frontiers | Frontiers in Applied Mathematics and Statistics

TYPE Original Research
PUBLISHED 04 September 2024
DOI 10.3389/fams.2024.1467426

OPEN ACCESS

EDITED BY
Kateryna Buryachenko,
Humboldt University of Berlin, Germany

REVIEWED BY
Serhii Bak,
Vinnytsia State Pedagogical University named after Mykhailo Kotsiubynsky, Ukraine
Mariia Savchenko,
Technical University of Braunschweig, Germany

*CORRESPONDENCE
Mykola Bokalo
✉ mm.bokalo@gmail.com

RECEIVED 19 July 2024
ACCEPTED 12 August 2024
PUBLISHED 04 September 2024

CITATION
Bokalo M, Skira I and Bokalo T (2024) Strong nonlinear functional-differential variational inequalities: problems without initial conditions.
Front. Appl. Math. Stat. 10:1467426.
doi: 10.3389/fams.2024.1467426

Strong nonlinear functional-differential variational inequalities: problems without initial conditions

Mykola Bokalo*, Iryna Skira and Taras Bokalo

Department of Mathematical Statistics and Differential Equations, Ivan Franko National University of Lviv, Lviv, Ukraine

Problems without initial conditions for evolution equations and variational inequalities appear in the modeling of different non-stationary processes within many fields of science, such as ecology, economics, physics, cybernetics, etc., if these processes started a long time ago and initial conditions do not affect them in the actual time moment. Thus, we can assume that the initial time is minus infinity. In the case of linear and weakly nonlinear evolution equations and variational inequalities, standard initial conditions should be replaced with the behavior of the solution as the time variable goes to minus infinity. However, for some strongly nonlinear evolution equations and variational inequalities, this problem has a unique solution in the class of functions without behavior restriction as the time variable goes to minus infinity. In this study, the correctness of the problem without initial conditions for such types of variational inequalities from a new class, or more precisely, for sub-differential inclusions with functionals, is investigated. Moreover, estimates of solutions are obtained. The results are new and mostly theoretical.

KEYWORDS

parabolic variational inequality, evolution variational inequality, evolution inclusion, sub-differential inclusion, Fourier problem, problem without initial conditions

1 Introduction

The aim of this study is to investigate problems without initial conditions for the evolution of functional-differential variational inequalities of a special form, so-called sub-differential inclusions with functionals. The partial case of this problem is a problem without initial conditions, or, in other words, the Fourier problem for integro-differential equations of the parabolic type.

Problem without initial conditions for evolution equations and variational inequalities (sub-differential inclusions) appear in the modeling of different non-stationary processes within many fields of science, such as ecology, economics, physics, cybernetics, etc., if these processes started a long time ago and initial conditions do not affect them in the actual time moment. Thus, we can assume that the initial time is minus infinity.

The research on the problem without initial conditions for the evolution equations and variational inequalities was conducted in the monographs [1–4], the papers [5–19], and others.

Note that the uniqueness of the solutions to the problem without initial conditions for linear and weak nonlinear evolution equations and variational inequalities is possible only under some restrictions on the behavior of solutions as the time variable changes to $-\infty$. Moreover, in this case, to prove the existence of a solution, it is necessary to impose certain restrictions on the growth of the input data when the time variable goes to $-\infty$. For the first time, it was strictly justified by Tychonoff [5] in the case of the heat equation. Later, similar results for various evolution equations and variational inequalities were obtained in monographs [1–4], papers [6–8, 12, 14, 16–19], and others.

However, as was shown by Bokalo [9], a problem without initial conditions for some strongly nonlinear parabolic equations has a unique solution in the class of functions without behavior restriction as the time variable changes to $-\infty$. Furthermore, similar results were obtained in studies [10, 13, 15] (see also references therein) for strongly nonlinear evolution equations and in Bokalo [11] for evolution variational inequalities.

Note that the problem without initial conditions for weakly nonlinear functional-differential variational inequalities was investigated only in the study [17]. There, the existence and uniqueness of the solution to this problem were proved under certain restrictions on its behavior and the growth of the input data when the time variable is directed to $-\infty$. As we know, the problem without initial conditions for strongly nonlinear functional-differential variational inequalities without restrictions on the behavior of the solution and the growth of the input data when the time variable is directed to $-\infty$ has not been considered in the literature, and this serves as one of the motivations for the study of such problems.

The outline of this study is as follows: Section 2 comprises notations, definitions of needed function spaces, and auxiliary results. In Section 3, we set the problem statement and provide our key findings. The proof of the main results is kept in Section 4. Comments on the main results are given in Section 5. Section 6 provides conclusions.

2 Preliminaries

Let V be a separable reflexive real Banach space with norm $\|\cdot\|$, and H be a real Hilbert space with the scalar products $(\cdot,\cdot)$ and norms $|\cdot|$, respectively. Suppose that $V \subset H$ with dense, continuous, and compact injection, i.e., the closure of V in H coincides with H, and there exists a constant $\lambda > 0$ such that $\lambda|v|^2 \leq \|v\|^2$ for all $v \in V$, and for every sequence $\{v_k\}_{k=1}^{\infty}$ bounded in V, there exists an element $v \in V$ and a subsequence $\{v_{k_j}\}_{j=1}^{\infty}$ such that $v_{k_j} \underset{j\to\infty}{\longrightarrow} v$ strongly in H.

Let V' and H' be the dual spaces of V and H, respectively. Suppose the space H' (after appropriate identification of functionals) is a subspace of V'. Identifying the spaces H and H' by the Riesz-Fréchet representation theorem, we obtain dense and continuous embeddings

$$V \subset H \subset V'. \tag{1}$$

Note that in this case $\langle g, v\rangle = (g, v)$ for every $v \in V, g \in H \subset V'$, where $\langle g, v\rangle$ is the means the action of an element $g \in V'$ on an element of $v \in V$, i.e., $\langle\cdot,\cdot\rangle$ is canonical product for the duality pair $[V', V]$. Therefore, we can use the notation $(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle$, and we will do it in the future.

Let $T > 0$ be an arbitrary fixed real number, and let $S := (-\infty, T]$, and $\mathrm{int}S := (-\infty, T)$.

We introduce some spaces for functions and distributions. Let X be an arbitrary Banach space with the norm $\|\cdot\|_X$. By $C(S; X)$ we mean the linear space of continuous functions defined on S with values in X. We say that $w_m \underset{m\to\infty}{\longrightarrow} w$ in $C(S; X)$ if for each $t_1, t_2 \in S$, $t_1 < t_2$, sequence $\{w_m|_{[t_1,t_2]}\}_{m=1}^{\infty}$ converges to $w|_{[t_1,t_2]}$ in $C([t_1, t_2]; X)$ (hereafter $\widetilde{w}|_{[t_1,t_2]}$ is restriction of a function $\widetilde{w} : S \to X$ to segment $[t_1, t_2] \subset S$).

Let $r \in [1,\infty]$, r' is dual to r, i.e., $1/r + 1/r' = 1$. Denote by $L^r_{\mathrm{loc}}(S; X)$ the linear space of classes of equivalent measurable functions $w : S \to X$ such that $w|_{[t_1,t_2]} \in L^r(t_1, t_2; X)$ for each $t_1, t_2 \in S$, $t_1 < t_2$. We say that a sequence $\{w_m\}$ is *bounded* (*strongly, weakly,* or $*$*-weakly* convergent, respectively, to w) in $L^r_{\mathrm{loc}}(S; X)$ if, for each $t_1, t_2 \in S$, $t_1 < t_2$, the sequence $\{w_m|_{[t_1,t_2]}\}$ is bounded (strongly, weakly, or $*$-weakly convergent, respectively, to $w|_{[t_1,t_2]}$) in $L^r(t_1, t_2; X)$.

By $D'(\mathrm{int}S; V'_w)$, we mean the space of continuous linear functionals on $D(\mathrm{int}S)$ with values in V'_w (hereafter, $D(\mathrm{int}S)$ is the space of test functions, i.e., the space of infinitely differentiable on $\mathrm{int}S$ functions with compact supports, equipped with the corresponding topology, and V'_w is the linear space V' equipped with weak topology). It is easy to see (using (1)) that spaces $L^r_{\mathrm{loc}}(S; V)$, $L^2_{\mathrm{loc}}(S; H)$, and $L^{r'}_{\mathrm{loc}}(S; V')$ can be identified with the corresponding subspaces of $D'(\mathrm{int}S; V'_w)$ by rule $\langle f, \varphi\rangle_D = \int_S f(t)\varphi(t)\,dt$, where $\langle\cdot,\cdot\rangle_D$ is the means the action of an element of $D'(\mathrm{int}S; V'_w)$ on an element of $D(\mathrm{int}S)$, f is an element of one of spaces $L^r_{\mathrm{loc}}(S; V)$, $L^2_{\mathrm{loc}}(S; H)$, $L^{r'}_{\mathrm{loc}}(S; V')$. In particular, this allows us to talk about derivatives w' of functions w from $L^r_{\mathrm{loc}}(S; V)$ or $L^2_{\mathrm{loc}}(S; H)$ in the perception of distributions $D'(\mathrm{int}S; V'_w)$ and the belonging of such derivatives to $L^{r'}_{\mathrm{loc}}(S; V')$ or $L^2_{\mathrm{loc}}(S; H)$.

