$, $\tilde g(t)=g_{(1)}(t)-g_{(2)}(t)$, satisfies the conditions $\tilde u(x,0)\equiv0$, $\tilde u_t(x,0)\equiv0$, the equality

$$\begin{aligned}\int_{Q_{T_0}}\Big(\tilde u_{tt}v+&\sum_{i,j=1}^n a_{ij}(x,t)\tilde u_{x_i}v_{x_j}+\sum_{i,j=1}^n b_{ij}(x)\tilde u_{x_it}v_{x_j}\\&+(\varphi_1(x,u_{(1)})-\varphi_1(x,u_{(2)}))v\\&+(\varphi_2(x,(u_{(1)})_t)-\varphi_2(x,(u_{(2)})_t))v\Big)\,dx\,dt\\&=\int_{Q_{T_0}}f_1(x)\tilde g(t)v\,dx\,dt,\end{aligned}\tag{59}$$

for all $v\in L^2(0,T_0;H^1_0(\Omega))$ and the equality

$$\begin{aligned}\tilde g(t)=\Big(\int_\Omega K(x)f_1(x)\,dx\Big)^{-1}\Big(\int_\Omega\Big(&\sum_{i,j=1}^n K_{x_j}(x)a_{ij}(x,t)\tilde u_{x_i}\\&-\sum_{i,j=1}^n(K_{x_j}(x)b_{ij}(x,t))_{x_i}\tilde u_t\\&+K(x)(\varphi_1(x,u_{(1)})-\varphi_1(x,u_{(2)}))+K(x)(\varphi_2(x,(u_{(1)})_t)\\&-\varphi_2(x,(u_{(2)})_t))\Big)\,dx\Big),\ t\in[0,T_0],\end{aligned}\tag{60}$$

holds.

After choosing $v=\tilde u_t$ in (59) we get

$$\begin{aligned}\int_{Q_{T_0}}\Big(\tilde u_{tt}\tilde u_t+&\sum_{i,j=1}^n a_{ij}(x,t)\tilde u_{x_i}\tilde u_{x_jt}+\sum_{i,j=1}^n b_{ij}(x,t)\tilde u_{x_it}\tilde u_{x_jt}\\&+(\varphi_1(x,u_{(1)})-\varphi_1(x,u_{(2)}))\tilde u_t\\&+(\varphi_2(x,(u_{(1)})_t)-\varphi_2(x,(u_{(2)})_t))\tilde u_t\Big)\,dx\,dt\\&=\int_{Q_{T_0}}f_1(x)\tilde g(t)\tilde u_t\,dx\,dt.\end{aligned}\tag{61}$$

It is easy to get from (60) and (H5) inequalities

$$\int_0^{T_0}(\tilde g(t))^2\,dt\le M_2\int_{Q_{T_0}}\Big((\tilde u_t)^2+\sum_{i=1}^n(\tilde u_{x_i})^2\Big)\,dx\,dt. \tag{62}$$

From (61) by the same way as from (39) we got (42), we find the following estimate

$$\int_{Q_{T_0}}\Big((\tilde u_t)^2+\sum_{i=1}^n(\tilde u_{x_i})^2\Big)\,dx\,dt\le M_3T_0\int_0^{T_0}(\tilde g(t))^2\,dt, \tag{63}$$

and taking into account (62) from (63), we obtain $(1-M_4T_0)\int_{Q_{T_0}}\Big((\tilde u_t)^2+\sum_{i=1}^n(\tilde u_{x_i})^2\Big)\,dx\,dt\le0$. Since $M_4T_0<1$, we conclude that $\int_{Q_{T_0}}\Big((\tilde u_t)^2+\sum_{i=1}^n(\tilde u_{x_i})^2\Big)\,dx\,dt=0$, hence, $u_{(1)}=u_{(2)}$ in Q_{T_0}. Then (62) implies $\tilde g(t)\equiv0$, and, therefore, $g_{(1)}(t)\equiv g_{(2)}(t)$ on $[0,T_0]$.

III. Let now $T>0$ be arbitrary number.

Let us divide the interval $[0,T]$ into a finite number of intervals $[0,T_1]$, $[T_1,2T_1]$, $\dots,[(N-1)T_1,NT_1]$, where $NT_1=T$, and $T_1\le T_0$. According to I and II, there exists a unique solution $(u_1(x,t),g_{0,1}(t))$ to the problem (2–5) in the domain Q_{T_1}.