Let us define the spaces

$$H^1_{\mathrm{loc}}(S; H) := \{w \in L^2_{\mathrm{loc}}(S; H) \mid w' \in L^2_{\mathrm{loc}}(S; H)\},$$

$$W^{1,r}_{\mathrm{loc}}(S; V) := \{w \in L^r_{\mathrm{loc}}(S; V) \mid w' \in L^{r'}_{\mathrm{loc}}(S; V')\}, \quad r > 1.$$

From known results [see, e.g., Gajewski et al. [20]] it follows that $H^1_{\mathrm{loc}}(S; H) \subset C(S; H)$ and $W^{1,r}_{\mathrm{loc}}(S; V) \subset C(S; H)$, and for every w in $H^1_{\mathrm{loc}}(S; H)$ or $W^{1,r}_{\mathrm{loc}}(S; V)$ the function $t \to |w(t)|^2$ is continuous on any segment of the interval S, and the following equality holds:

$$\frac{d}{dt}|w(t)|^2 = 2(w'(t), w(t)) \quad \text{for almost every (a.e.)} \quad t \in S. \tag{2}$$

In this study, we use the following well-known facts:

Proposition 2.1 [Corollaries from Young's inequality, Gajewski et al. [20]]. *Let $r > 1$, $\varepsilon > 0$ be arbitrary, and r' such that $1/r + 1/r' = 1$. Then, for all $a, b \in \mathbb{R}$, following inequality holds:*

$$a\,b \leq \varepsilon|a|^r + \varepsilon^{-1/(r-1)}\,|b|^{r'}. \tag{3}$$

In particular,

$$ab \leq \varepsilon|a|^2 + \varepsilon^{-1}|b|^2. \tag{4}$$

Proof. Inequality (3) is a corollary from standard Young's inequality: $ab \le |a|^r/r + |b|^{r'}/r'$, if we note that $r > 1$ and $r' > 1$. Inequality (4) we get from inequality (3) with $r = 2$. □

PROPOSITION 2.2 [Cauchy-Bunyakovsky-Schwarz inequality, Gajewski et al. [20]]. *Let t_1, $t_2 \in \mathbb{R}$, and $t_1 < t_2$. Then, for $v, w \in L^2(t_1, t_2; H)$, we have $\big(v(\cdot), w(\cdot)\big) \in L^1\big(t_1, t_2\big)$ and*

$$\int_{t_1}^{t_2} (w(t), v(t))\, dt \le \Big(\int_{t_1}^{t_2} |v(t)|^2\, dt\Big)^{1/2} \Big(\int_{t_1}^{t_2} |w(t)|^2\, dt\Big)^{1/2}.$$

PROPOSITION 2.3 [Hölder's inequality, Gajewski et al. [20]]. *Let $r \in [1, \infty]$, r' be a conjugated to r (i.e., $1/r + 1/r' = 1$), t_1, $t_2 \in \mathbb{R}$, $t_1 < t_2$. Suppose that X is a Banach space and X' is a dual of X, $\langle \cdot, \cdot \rangle_X$ is the action of an element of X' on an element of X. Then, for $v \in L^r(t_1, t_2; X)$ and $w \in L^{r'}(t_1, t_2; X')$, we have $\langle w(\cdot), v(\cdot)\rangle_X \in L^1\big(t_1, t_2\big)$ and*

$$\int_{t_1}^{t_2} \langle w(t), v(t)\rangle_X\, dt \le \|w\|_{L^{r'}(t_1,t_2;X')} \|v\|_{L^r(t_1,t_2;X)}.$$

PROPOSITION 2.4 [Lemma 1.1 [9]]. *Let $z : S \to \mathbb{R}$ be a nonnegative and absolutely continuous on each interval of S function that satisfies differential inequality*

$$z'(t) + \beta(t)\chi\big(z(t)\big) \le 0 \quad \textit{for a.e. } t \in S,$$

where $\beta \in L^1_{\mathrm{loc}}(S; \mathbb{R})$, $\beta(t) \ge 0$ for a.e. $t \in S$, $\int_S \beta(t)\, dt = +\infty$; $\chi \in C\big([0, +\infty)\big)$, $\chi(0) = 0$, $\chi(s) > 0$ if $s > 0$ and $\int_1^{+\infty} \frac{ds}{\chi(s)} < \infty$. Then $z \equiv 0$ on S.

PROPOSITION 2.5 [25]. *Let Y be a Banach space with the norm $\|\cdot\|_Y$, and $\{v_k\}_{k=1}^{\infty}$ be a sequence of elements of Y that is weakly or $*$-weakly convergent to v in Y. Then $\varliminf_{k\to\infty} \|v_k\|_Y \ge \|v\|_Y$.*

PROPOSITION 2.6 [Aubin theorem, Aubin [21]]. *Let $r > 1$ and $q > 1$ be given numbers. Suppose that B_0, B_1, and B_2 are Banach spaces such that $B_0 \overset{c}{\subset} B_1 \subset B_2$ (symbol $\subset$ means continuous embedding and symbol $\overset{c}{\subset}$ means compact embedding). Then*

$$\{w \in L^r(0, T; B_0) \mid w' \in L^q(0, T; B_2)\} \overset{c}{\subset} \big(L^r(0, T; B_1) \cap C([0, T]; B_2)\big). \tag{5}$$

Note that we understand embedding (5) as follows: if a sequence $\{w_m\}_{m=1}^{\infty}$ is bounded in the space $L^r(0, T; B_0)$, and the sequence $\{w'_m\}_{m=1}^{\infty}$ is bounded in the space $L^q(0, T; B_2)$, then there exists a function $w \in L^r(0, T; B_1) \cap C([0, T]; B_2)$ and the subsequence $\{w_{m_j}\}_{j=1}^{\infty}$ of the sequence $\{w_m\}_{m=1}^{\infty}$ such that $w_{m_j} \underset{j\to\infty}{\longrightarrow} w$ in $C([0, T]; B_2)$ and strongly in $L^r(0, T; B_1)$.

PROPOSITION 2.7. *Let a sequence $\{w_m\}_{m=1}^{\infty}$ be bounded in the space $L^r_{\mathrm{loc}}(S; V)$, where $r > 1$, and the sequence $\{w'_m\}$ be bounded in the space $L^2_{\mathrm{loc}}(S; H)$. Then there exists a function $w \in L^r_{\mathrm{loc}}(S; V)$, $w' \in L^2_{\mathrm{loc}}(S; H)$, and a subsequence $\{w_{m_j}\}_{j=1}^{\infty}$ of the sequence $\{w_m\}_{m=1}^{\infty}$ such that $w_{m_j} \underset{j\to\infty}{\longrightarrow} w$ in $C(S; H)$ and weakly in $L^r_{\mathrm{loc}}(S; V)$, and $w'_{m_j} \underset{j\to\infty}{\longrightarrow} w'$ weakly in $L^2_{\mathrm{loc}}(S; H)$.*

Proof. From Proposition 2.6 for $q = 2$, $B_0 = V$, $B_1 = B_2 = H$, we have that, for every $t_1, t_2 \in S$, $t_1 < t_2$, from the sequence of restrictions of the elements $\{w_m\}_{m=1}^{\infty}$ to the segment $[t_1, t_2]$, one can choose a subsequence that is convergent in $C([t_1, t_2]; H)$ and weakly in $L^r(t_1, t_2; V)$, and the sequence of derivatives of the elements of this subsequence is weakly convergent in $L^2(t_1, t_2; H)$. For each $k \in \mathbb{N}$, we choose a subsequence $\{w_{m_{k,j}}\}_{j=1}^{\infty}$ of the given sequence that is convergent in $C([T - k, T]; H)$ and weakly in $L^r(T - k, T; V)$ to some function $\widehat{w}_k \in C([T - k, T]; H) \cap L^r(T - k, T; V)$, and the sequence $\{w'_{m_{k,j}}\}_{j=1}^{\infty}$ is weakly convergent to the derivative $\widehat{w}'_k$ in $L^2(T - k, T; H)$. Making this choice, we ensure that the sequence $\{w_{m_{k+1,j}}\}_{j=1}^{\infty}$ was a subsequence of the sequence $\{w_{m_{k,j}}\}_{j=1}^{\infty}$. Now, according to the diagonal process, we select the desired subsequence as $\{w_{m_{j,j}}\}_{j=1}^{\infty}$, and we define the function w as follows: for each $k \in \mathbb{N}$, we take $w(t) := \widehat{w}_k(t)$ for $t \in (T - k, T - k + 1)$.

3 Statement of the problem and formulation of main results

Let $\Phi : V \to \mathbb{R}_\infty := (-\infty, +\infty]$ be a proper functional, i.e., $\mathrm{dom}(\Phi) := \{v \in V : \Phi(v) < +\infty\} \ne \emptyset$, which satisfies the conditions:

$(\mathcal{A}_1)$ $\Phi\big(\alpha v + (1 - \alpha)w\big) \le \alpha\Phi(v) + (1 - \alpha)\Phi(w)$ $\forall\, v, w \in V$, $\forall\, \alpha \in [0, 1]$,

i.e., the functional Φ is *convex*;

$(\mathcal{A}_2)$ $v_k \underset{k\to\infty}{\longrightarrow} v$ in V $\Longrightarrow$ $\varliminf_{k\to\infty} \Phi(v_k) \ge \Phi(v)$,

i.e., the functional Φ is *lower semicontinuous*;

$(\mathcal{A}_3)$ there exist the constants $p > 2$ and $K_1 > 0$ such that

$$\Phi(v) \ge K_1\|v\|^p \quad \forall\, v \in \mathrm{dom}(\Phi);$$

moreover, $\Phi(0) = 0$.

Recall [see, e.g., Showalter [4]] that for a functional Φ satisfying the conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ its *sub-differential* is a mapping $\partial\Phi : V \to 2^{V'}$, defined as follows:

$$\partial\Phi(v) := \{v^* \in V' \mid \Phi(w) \ge \Phi(v) + (v^*, w - v) \quad \forall\, w \in V\}, \quad v \in V,$$

and the *domain* of the sub-differential $\partial\Phi$ is the set $D(\partial\Phi) := \{v \in V \mid \partial\Phi(v) \ne \varnothing\}$. We identify the subdifferential $\partial\Phi$ with its graph, assuming that $[v, v^*] \in \partial\Phi$ if and only if $v^* \in \partial\Phi(v)$, i.e., $\partial\Phi = \{[v, v^*] \mid v \in D(\partial\Phi),\ v^* \in \partial\Phi(v)\}$. R. Rockafellar in study [22, Theorem A] proves that the sub-differential $\partial\Phi$ is a *maximal monotone operator*, i.e.,

$$(v_1^* - v_2^*, v_1 - v_2) \ge 0 \quad \forall\, [v_1, v_1^*],\ [v_2, v_2^*] \in \partial\Phi$$

and for every element $[v_1, v_1^*] \in V \times V'$ we have the implication

$$(v_1^* - v_2^*, v_1 - v_2) \ge 0 \quad \forall\, [v_2, v_2^*] \in \partial\Phi \quad \Longrightarrow \quad [v_1, v_1^*] \in \partial\Phi.$$

Suppose that the following condition holds:

$(\mathcal{A}_4)$ there exist the constants $q > 2$ and $K_2 > 0$, $K_3 > 0$ such that

$$(v_1^* - v_2^*, v_1 - v_2) \ge K_2|v_1 - v_2|^2 + K_3|v_1 - v_2|^q \quad \forall\, [v_1, v_1^*],\ [v_2, v_2^*] \in \partial\Phi.$$

Assume that $B(t,\cdot) : H \to H$, $t \in S$, is a given family of operators that satisfy the condition:

($\mathcal{B}$) for any $v \in H$ the mapping $B(\cdot, v) : S \to H$ is measurable, and there exists a constant $L \geq 0$ such that following inequality holds:

$$|B(t,v_1) - B(t,v_2)| \leq L|v_1 - v_2|$$

for a.e. $t \in S$, and all $v_1, v_2 \in H$; in addition, $B(t,0) = 0$ for a.e. $t \in S$.

Remark 3.1. From the condition ($\mathcal{B}$) it follows that

$$|B(t,v)| \leq L|v| \tag{6}$$

for a.e. $t \in S$ and for all $v \in H$.

Next, we will assume that the conditions ($\mathcal{A}_1$)—($\mathcal{A}_4$) and ($\mathcal{B}$) are fulfilled, and p' and q' are such that $1/p + 1/p' = 1$, $1/q + 1/q' = 1$.

Let us consider the **evolution variational inequality**, or, in other words, **subdifferential inclusion**

$$u'(t) + \partial\Phi(u(t)) + B(t,u(t)) \ni f(t), \quad t \in S, \tag{7}$$

where $f \in L^{p'}_{\text{loc}}(S; V') + L^{q'}_{\text{loc}}(S; H)$ is given function.

Definition 3.1. *The **solution** of variational inequality (7) is called a function $u : S \to V$ that satisfies the following conditions:*

1) $u \in W^{1,p}_{\text{loc}}(S; V) \cap L^q_{\text{loc}}(S; H)$;

2) $u(t) \in D(\partial\Phi)$ *for a.e.* $t \in S$;

3) *there exists a function* $g \in L^{p'}_{\text{loc}}(S; V') + L^{q'}_{\text{loc}}(S; H)$ *such that, for a.e.* $t \in S$, $g(t) \in \partial\Phi\big(u(t)\big)$ *and*

$$u'(t) + g(t) + B(t,u(t)) = f(t) \quad \text{in } V'.$$

The problem of finding a solution to variational inequality (7) for given Φ, B, and f is called the problem $\mathbf{P}(\Phi, B, f)$, and the function u is called its solution.

We consider the existence and uniqueness of the solution to the problem $\mathbf{P}(\Phi, B, f)$. The main results of this study are the following two theorems:

THEOREM 3.1. *Suppose that*

$$L < K_2. \tag{8}$$

Then the problem $\mathbf{P}(\Phi, B, f)$ *has at most one solution.*

THEOREM 3.2. *Let inequality (8) hold, and let* $f \in L^2_{\text{loc}}(S; H)$. *Then the problem* $\mathbf{P}(\Phi, B, f)$ *has a unique solution. In addition, this solution belongs to the space* $L^\infty_{\text{loc}}(S; V) \cap H^1_{\text{loc}}(S; H)$, *and for arbitrary* $t_1, t_2 \in S$, $t_1 < t_2$, $\delta > 0$ *satisfies the estimates:*

$$\max_{t\in[t_1,t_2]} |u(t)|^2 + \int_{t_1}^{t_2} \big[|u(t)|^2 + |u(t)|^q + \|u(t)\|^p\big]\,dt \leq C_1\Big[\delta^{-\frac{2}{q-2}} + \int_{t_1-\delta}^{t_2} |f(t)|^2\,dt\Big], \tag{9}$$

$$\operatorname*{ess\,sup}_{t\in[t_1,t_2]} \|u(t))\|^p + \int_{t_1}^{t_2} |u'(t)|^2\,dt \leq C_2\Big[\max\{\delta^{-\frac{2}{q-2}}, \delta^{-\frac{q}{q-2}}\} + \int_{t_1-2\delta}^{t_2} |f(t)|^2\,dt + \delta^{-1}\int_{t_1-2\delta}^{t_1} |f(t)|^2\,dt\Big], \tag{10}$$

where C_1, C_2 *are positive constants depending on* K_1, K_2, K_3, *and* q *only.*

Remark 3.2. If Φ is such that $\text{dom}(\Phi) := V$ and $\partial\Phi(v) = \{A(v)\}$, $v \in V$, where $A : V \to V'$ is some operator, then variational inequality (7) will be functional-differential equation

$$u'(t) + A(u(t)) + B(t,u(t)) = f(t), \quad t \in S. \tag{11}$$

Note that condition ($\mathcal{A}_3$) implies the coercivity of operator A, i.e.,

$$(A(v), v) \geq K_1\|v\|^p, \quad v \in V.$$

In addition, from condition ($\mathcal{A}_4$) follows the strong monotonicity of the operator A, i.e.,

$$(A(v_1) - A(v_2), v_1 - v_2) \geq K_2|v_1 - v_2|^2 + K_3|v_1 - v_2|^q \quad \forall\, v_1, v_2 \in V.$$

4 Proof of the main results

Proof. [Proof of the Theorem 3.1] Assume the contrary. Let u_1 and u_2 be two solutions to the problem $\mathbf{P}(\Phi, B, f)$. Then for every $i \in \{1, 2\}$ there exists function $g_i \in L^{p'}_{\text{loc}}(S; V') + L^{q'}_{\text{loc}}(S; H)$ such that, for a.e. $t \in S$, $g_i(t) \in \partial\Phi\big(u_i(t)\big)$ and

$$u_i'(t) + g_i(t) + B(t,u_i(t)) = f(t) \quad \text{in } V', \quad i = 1, 2. \tag{12}$$

We put $w := u_1 - u_2$. From equalities (12), for a.e. $t \in S$, we obtain

$$w'(t) + g_1(t) - g_2(t) + B(t,u_1(t)) - B(t,u_2(t)) = 0 \quad \text{in } V'. \tag{13}$$

Multiplying equality (13) scalar by $w(t)$, for a.e. $t \in S$, we obtain

$$(w'(t), w(t)) + (g_1(t) - g_2(t), u_1(t) - u_2(t)) + \big(B(t,u_1(t)) - B(t,u_2(t)), u_1(t) - u_2(t)\big) = 0. \tag{14}$$

By condition ($\mathcal{A}_4$) and the fact that $g_i(t) \in \partial\Phi(u_i(t))$, $i = 1, 2$, we have the inequality

$$(g_1(t) - g_2(t), u_1(t) - u_2(t)) \geq K_2|w(t)|^2 + K_3|w(t)|^q \quad \text{for a.e. } t \in S. \tag{15}$$

By condition ($\mathcal{B}$), for a.e. $t \in S$, we obtain

$$\big(B(t,u_1(t)) - B(t,u_2(t)), u_1(t) - u_2(t)\big) \geq -L|w(t)|^2. \tag{16}$$

By Equations (2), (8), (15), and (16), from Equation (14) we get such differential inequality

$$(|w(t)|^2)' + 2K_3\big(|w(t)|^2\big)^{q/2} \leq 0 \quad \text{for a.e. } t \in S. \tag{17}$$

From Equation (17), taking into account the condition $q/2 > 1$ and using Proposition 2.4 with $z(t) := |w(t)|^2$, $\beta(t) := 2K_3$ for all $t \in S$, and $\chi(s) := s^{q/2}$ for all $s \in [0, +\infty)$, we receive $|w(t)|^2 = 0$ for all $t \in S$, i.e., $u_1 = u_2$ a.e. on S. The resulting contradiction completes the proof of the uniqueness of the solution to the problem $\mathbf{P}(\Phi, B, f)$.