Now, we will prove that there exists a unique solution in the domain $Q_{T_1,2T_1} := \Omega \times (T_1; 2T_1)$ for the problem for the Equation 2 with conditions (4) and (5) as $t \in [T_1; 2T_1]$, and with the initial condition $u(x, T_1) = u_1(x, T_1)$, $u_t(x, T_1) = u_{1t}(x, T_1)$, and $x \in \Omega$.

Let us change the variables $t = \tau + T_1$, $\tau \in [0; T_1]$ in this problem. We will denote $G_0(\tau) = g(\tau + T_1)$, $U(x,\tau) = u(x, \tau + T_1)$, $a_{ij}^{(1)}(x,\tau) = a_{ij}(x, \tau + T_1)$, $b_{ij}^{(1)}(x,\tau) = b_{ij}(x, \tau + T_1)$, $f_2^{(1)}(x,\tau) = f_2(x, \tau + T_1)$, and $E^{(1)}(\tau) = E(\tau + T_1)$. For the pair $(U(x,\tau), G_0(\tau))$ we obtain the problem:

$$U_{\tau\tau} - \sum_{i,j=1}^{n}(a_{ij}^{(1)}(x,\tau)U_{x_i})_{x_j} - \sum_{i,j=1}^{n}(b_{ij}^{(1)}(x,\tau)U_{x_i\tau})_{x_j} + \varphi_1(x,U) + \varphi_2(x,U_\tau) = f_1^{(1)}(x)G_0(\tau) + f_2^{(1)}(x,\tau), \quad (x,\tau) \in Q_{T_1} \tag{64}$$

$$U(x,0) = u_1(x,T_1),\ U_t(x,0) = u_{1t}(x,T_1),\ \ x \in \Omega, \tag{65}$$

$$U|_{\partial\Omega\times(0,T_1)} = 0, \tag{66}$$

$$\int_\Omega K(x)U(x,\tau)\,dxdy = E^{(1)}(\tau), \qquad \tau \in [0,T_1]. \tag{67}$$

It is obvious that all coefficients of the Equation 64 and the functions $f_2^{(1)}(x,\tau)$, $u_1(x,T_1)$, $u_{1t}(x,T_1)$, $E^{(1)}(\tau)$ satisfy the same conditions as the functions from (2) and (5). According to I and II, there exists a unique solution to the problem (64–67) in Q_{T_1}, and, thus for the problems for the Equation 2 with conditions (4) and (5) as $t \in [T_1; 2T_1]$ and with the initial condition $u(x,T_1) = u_1(x,T_1)$, $u_t(x,T_1) = u_{1t}(x,T_1)$, and $x \in \Omega$, in the domain $Q_{T_1,2T_1}$. Denote it by $(u_2(x,t), g_{0,2}(t))$. By following similar reasoning on the intervals $[2T_1; 3T_1], \dots, [(N-1)T_1; NT_1]$, we can prove the existence and uniqueness of weak solutions $(u_k(x,kt), g_{0,k}(t))$, $k = 3,\dots,N$, in the domain $Q_{(k-1)T_1,kT_1} := \Omega \times ((k-1)T_1, kT_1)$ of the inverse problem for the Equation 2 with conditions (4) and (5) as $t \in [(k-1)T_1; kT_1]$ and the initial condition $u(x,(k-1)T_1) = u_{k-1}(x,(k-1)T_1)$, $x \in \Omega$. It is clear that a pair of functions $(u(x,t), g_0(t))$, where

$$u(x,t) = \begin{cases} u_1(x,t), & \text{if } (x,t) \in Q_{T_1}; \\ u_2(x,t), & \text{if } (x,t) \in Q_{T_1,2T_1}; \\ \dots & \dots \\ u_N(x,t), & \text{if } (x,t) \in Q_{(N-1)T_1,NT_1}, \end{cases}$$

$$g_0(t) = \begin{cases} g_{0,1}(t), & \text{if } t \in [0,T_1]; \\ g_{0,2}(t), & \text{if } t \in [T_1,2T_1]; \\ \dots & \dots \\ g_{0,N}(t), & \text{if } t \in [(N-1)T_1, NT_1], \end{cases}$$

is a solution for the problem (2–5) in the domain Q_T.