Proof. [Proof of the Theorem 3.2] We divide the proof into seven steps.

Step 1 (*auxiliary statements*). We define the functional $\Phi_H : H \to \mathbb{R}_\infty$ by the rule: $\Phi_H(v) := \Phi(v)$ if $v \in V$, and $\Phi_H(v) := +\infty$ otherwise. Note that conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, Lemma IV.5.2, and Proposition IV.5.2 of the monograph [4] imply that Φ_H is a proper, convex, and lower semicontinuous functional on H, $\mathrm{dom}(\Phi_H) = \mathrm{dom}(\Phi) \subset V$ and $\partial\Phi_H = \partial\Phi \cap (V \times H)$, where $\partial\Phi_H : H \to 2^H$ is the sub-differential of the functional Φ_H. In addition, the condition $(\mathcal{A}_3)$ implies that $0 \in \partial\Phi_H(0)$.

The following statements will be used in the sequel:

LEMMA 4.1 [[4, Lemma IV.4.3]]. *Let $-\infty < a < b < +\infty$, and $w \in H^1(a,b;H)$, $g \in L^2(a,b;H)$ such that $g(t) \in \partial\Phi_H(w(t))$ for a.e. $t \in (a,b)$. Then the function $\Phi_H(w(\cdot))$ is absolutely continuous on the interval $[a,b]$ and for any function $h:[a,b] \to H$ such that, for a.e. $t \in (a,b)$, $h(t) \in \partial\Phi_H\big(w(t)\big)$, and the following equality holds:*

$$\frac{d}{dt}\Phi_H\big(w(t)\big) = (h(t), w'(t)).$$

LEMMA 4.2 ([23, Proposition 3.12], [4, Proposition IV.5.2]). *Let $\widetilde{f} \in L^2(0,T;H)$ and $w_0 \in \mathrm{dom}(\Phi)$. Then there exists a unique function $w \in C([0,T];H) \cap H^1(0,T;H)$ such that $w(0) = w_0$ and, for a.e., $t \in (0,T]$, $w(t) \in D(\partial\Phi_H)$ and*

$$w'(t) + \partial\Phi_H\big(w(t)\big) \ni \widetilde{f}(t) \quad \text{in } H. \tag{18}$$

LEMMA 4.3. *Let $\widetilde{f} \in L^2(0,T;H)$ and $w_0 \in \mathrm{dom}(\Phi)$. Then there exists a unique function $w \in C([0,T];H) \cap H^1(0,T;H)$ such that $w(0) = w_0$ and, for a.e. $t \in (0,T]$, $w(t) \in D(\partial\Phi_H)$ and*

$$w'(t) + \partial\Phi_H\big(w(t)\big) + B(t,w(t)) \ni \widetilde{f}(t) \quad \text{in } H, \tag{19}$$

i.e., there exists $\widetilde{g} \in L^2(0,T;H)$ such that, for a.e. $t \in (0,T]$, we have $\widetilde{g}(t) \in \partial\Phi_H(w(t))$ and

$$w'(t) + \widetilde{g}(t) + B(t,w(t)) = \widetilde{f}(t) \quad \text{in } H. \tag{20}$$

Proof. [Proof of Lemma 4.3] Let $\alpha > 0$ be an arbitrary fixed number, and set

$$M := \{w \in C([0,T];H) \mid w(0) = w_0\}.$$

Consider M with the metric

$$\rho(w_1,w_2) = \max_{t\in[0,T]} \left[e^{-\alpha t}|w_1(t) - w_2(t)|\right], \quad w_1,w_2 \in M.$$

The metric space (M,ρ) is complete. Now let us consider an operator $A : M \to M$ defined as follows: for any given function $\widetilde{w} \in M$, it defines a function $\widehat{w} \in M \cap H^1(0,T;H)$ such that, for a.e. $t \in (0,T]$, $\widehat{w}(t) \in D(\partial\Phi_H)$ and

$$\widehat{w}'(t) + \partial\Phi_H(\widehat{w}(t)) \ni \widetilde{f}(t) - B(t,\widetilde{w}(t)) \quad \text{in} \quad H. \tag{21}$$

Clearly, variational inequality (21) coincides with variational inequality (18) after replacing $\widetilde{f}(t)$ by $\widetilde{f}(t) - B(t,\widetilde{w}(t))$, and $w(0) = w_0$ by $\widehat{w}(0) = w_0$. Thus, using Lemma 4.2, we get that operator A is well-defined. Let us demonstrate that the operator A is a contraction for some $\alpha > 0$. Indeed, let $\widetilde{w}_1, \widetilde{w}_2$ be arbitrary functions from M, and $\widehat{w}_1 := A\widetilde{w}_1$, $\widehat{w}_2 := A\widetilde{w}_2$. According to Equation (21) there exist functions $\widehat{g}_1$ and $\widehat{g}_2$ from $L^2(0,T;H)$ such that for every $j \in \{1,2\}$ and for a.e. $t \in (0,T]$ we have $\widehat{g}_j(t) \in \partial\Phi_H(\widehat{w}_j(t))$ and

$$\widehat{w}_j'(t) + \widehat{g}_j(t) = \widetilde{f}(t) - B(t,\widetilde{w}_j(t)), \tag{22}$$

while $\widehat{w}_j(0) = w_0$.

Subtracting identity (22) for $j = 2$ from identity (22) for $j = 1$, and, for a.e. $t \in (0,T]$, multiplying the obtained identity by $\widehat{w}_1(t) - \widehat{w}_2(t)$, we get

$$\big((\widehat{w}_1(t) - \widehat{w}_2(t))', \widehat{w}_1(t) - \widehat{w}_2(t)\big) + (\widehat{g}_1(t) - \widehat{g}_2(t), \widehat{w}_1(t) - \widehat{w}_2(t))$$
$$= -(B(t,\widetilde{w}_1(t)) - B(t,\widetilde{w}_2(t)), \widehat{w}_1(t) - \widehat{w}_2(t)) \quad \text{for a.e.} \quad t \in (0,T], \tag{23}$$
$$\widehat{w}_1(0) - \widehat{w}_2(0) = 0. \tag{24}$$

We integrate equality (23) by t from 0 to $\sigma \in (0,T]$, taking into account (24) and that [see Equation (2)] for a.e. $t \in (0,T]$. The following holds:

$$\big((\widehat{w}_1(t) - \widehat{w}_2(t))', \widehat{w}_1(t) - \widehat{w}_2(t)\big) = \frac{1}{2}\big(|\widehat{w}_1(t) - \widehat{w}_2(t)|^2\big)'.$$

As a result, we get the equality

$$\frac{1}{2}|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 + \int_0^\sigma (\widehat{g}_1(t) - \widehat{g}_2(t), \widehat{w}_1(t) - \widehat{w}_2(t))\,dt$$
$$= -\int_0^\sigma \big(B(t,\widetilde{w}_1(t)) - B(t,\widetilde{w}_2(t)), \widehat{w}_1(t) - \widehat{w}_2(t)\big)\,dt. \tag{25}$$

By condition $(\mathcal{A}_4)$, for a.e. $t \in (0,T]$, we have the inequality

$$(\widehat{g}_1(t) - \widehat{g}_2(t), \widehat{w}_1(t) - \widehat{w}_2(t)) \ge K_2|\widehat{w}_1(t) - \widehat{w}_2(t)|^2. \tag{26}$$

Taking into account condition $(\mathcal{B})$ and inequality (4) for a.e. $t \in (0,T]$, we obtain

$$\big|\big(B(t,\widetilde{w}_1(t)) - B(t,\widetilde{w}_2(t)), \widehat{w}_1(t) - \widehat{w}_2(t)\big)\big|$$
$$\le \big|B(t,\widetilde{w}_1(t)) - B(t,\widetilde{w}_2(t))\big|\,\big|\widehat{w}_1(t) - \widehat{w}_2(t)\big|$$
$$\le L|\widetilde{w}_1(t) - \widetilde{w}_2(t)|\,|\widehat{w}_1(t) - \widehat{w}_2(t)| \le \varepsilon|\widehat{w}_1(t) - \widehat{w}_2(t)|^2$$
$$+\varepsilon^{-1}L^2|\widetilde{w}_1(t) - \widetilde{w}_2(t)|^2, \tag{27}$$

where $\varepsilon > 0$ is an arbitrary.