IV. The uniqueness of solution is proved similar as in II, III: Assume that $(u_{(1)}(x,t), g_{(1)}(t))$ and $(u_{(2)}(x,t), g_{(2)}(t))$ be two solutions of problem (2–5). Then the pair of functions $(\tilde{u}(x,t), \tilde{g}(t))$, where $\tilde{u}(x,t) = u_{(1)}(x,t) - u_{(2)}(x,t)$, $\tilde{g}(t) = g_{(1)}(t) - g_{(2)}(t)$, satisfies the conditions $\tilde{u}(x,0) \equiv 0$, $\tilde{u}_t(x,0) \equiv 0$, the equality

$$\int_{Q_\tau} \Big(\tilde{u}_{tt}v + \sum_{i,j=1}^{n} a_{ij}(x,t)\tilde{u}_{x_i}v_{x_j} + \sum_{i,j=1}^{n} b_{ij}(x)\tilde{u}_{x_it}v_{x_j} + (\varphi_1(x,u_{(1)}) - \varphi_1(x,u_{(2)}))v + (\varphi_2(x,(u_{(1)})_t) - \varphi_2(x,(u_{(2)})_t))v\Big)\,dx\,dt = \int_{Q_\tau} f_1(x)\tilde{g}(t)v\,dx\,dt, \tag{68}$$

for all $v \in L^2(0,T;H_0^1(\Omega))$, $\tau \in (0,T]$, and the equality

$$\tilde{g}(t) = \Big(\int_\Omega K(x)f_1(x)\,dx\Big)^{-1}\Big(\int_\Omega \Big(\sum_{i,j=1}^{n} K_{x_j}(x)a_{ij}(x,t)\tilde{u}_{x_i} - \sum_{i,j=1}^{n}(K_{x_j}(x)b_{ij}(x,t))_{x_i}\tilde{u}_t + K(x)(\varphi_1(x,u_{(1)}) - \varphi_1(x,u_{(2)})) + K(x)(\varphi_2(x,(u_{(1)})_t) - \varphi_2(x,(u_{(2)})_t))\Big)\,dx\Big),\ t \in [0,T], \tag{69}$$

holds.

Let us divide the interval $[0,T]$ into a finite number of intervals $[0,T_1]$, $[T_1, 2T_1]$, $\dots, [(N-1)T_1, NT_1]$, where $NT_1 = T$, and $T_1 \le T_0$.

Let us choose $\tau \in [0,T_1]$ in (68). After choosing here $v = \tilde{u}_t$, we get

$$\int_{Q_\tau} \Big(\tilde{u}_{tt}\tilde{u}_t + \sum_{i,j=1}^{n} a_{ij}(x,t)\tilde{u}_{x_i}\tilde{u}_{x_jt} + \sum_{i,j=1}^{n} b_{ij}(x,t)\tilde{u}_{x_it}\tilde{u}_{x_jt} + (\varphi_1(x,u_{(1)}) - \varphi_1(x,u_{(2)}))\tilde{u}_t + (\varphi_2(x,(u_{(1)})_t) - \varphi_2(x,(u_{(2)})_t))\tilde{u}_t\Big)\,dx\,dt = \int_{Q_\tau} f_1(x)\tilde{g}(t)\tilde{u}_t\,dx\,dt,\ \tau \in [0;T_1]. \tag{70}$$

It is easy to get from (69) and (H5) inequalities

$$\int_0^{T_1} (\tilde{g}(t))^2\,dt \le M_2 \int_{Q_{T_1}} \Big((\tilde{u}_t)^2 + \sum_{i=1}^{n}(\tilde{u}_{x_i})^2\Big)\,dx\,dt. \tag{71}$$

From (70), by the same way as from (39), we got (42). We find the following estimate:

$$\int_{Q_{T_1}} \Big((\tilde{u}_t)^2 + \sum_{i=1}^{n}(\tilde{u}_{x_i})^2\Big)\,dx\,dt \le M_3T_1 \int_0^{T_1} (\tilde{g}(t))^2\,dt, \tag{72}$$

and taking into account (71) from (72), we obtain $(1 - M_4T_1)\int_{Q_{T_1}} \Big((\tilde{u}_t)^2 + \sum_{i=1}^{n}(\tilde{u}_{x_i})^2\Big)\,dx\,dt \le 0$. Since $M_4T_1 < 1$, we conclude that $\int_{Q_{T_1}} \Big((\tilde{u}_t)^2 + \sum_{i=1}^{n}(\tilde{u}_{x_i})^2\Big)\,dx\,dt = 0$, hence, $u_{(1)} = u_{(2)}$ in Q_{T_1}. Then (71) implies $\tilde{g}(t) \equiv 0$, and, therefore, $g_{(1)}(t) \equiv g_{(2)}(t)$ on $[0,T_1]$.