From Equation (25), according to Equations (26) and (27), we have

$$|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 + 2(K_2 - \varepsilon)\int_0^\sigma |\widehat{w}_1(t) - \widehat{w}_2(t)|^2\,dt$$
$$\le 2\varepsilon^{-1}L^2 \int_0^\sigma \big|\widetilde{w}_1(t) - \widetilde{w}_2(t)\big|^2 dt. \tag{28}$$

Choosing $\varepsilon = K_2$, from Equation (28) we obtain

$$|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 \le C_3 \int_0^\sigma |\widetilde{w}_1(t) - \widetilde{w}_2(t)|^2\,dt, \quad \sigma \in (0,T], \tag{29}$$

where $C_3 := 2K_2^{-1}L^2$.

After multiplying inequality (30) by $e^{-2\alpha\sigma}$, we obtain

$$
\begin{aligned}
e^{-2\alpha\sigma}|\widehat{w}_1(\sigma)-\widehat{w}_2(\sigma)|^2 &\le C_3 e^{-2\alpha\sigma}\textstyle\int_0^\sigma e^{2\alpha t}e^{-2\alpha t}|\widetilde{w}_1(t)-\widetilde{w}_2(t)|^2\,dt\\
&\le C_3 e^{-2\alpha\sigma}\max_{t\in[0,T]}\left[e^{-\alpha t}|\widetilde{w}_1(t)-\widetilde{w}_2(t)|\right]^2\textstyle\int_0^\sigma e^{2\alpha t}\,dt\\
&= \tfrac{C_3}{2\alpha}\left(1-e^{-2\alpha\sigma}\right)\left[\rho(\widetilde{w}_1,\widetilde{w}_2)\right]^2 \le \tfrac{C_3}{2\alpha}\left[\rho(\widetilde{w}_1,\widetilde{w}_2)\right]^2,\\
&\qquad \sigma\in(0,T].
\end{aligned} \tag{30}
$$

From Equation (30), it easily follows that

$$\rho(\widehat{w}_1,\widehat{w}_2)\le\sqrt{C_3/(2\alpha)}\,\rho(\widetilde{w}_1,\widetilde{w}_2).$$

From this, choosing $\alpha > 0$ such that inequality $C_3/(2\alpha) < 1$ holds, we obtain that operator $A: M \to M$ is a contraction. Hence, we may apply the Banach fixed-point theorem [24, Theorem 5.7] and deduce that there exists a unique function $w \in M\cap H^1(0,T;H)$ such that $Aw = w$, i.e., we have proved over the statement, i.e., Lemma 4.3.

Step 2 (*solution approximations*). Let us consider the next ***problem***: *to find a function $u \in H^1_{\rm loc}(S;H)$ such that, for a.e., $t \in S$, $u(t) \in D(\partial\Phi_H)$ and*

$$u'(t)+\partial\Phi_H\big(u(t)\big)+B(t,u(t))\ni f(t)\quad in\ \ H. \tag{31}$$

We call this problem the problem $\mathbf{P}(\Phi_H,B,f)$. The solution of the problem $\mathbf{P}(\Phi_H,B,f)$ is the solution of the problem $\mathbf{P}(\Phi,B,f)$. We prove the existence of a solution to the problem $\mathbf{P}(\Phi_H,B,f)$.

At first, we construct a sequence of functions, that, in some perception, approximates the solution of the problem $\mathbf{P}(\Phi_H,B,f)$. For each $k\in\mathbb{N}$ we put $\widehat{f}_k(t) := f(t)$ for $t\in S_k := (T-k,T]$ and let us consider the problem of finding a function $\widehat{u}_k \in H^1(S_k;H)$ such that $\widehat{u}_k(T-k) = 0$ and, for a.e. $t\in S_k$, we have $\widehat{u}_k(t)\in D(\partial\Phi_H)$ and

$$\widehat{u}_k'(t)+\partial\Phi_H\big(\widehat{u}_k(t)\big)+B(t,\widehat{u}_k(t))\ni\widehat{f}_k(t)\quad\text{in}\ \ H. \tag{32}$$

The existence of a unique solution to problem (32) implies Lemma 4.3. Note that sub-differential inclusion in (32) means that there exists a function $\widehat{g}_k \in L^2(S_k;H)$ such that, for a.e., $t\in S_k$, we have $\widehat{g}_k(t)\in\partial\Phi_H(\widehat{u}_k(t))$ and

$$\widehat{u}_k'(t)+\widehat{g}_k(t)+B(t,\widehat{u}_k(t))=\widehat{f}_k(t)\quad\text{in}\ \ H. \tag{33}$$

Note that $D(\partial\Phi_H)\subset\operatorname{dom}(\Phi_H)=\operatorname{dom}(\Phi)\subset V$, and thus $\widehat{u}_k(t)\in V$ for a.e. $t\in S_k$. According to the definition of the subdifferential of a functional and the fact that $\widehat{g}_k(t)\in\partial\Phi(\widehat{u}_k(t))$, we have

$$\Phi(0)\ge\Phi(\widehat{u}_k(t))+(\widehat{g}_k(t),0-\widehat{u}_k(t))\quad\text{for a.e. }\ t\in S_k.$$

From this and condition $(\mathcal{A}_3)$ we obtain

$$(\widehat{g}_k(t),\widehat{u}_k(t))\ge\Phi(\widehat{u}_k(t))\ge K_1\|\widehat{u}_k(t)\|^p\quad\text{for a.e. }\ t\in S_k. \tag{34}$$

Since the left side of this chain of inequalities belongs to $L^1(S_k)$, then $\widehat{u}_k$ belongs to $L^p(S_k;V)$.

For each $k\in\mathbb{N}$, we extend functions $\widehat{f}_k,\widehat{u}_k$, and $\widehat{g}_k$ by zero for the entire interval S and denote these extensions by f_k, u_k, and g_k, respectively. From the above, it follows that, for each $k\in\mathbb{N}$, the function u_k belongs to $L^p(S;V)$, its derivative u_k' belongs to $L^2(S;H)$, and, for a.e. $t\in S$, $g_k(t)\in\partial\Phi_H\big(u_k(t)\big)$ and [see Equation (33)],

$$u_k'(t)+g_k(t)+B(t,u_k(t))=f_k(t)\quad\text{in}\quad H. \tag{35}$$

Step 3 (*estimates of solution approximations*). To demonstrate the convergence $\{u_k\}_{k=1}^\infty$ to the solution of the problem $\mathbf{P}(\Phi_H,B,f)$, we need some estimates of the functions u_k, $k\in\mathbb{N}$.

Let the function $\theta_*\in C^1(\mathbb{R})$ such that $\theta_*(t)=0$ if $t\in(-\infty,-1]$, $\theta_*(t)=e^{\frac{t^2}{t^2-1}}$ if $t\in(-1,0)$, $\theta_*(t)=1$ if $t\in[0,+\infty)$ [see Bokalo [9]]. Obviously, $\theta_*'(t)\ge0$ for arbitrary $t\in\mathbb{R}$, and for any $0<\nu<1$, we have

$$\sup_{t\in(-1,0)}\frac{\theta_*'(t)}{\theta_*^\nu(t)}=C_4, \tag{36}$$

where $C_4>0$ is a constant depending on ν only.

Let t_1, t_2, and δ be arbitrary real fixed numbers such that $t_1,t_2\in S$, $t_1<t_2$, $\delta>0$. We put

$$\theta(t):=\theta_*\Big(\frac{t-t_1}{\delta}\Big),\qquad t\in S. \tag{37}$$

It is clear that $\theta(t)=0$ if $t\in(-\infty,t_1-\delta]$, $0<\theta(t)<1$ if $t\in(t_1-\delta,t_1)$, $\theta(t)=1$ if $t\in[t_1,+\infty)$, and $\theta'(t)=\delta^{-1}\theta_*'((t-t_1)/\delta)\ge0$ for every $t\in\mathbb{R}$.

Let $k\in\mathbb{N}$. Obviously, $\theta u_k\in H^1(S;H)$. For each $t\in S$, multiply the identity (35) scalar by $\theta(t)u_k(t)$ and integrate from $t_1-\delta$ to $\tau\in[t_1,t_2]$. As a result, we obtain

$$
\begin{aligned}
&\int_{t_1-\delta}^{\tau}\theta(t)(u_k'(t),u_k(t))\,dt+\int_{t_1-\delta}^{\tau}\theta(t)(g_k(t),u_k(t))\,dt\\
&+\int_{t_1-\delta}^{\tau}\theta(t)\big(B(t,u_k(t)),u_k(t)\big)\,dt=\int_{t_1-\delta}^{\tau}\theta(t)(f_k(t),u_k(t))\,dt.
\end{aligned} \tag{38}
$$

From this, taking into account (2) and using the integration-by-parts formula, we transform the first term on the left side of the equality (38) as follows:

$$
\begin{aligned}
&\int_{t_1-\delta}^{\tau}\theta(t)(u_k'(t),u_k(t))\,dt=\frac12\int_{t_1-\delta}^{\tau}\theta(t)\big(|u_k(t)|^2\big)'\,dt=\frac12|u_k(\tau)|^2\\
&-\frac12\int_{t_1-\delta}^{t_1}\theta'(t)|u_k(t)|^2\,dt.
\end{aligned} \tag{39}
$$