Let us choose $\tau \in [0, 2T_1]$ in (68). After choosing here $v = \tilde{u}_t$, we get

$$\int_{Q_\tau} \Big(\tilde{u}_{tt}\tilde{u}_t + \sum_{i,j=1}^{n} a_{ij}(x,t)\tilde{u}_{x_i}\tilde{u}_{x_j t} + \sum_{i,j=1}^{n} b_{ij}(x,t)\tilde{u}_{x_i t}\tilde{u}_{x_j t} + (\varphi_1(x,u_{(1)}) - \varphi_1(x,u_{(2)}))\tilde{u}_t + (\varphi_2(x,(u_{(1)})_t) - \varphi_2(x,(u_{(2)})_t))\tilde{u}_t\Big)\,dx\,dt = \int_{Q_\tau} f_1(x)\tilde{g}(t)\tilde{u}_t\,dx\,dt,\ \ \tau \in [0; 2T_1]. \tag{73}$$

Note that $\tilde{u} \equiv 0$ in Q_{T_1} and $\tilde{g} \equiv 0$ on $[0; T_1]$, therefore, from (73) it follows that

$$\int_{Q_{T_1,\tau}} \Big(\tilde{u}_{tt}\tilde{u}_t + \sum_{i,j=1}^{n} a_{ij}(x,t)\tilde{u}_{x_i}\tilde{u}_{x_j t} + \sum_{i,j=1}^{n} b_{ij}(x,t)\tilde{u}_{x_i t}\tilde{u}_{x_j t} + (\varphi_1(x,u_{(1)}) - \varphi_1(x,u_{(2)}))\tilde{u}_t + (\varphi_2(x,(u_{(1)})_t) - \varphi_2(x,(u_{(2)})_t))\tilde{u}_t\Big)\,dx\,dt = \int_{Q_{T_1,\tau}} f_1(x)\tilde{g}(t)\tilde{u}_t\,dx\,dt,\ \ \tau \in [T_1; 2T1]. \tag{74}$$

It is easy to get from (69) and (H5) inequalities

$$\int_{T_1}^{2T_1} (\tilde{g}(t))^2\,dt \le M_2 \int_{Q_{T_1,2T_1}} \left((\tilde{u}_t)^2 + \sum_{i=1}^{n}(\tilde{u}_{x_i})^2\right)dx\,dt. \tag{75}$$

From (74), by the same way as from (39), we got (42). We find the following estimate

$$\int_{Q_{T_1,2T_1}} \left((\tilde{u}_t)^2 + \sum_{i=1}^{n}(\tilde{u}_{x_i})^2\right)dx\,dt \le M_3 T_1 \int_{T_1}^{2T_1} (\tilde{g}(t))^2\,dt, \tag{76}$$

and taking into account (75) from (76), we obtain $(1 - M_4T_1)\int_{Q_{T_1,2T_1}} \left((\tilde{u}_t)^2 + \sum_{i=1}^{n}(\tilde{u}_{x_i})^2\right)dx\,dt \le 0$. Since $M_4T_1 < 1$, we conclude that $\int_{Q_{T_1,2T_1}} \left((\tilde{u}_t)^2 + \sum_{i=1}^{n}(\tilde{u}_{x_i})^2\right)dx\,dt = 0$, hence, $u_{(1)} = u_{(2)}$ in $Q_{T_1,2T_1}$. Then, (75) implies $\tilde{g}(t) \equiv 0$, and, therefore, $g_{(1)}(t) \equiv g_{(2)}(t)$ on $[T_1, 2T_1]$. Therefore, $u_{(1)} = u_{(2)}$ in Q_{2T_1}, $g_{(1)}(t) \equiv g_{(2)}(t)$ on $[0, 2T_1]$.

Considering $\tau \in [0, 3T_1], \ldots, \tau \in [0, NT_1]$ in (68), by the same arguments, we find that $u_{(1)} = u_{(2)}$ in $Q_{(k-1)T_1,kT_1}$, $g_{(1)}(t) \equiv g_{(2)}(t)$ on $[(k-1)T_1, kT_1]$, $k = 1, 2, \ldots, N$. Therefore, $u_{(1)} = u_{(2)}$ in Q_T, $g_{(1)}(t) \equiv g_{(2)}(t)$ on $[0, T]$.

5 Conclusions

In this study, we have derived the necessary conditions for the existence and the uniqueness of the solution for the initial-boundary value problem, as well as the inverse problem, for semilinear hyperbolic equation of the third order with an unknown parameter in its right-hand side function. To determine the unknown function, an additional integral-type overdetermination condition have been introduced. These results were obtained by utilizing the properties of the solutions to the initial-boundary value problem and the method of successive approximations.

Data availability statement

Publicly available datasets were analyzed in this study. This data can be found at: no.

Author contributions

NP: Writing – original draft, Writing – review & editing.

Funding

The author(s) declare that no financial support was received for the research, authorship, and/or publication of this article.

Conflict of interest

The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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