Then from Equation (38), using Equation (39), we receive

$$
\begin{aligned}
&|u_k(\tau)|^2+2\int_{t_1-\delta}^{\tau}\theta(t)(g_k(t),u_k(t))\,dt=\int_{t_1-\delta}^{t_1}\theta'(t)|u_k(t)|^2\,dt\\
&-2\int_{t_1-\delta}^{\tau}\theta(t)\big(B(t,u_k(t)),u_k(t)\big)\,dt+2\int_{t_1-\delta}^{\tau}\theta(t)(f_k(t),u_k(t))\,dt.
\end{aligned} \tag{40}
$$

Since $(0,0)\in\partial\Phi_H$ and $(g_k(t),u_k(t))\in\partial\Phi_H$ for a.e. $t\in S$, from condition $(\mathcal{A}_4)$ we get

$$(g_k(t),u_k(t))\ge K_2|u_k(t)|^2+K_3|u_k(t)|^q\quad\text{for a.e. }\ t\in S. \tag{41}$$

According to the definition of u_k and g_k and using the inequality (34), we obtain

$$(g_k(t), u_k(t)) \geq \Phi\big(u_k(t)\big) \geq K_1 \|u_k(t)\|^p \quad \text{for a.e. } t \in S. \tag{42}$$

Let us estimate the second term on the left-hand side of equality (40), using inequalities (41) and (42), in this way:

$$2\int_{t_1-\delta}^{\tau} \theta(t)(g_k(t), u_k(t))\, dt \geq 2(\sigma + (1-\sigma)) \int_{t_1-\delta}^{\tau} \theta(t)(g_k(t), u_k(t))dt$$
$$\geq 2\sigma K_2 \int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)|^2\, dt + 2\sigma K_3 \int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)|^q\, dt$$
$$+ 2(1-\sigma)K_1 \int_{t_1-\delta}^{\tau} \theta(t)\|u_k(t)\|^p\, dt + 2(1-\sigma)\int_{t_1-\delta}^{\tau} \theta(t)\Phi\big(u_k(t)\big)\, dt, \tag{43}$$

where $\sigma \in (0,1)$ is arbitrary.

Using the inequality (34) (with $r = q/2$, $r' = q/(q-2)$), we estimate the first term on the right-hand side of Equation (40) as follows:

$$\int_{t_1-\delta}^{\tau} \theta'(t)|u_k(t)|^2\, dt = \int_{t_1-\delta}^{t_1} \theta'(t)\theta^{-\frac{2}{q}}(t)\cdot\theta^{\frac{2}{q}}(t)|u_k(t)|^2\, dt$$
$$\leq \varepsilon_1 \int_{t_1-\delta}^{t_1} \theta(t)|u_k(t)|^q\, dt + \varepsilon_1^{-\frac{2}{q-2}} \int_{t_1-\delta}^{t_1} \left(\theta'(t)\theta^{-\frac{2}{q}}(t)\right)^{\frac{q}{q-2}} dt, \tag{44}$$

where $\varepsilon_1 > 0$ is an arbitrary number.

Based on Equation (36), it is easy to demonstrate that

$$\int_{t_1-\delta}^{t_1} \left(\theta'(t)\cdot\theta^{-\frac{2}{q}}(t)\right)^{\frac{q}{q-2}} dt =$$
$$\int_{t_1-\delta}^{t_1} \left(\delta^{-1}\cdot\theta_*'((t-t_1)/\delta)\cdot\theta_*^{-\frac{2}{q}}((t-t_1)/\delta)\right)^{\frac{q}{q-2}} dt$$
$$= \Big[(t-t_1)/\delta = s,\ t = \delta s + t_1,\ dt = \delta ds\Big] = \delta^{-\frac{2}{q-2}}$$
$$\int_{-1}^{0} \left(\theta_*'(s)\cdot\theta_*^{-\frac{2}{q}}(s)\right)^{\frac{q}{q-2}} ds$$
$$\leq C_4^{\frac{q}{q-2}}\cdot\delta^{-\frac{2}{q-2}}, \tag{45}$$

where C_4 is constant from Equation (36) with $\nu = 2/q$ (note that C_4 depends on q only).

So from Equation (44) using Equation (45), we obtained

$$\int_{t_1-\delta}^{\tau} \theta'(t)|u_k(t)|^2\, dt \leq \varepsilon_1 \int_{t_1-\delta}^{t_1} \theta(t)|u_k(t)|^q\, dt + C_5\,(\varepsilon_1\delta)^{-\frac{2}{q-2}}, \tag{46}$$

where $C_5 := C_4^{\frac{q}{q-2}}$ depends on q only.

Let us estimate the second term on the right-hand side of equality (40). Using (6), we receive

$$\left|\int_{t_1-\delta}^{\tau} \theta(t)\big(B(t,u_k(t)), u_k(t)\big)dt\right| \leq \int_{t_1-\delta}^{\tau} \theta(t)\big|B(t,u_k(t))\big||u_k(t)|\, dt$$
$$\leq L\int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)|^2\, dt. \tag{47}$$

Let us estimate the third term on the right-hand side of equality (40), using inequality (4):

$$\int_{t_1-\delta}^{\tau} \theta(t)(f_k(t), u_k(t))\, dt \leq \int_{t_1-\delta}^{\tau} \theta(t)|f_k(t)||u_k(t))|\, dt$$
$$\leq \varepsilon_2 \int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)|^2\, dt + \varepsilon_2^{-1}\int_{t_1-\delta}^{\tau} \theta(t)|f_k(t)|^2\, dt, \tag{48}$$

where $\varepsilon_2 > 0$ is an arbitrary constant.

From Equation (40), using Equations (43), and (46)–(48), we receive

$$|u_k(\tau)|^2 + 2(\sigma K_2 - L - \varepsilon_2)\int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)|^2\, dt + (2\sigma K_3 - \varepsilon_1)$$
$$\int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)|^q\, dt + 2(1-\sigma)K_1\int_{t_1-\delta}^{\tau} \theta(t)\|u_k(t)\|^p\, dt$$
$$+ 2(1-\sigma)\int_{t_1-\delta}^{\tau} \theta(t)\Phi\big(u_k(t)\big)\, dt$$
$$\leq C_5(\varepsilon_1\delta)^{-\frac{2}{q-2}} + 2\varepsilon_2^{-1}\int_{t_1-\delta}^{\tau} \theta(t)|f_k(t)|^2\, dt. \tag{49}$$

In Equation (49), using condition (8), we choose $\sigma \in (0,1)$ such that the inequality $\sigma K_2 - L > 0$ holds, and then we take $\varepsilon_1 = \sigma K_3$, $\varepsilon_2 = (\sigma K_2 - L)/2$. As a result, we get

$$|u_k(\tau)|^2 + \int_{t_1-\delta}^{\tau} \theta(t)\big[|u_k(t)|^2 + |u_k(t)|^q + \|u_k(t)\|^p + \Phi\big(u_k(t)\big)\big]\, dt$$
$$\leq C_6\delta^{-\frac{2}{q-2}} + C_7\int_{t_1-\delta}^{\tau} \theta(t)|f_k(t)|^2\, dt, \tag{50}$$

where C_6, C_7 are positive constants dependent on K_1, K_2, K_3, L, and q only.

Since $\tau \in [t_1, t_2]$ is arbitrary, from Equation (50) and the definition of θ, we obtain

$$\max_{t\in[t_1,t_2]} |u_k(t)|^2 + \int_{t_1}^{t_2} |u_k(t)|^2\, dt +$$
$$\int_{t_1}^{t_2} |u_k(t)|^q\, dt + \int_{t_1}^{t_2} \|u_k(t)\|^p\, dt + \int_{t_1}^{t_2} \Phi\big(u_k(t)\big)\, dt$$
$$\leq 2\,C_6\delta^{-\frac{2}{q-2}} + 2\,C_7\int_{t_1-\delta}^{t_2} |f_k(t)|^2\, dt. \tag{51}$$

From Equation (50) and the definition of f_k, since $t_1, t_2 \in S$ and $\delta > 0$ are all arbitrary, it follows that

the sequence $\{u_k\}$ is bounded in $L^{\infty}_{\mathrm{loc}}(S;H)$, $L^2_{\mathrm{loc}}(S;H)$, $L^q_{\mathrm{loc}}(S;H)$, and $L^p_{\mathrm{loc}}(S;V)$, *and* (52)

the sequence $\big\{\Phi(u_k)\big\}$ is bounded in $L^1_{\mathrm{loc}}(S)$. (53)

Step 4 (estimates of derivatives of solution approximations). Now let us find estimates of u_k', $k \in \mathbb{N}$. Let t_1, t_2, and δ be arbitrary real numbers such that $t_1, t_2 \in S$, $t_1 < t_2$, and $\delta > 0$. θ is a function defined above. We multiply equality (35) for almost every $t \in S$

scalar by $\theta(t)u_k'(t)$ and integrate the resulting equality from $t_1-\delta$ to $\tau \in [t_1,t_2]$:

$$\int_{t_1-\delta}^{\tau} \theta(t)|u_k'(t)|^2\,dt + \int_{t_1-\delta}^{\tau} \theta(t)(g_k(t),u_k'(t))\,dt + \int_{t_1-\delta}^{\tau} \theta(t)(B(t,u_k(t)),u_k'(t))\,dt = \int_{t_1-\delta}^{\tau} \theta(t)(f_k(t),u_k'(t))\,dt. \tag{54}$$

Since $g_k \in L^2(t_1-\delta,t_2;H)$ and $g_k(t) \in \partial\Phi_H(u_k(t))$ for a. e. $t \in (t_1-\delta,t_2)$, Lemma 4.1 implies that the function $\Phi_H\big(u_k(\cdot)\big)$ is continuous on $[t_1-\delta,t_2]$ and

$$\big(\Phi_H(u_k(t))\big)' = (g_k(t),u_k'(t)) \quad \text{for a.e. } t \in (t_1-\delta,t_2). \tag{55}$$

Taking into account Equation (55), we can estimate the second term on the left side of Equation (54) as follows:

$$\int_{t_1-\delta}^{\tau} \theta(t)(g_k(t),u_k'(t))\,dt = \int_{t_1-\delta}^{\tau} \theta(t)\big(\Phi_H(u_k(t))\big)'\,dt = \Phi_H\big(u_k(\tau)\big) - \int_{t_1-\delta}^{\tau} \theta'(t)\Phi_H\big(u_k(t)\big)\,dt \geq \Phi_H\big(u_k(\tau)\big) - \max_{t\in[t_1-\delta,t_1]} \theta'(t) \int_{t_1-\delta}^{t_1} \Phi_H\big(u_k(t)\big)\,dt. \tag{56}$$

By inequality (4) with $\varepsilon = 4$, taking into Equation (6), we receive

$$\Big|\int_{t_1-\delta}^{\tau} \theta(t)\big(B(t,u_k(t)),u_k'(t)\big)\,dt\Big| \leq \int_{t_1-\delta}^{\tau} \theta(t)\big|B(t,u_k(t))\big||u_k'(t)|\,dt \leq L\int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)||u_k'(t)|\,dt \leq 4L^2\int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)|^2 dt + \frac{1}{4}\int_{t_1-\delta}^{\tau} \theta(t)|u_k'(t)|^2\,dt, \tag{57}$$

$$\int_{t_1-\delta}^{\tau} \theta(t)(f_k(t),u_k'(t))\,dt \leq 4\int_{t_1-\delta}^{\tau} \theta(t)|f_k(t)|^2\,dt + \frac{1}{4}\int_{t_1-\delta}^{\tau} \theta(t)|u_k'(t)|^2\,dt. \tag{58}$$

From Equation (54), using Equations (56)–(58) and

$$\max_{t\in[t_1-\delta,t_1]} \theta'(t) = \delta^{-1} \max_{t\in[t_1-\delta,t_1]} \theta_*'((t-t_1)/\delta) \leq C_8\delta^{-1}, \quad C_8 := \max_{s\in[-1,0]} \theta_*'(s),$$

we have

$$\frac{1}{2}\int_{t_1}^{\tau} |u_k'(t)|^2\,dt + \Phi_H\big(u_k(\tau)\big) \leq 4\int_{t_1-\delta}^{\tau} |f_k(t)|^2\,dt + 4L^2\int_{t_1-\delta}^{\tau} |u_k(t)|^2\,dt + C_8\delta^{-1}\int_{t_1-\delta}^{t_1} \Phi_H\big(u_k(t)\big)\,dt. \tag{59}$$

Since $\tau \in [t_1,t_2]$ is arbitrary, from Equation (59) by the definition of Φ_H and condition $(\mathcal{A}_3)$ (remind that $u_k(t) \in V$ for a.e. $t \in S$), we have

$$\operatorname{less\,sup}_{t\in[t_1,t_2]} \|u_k(t)\|^p + \int_{t_1}^{t_2} |u_k'(t)|^2\,dt \leq C_9\Big[\int_{t_1-\delta}^{t_2} |f_k(t)|^2\,dt + \int_{t_1-\delta}^{t_2} |u_k(t)|^2\,dt + \delta^{-1}\int_{t_1-\delta}^{t_1} \Phi\big(u_k(t)\big)\,dt\Big], \tag{60}$$

where $C_9 > 0$ is a positive constant dependent on K_1 and L only.

From Equation (60), taking into account (51), we obtain

$$\operatorname{ess\,sup}_{t\in[t_1,t_2]} \|u_k(t)\|^p + \int_{t_1}^{t_2} |u_k'(t)|^2\,dt \leq C_{10}\Big[\delta^{-\frac{2}{q-2}} + \delta^{-\frac{q}{q-2}} + \int_{t_1-2\delta}^{t_2} |f_k(t)|^2\,dt + \delta^{-1}\int_{t_1-2\delta}^{t_1} |f_k(t)|^2\,dt\Big], \tag{61}$$

where $C_{10} > 0$ is a positive constant dependent on K_1, K_2, K_3, L, and q only.

From the estimate (4) and the definition of f_k, since $t_1, t_2 \in S$ and $\delta > 0$ are arbitrary, it implies that

$$\text{the sequence } \{u_k\}_{k=1}^{+\infty} \text{ is bounded in } L^\infty_{\mathrm{loc}}(S;V), \tag{62}$$

$$\text{the sequence } \{u_k'\}_{k=1}^{+\infty} \text{ is bounded in } L^2_{\mathrm{loc}}(S;H). \tag{63}$$

From Equations (6) and (51) we have

$$\int_{t_1}^{t_2} \big|B(t,u_k(t))\big|^2\,dt \leq L^2\int_{t_1}^{t_2} |u_k(t)|^2\,dt \leq C_{11}\Big(1 + \int_{t_1-1}^{t_2} |f_k(t)|^2\,dt\Big) \leq C_{12}, \tag{64}$$

where C_{11}, C_{12} are positive constants independent on $k \in \mathbb{N}$.

From Equations (35), (63), and (64) and the definition of f_k, we get that

$$\text{the sequence } \{g_k\}_{k=1}^{+\infty} \text{ is bounded in } L^2_{\mathrm{loc}}(S;H). \tag{65}$$

Step 5 (*passing the limit*). Since V is reflexive Banach space, H is Hilbert space, and V embeds in H by compact injection, from Equations (52), (62), (63), (65), and Proposition 2.7, we have the existence of functions $u \in L^\infty_{\mathrm{loc}}(S;V) \cap L^q_{\mathrm{loc}}(S;H) \cap H^1_{\mathrm{loc}}(S;H)$, $g \in L^2_{\mathrm{loc}}(S;H)$, and a subsequence of the sequence $\{u_k,g_k\}_{k=1}^{+\infty}$ (until denoted by $\{u_k,g_k\}_{k=1}^{+\infty}$) such that

$$u_k \underset{k\to\infty}{\longrightarrow} u \quad *\text{-weakly in } L^\infty_{\mathrm{loc}}(S;V), \text{ and weakly in } L^p_{\mathrm{loc}}(S;V), \tag{66}$$

$$u_k \underset{k\to\infty}{\longrightarrow} u \quad \text{weakly in } L^q_{\mathrm{loc}}(S;H), \text{ and weakly in } H^1_{\mathrm{loc}}(S;H), \tag{67}$$

$$u_k \underset{k\to\infty}{\longrightarrow} u \quad \text{in } C(S;H), \tag{68}$$

$$g_k \underset{k\to\infty}{\longrightarrow} g \quad \text{weakly in } L^2_{\mathrm{loc}}(S;H). \tag{69}$$

From Equation (68) and condition $(\mathcal{B})$, for each $t_0 < T$, we have

$$\int_{t_0}^{T} \big|B(t,u_k(t)) - B(t,u(t))\big|^2\,dt \leq L^2\int_{t_0}^{T} |u_k(t)-u(t)|^2\,dt \underset{k\to\infty}{\longrightarrow} 0.$$

Thus, we obtain

$$B(\cdot,u_k(\cdot)) \underset{k\to\infty}{\longrightarrow} B(\cdot,u(\cdot)) \quad \text{strongly in} \quad L^2_{\mathrm{loc}}(S;H). \tag{70}$$

Let $v \in H$, $\varphi \in C(S)$ be arbitrary while supp φ is compact. For a.e. $t \in S$, we multiply equality (35) by v and $\varphi(t)$, and then integrate in t on S. As a result, we obtain equality

$$\int_S (u_k'(t), v)\varphi(t)\,dt + \int_S (g_k(t), v)\varphi(t) + \int_S (B(t, u_k(t)), v)\varphi(t)\,dt = \int_S (f_k(t), v)\varphi(t)\,dt, \quad k \in \mathbb{N}. \tag{71}$$

We pass to the limit in Equation (71) as $k \to \infty$, taking into account (67), (69), (70), and the convergence of $\{f_k\}_{k=1}^{\infty}$ to f in $L^2_{\rm loc}(S; H)$. As a result, since v, φ are arbitrary, for a.e. $t \in S$, we obtain the equality

$$u'(t) + g(t) + B(t, u(t)) = f(t) \quad \text{in } H.$$

Step 6 (proof that $u(t) \in D(\partial\Phi_H)$ and $g(t) \in \partial\Phi_H\big(u(t)\big)$ for a. e. $t \in S$). Let $k \in \mathbb{N}$ be an arbitrary number. Since $u_k(t) \in D(\partial\Phi_H)$ and $g_k(t) \in \partial\Phi_H\big(u_k(t)\big)$ for a.e. $t \in S$, applying the monotonicity of the sub-differential $\partial\Phi_H$, we obtain that for a.e. $t \in S$ the following inequality holds:

$$(g_k(t) - v^*, u_k(t) - v) \geq 0 \quad \forall\,[v, v^*] \in \partial\Phi_H. \tag{72}$$

Let $\tau \in S$ and $h > 0$ be arbitrary numbers. We integrate (72) in t from $\tau - h$ to τ:

$$\int_{\tau-h}^{\tau} (g_k(t) - v^*, u_k(t) - v)\,dt \geq 0 \quad \forall\,[v, v^*] \in \partial\Phi_H. \tag{73}$$

Now we pass to the limit in Equation (73) as $k \to \infty$, according to Equations (68) and (69). As a result, we obtain

$$\int_{\tau-h}^{\tau} (g(t) - v^*, u(t) - v)\,dt \geq 0 \quad \forall\,[v, v^*] \in \partial\Phi_H. \tag{74}$$

The monograph [25, Theorem 2] and Equation (74) imply that for every $[v, v^*] \in \partial\Phi_H$ there exists a set of measure zero $R_{[v,v_*]} \subset S$ such that for all $\tau \in S \setminus R_{[v,v^*]}$ we have $u(\tau) \in V$, $g(\tau) \in H$

$$0 \leq \lim_{h\to+0} \frac{1}{h}\int_{\tau-h}^{\tau} \big(g(t)-v^*, u(t)-v\big)\,dt = \big(g(\tau)-v^*, u(\tau)-v\big) \geq 0. \tag{75}$$

Let us demonstrate that there exists a set of measure zero $R \subset S$ such that

$$\forall \tau \in S\setminus R: \qquad \big(g(\tau)-v^*, u(\tau)-v\big) \geq 0 \quad \forall\,[v, v^*] \in \partial\Phi_H. \tag{76}$$

Since V and H are separable spaces, there exists a countable set $F \subset \partial\Phi_H$, which is dense in $\partial\Phi_H$. Denote $R := \bigcup_{[v,v^*]\in F} R_{[v,v_*]}$. Since the set F is countable and any countable union of sets of measure zero is a set of measure zero, then R is a set of measure zero.

Therefore, for any $\tau \in S \setminus R$ inequality (76) holds for every $[v, v^*] \in F$. Let $[\widehat{v}, \widehat{v}^*]$ be an arbitrary element from $\partial\Phi_H$. Then from the density F in $\partial\Phi_H$ we have the existence of a sequence $\{[v_l, v_l^*]\}_{l=1}^{\infty} \subset F$ such that $v_l \to v$ in V, $v_l^* \to v^*$ in H, and for every $\tau \in S \setminus R$

$$(g(\tau) - v_l^*, u(\tau) - v_l) \geq 0 \quad \forall l \in \mathbb{N}. \tag{77}$$

Thus, passing to the limit in inequality (77) as $l \to \infty$, we obtain $(g(\tau) - v^*, u(\tau) - v) \geq 0$ for every $\tau \in S \setminus R$. Hence, we have Equation (76), i.e., for a.e. $t \in S$, the following holds:

$$(g(t) - v^*, u(t) - v) \geq 0 \quad \forall\,[v, v^*] \in \partial\Phi_H.$$

From this, according to the maximal monotonicity of $\partial\Phi_H$, we obtain that $[u(t), g(t)] \in \partial\Phi_H$ for a.e. $t \in S$, i.e., $u(t) \in D(\partial\Phi_H)$ and $g(t) \in \partial\Phi_H(u(t))$ for a.e. $t \in S$. Thus, function u is the solution of the problem $\mathbf{P}(\Phi, B, f)$, and therefore $\mathbf{P}(\Phi_H, B, f)$.

Step 7 (completion of proof). Estimates (9) and (10) of the solution of the problem $\mathbf{P}(\Phi, B, f)$ follow directly from estimates (51) (given that $\int_{t_1}^{t_2} \Phi\big(u_k(t)\big)\,dt \geq 0$) and (4), convergence (66)–(68) and Proposition 2.5. □

5 Comments on the main results

Let us introduce an example of the problem that is studied here. Let $n \in \mathbb{N}$, Ω be a bounded domain in $\mathbb{R}^n$, $\partial\Omega$ be the boundary of Ω, and $\partial\Omega$ be the piecewise surface. We put $Q := \Omega \times S$, $\Sigma := \partial\Omega \times S$, and $\Omega_t := \Omega \times \{t\}$ $\forall t \in S$. For an arbitrary measurable set $F \subset \mathbb{R}^k$, where $k = n$ or $k = n + 1$, and $r \in [1, \infty]$, let $L^r(F)$ be the standard Lebesgue space with norm $\|\cdot\|_{L^r(F)}$. Let $L^r_{\rm loc}(\overline{Q})$ be the linear space of classes of equivalent functions defined on Q such that their restrictions on any bounded measurable set $Q' \subset Q$ belong to $L^r(Q')$. For $r \in (1, \infty)$, we denote by $W^{1,r}(\Omega) = \{v \in L^r(\Omega) \mid v_{x_i} \in L^r(\Omega), i = \overline{1, n}\}$ the standard Sobolev space with norm $\|v\|_{W^{1,r}(\Omega)} := \big(\|v\|^r_{L^r(\Omega)} + \|\nabla v\|^r_{L^r(\Omega)}\big)^{1/r}$, where $\nabla u := (u_{x_1}, \ldots, u_{x_n})$ [see, e.g., Brezis [24]].

Let $p > 2$ and K be a nonempty convex closed set in $W^{1,p}(\Omega)$, which contains 0. We consider the ***problem***: *find a function $u \in L^p_{\rm loc}(\overline{Q})$ such that $u_{x_i} \in L^p_{\rm loc}(\overline{Q})$, $i = \overline{1, n}$, $u_t \in L^2_{\rm loc}(\overline{Q})$, and, for a.e. $t \in S$, we have $u(\cdot, t) \in K$ and*

$$\int_{\Omega_t} \Big[u_t(v-u) + |\nabla u|^{p-2}\nabla u \nabla(v-u) + |u|^{p-2}u(v-u) + a(x)u(v-u)$$

$$+ (v-u)\int_{\Omega} b(x, y, t)u(y, t)\,dy\Big]\,dx \geq \int_{\Omega_t} f(v-u)\,dx \quad \forall v \in K, \tag{78}$$

where $f \in L^2_{\rm loc}(\overline{Q})$, $a \in L^\infty(\Omega)$, and $S \ni t \to b(\cdot, \cdot, t) \in L^2(\Omega \times \Omega)$ are given.

This problem is called problem (78), and a function u is its solution.

Note that in cases $K = W^{1,p}(\Omega)$, this problem is equivalent to the problem of finding a weak solution to a problem without initial conditions for a nonlinear integro-differential parabolic equation:

$$u_t - \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + |u|^{p-2}u + a(x)u + \int_\Omega b(x, y, t)u(y, t)\,dy = f(x, t), \quad (x, t) \in Q,$$

$$\frac{\partial u}{\partial \nu} = 0.$$

We remark that problem (78) can be written more abstractly. Indeed, after appropriate identification of functions and functionals, we have continuous and dense embedding

$$W^{1,p}(\Omega) \subset L^2(\Omega) \subset (W^{1,p}(\Omega))',$$

where $(W^{1,p}(\Omega))'$ is dual to $W^{1,p}(\Omega)$ space. Clearly, for any $h \in L^2(\Omega)$ and $v \in W^{1,p}(\Omega)$, we have $\langle h, v\rangle = (h, v)$, where $\langle\cdot,\cdot\rangle$ is the notation for action of element of $(W^{1,p}(\Omega))'$ on element of $W^{1,p}(\Omega)$, and $(\cdot,\cdot)$ is a scalar product in $L^2(\Omega)$. Thus, we can use the notation $(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle$.

Now, we denote $V := W^{1,p}(\Omega)$, $H := L^2(\Omega)$ and define operators $A : V \to V'$ and $B(t,\cdot) : H \to H$, $t \in S$, as follows:

$$(A(v), w) = \int\limits_{\Omega} \left[|\nabla v|^{p-2}\nabla v \nabla w + |v|^{p-2} v w + a v w\right] dx, \quad v, w \in V, \tag{79}$$

$$B(t, v)(\cdot) := \int\limits_{\Omega} b(\cdot, y, t) v(y)\, dy, \quad v \in H,\ t \in S. \tag{80}$$

Then problem (78) can be rewritten as follows: *find a function $u \in L^p_{\text{loc}}(S; V)$ such that $u' \in L^2_{\text{loc}}(S; H)$ and, for a.e. $t \in S$, we have $u(t) \in K$ and*

$$(u'(t) + A(u(t)) + B(t, u(t)), v - u(t)) \geq (f(t), v - u(t)) \quad \forall\, v \in K, \tag{81}$$

where $f \in L^2_{\text{loc}}(S; H)$ is given function.

We remark that, for a.e. $t \in S$, variational inequality (81) can be written as

$$(u'(t) + A(u(t)) + B(t, u(t)) - f(t), v - u(t)) + I_K(v) - I_K(u(t)) \geq 0 \quad \forall\, v \in V, \tag{82}$$

where

$$I_K(v) := \begin{cases} 0, & \text{if} \quad v \in K, \\ +\infty, & \text{if} \quad v \in V \setminus K. \end{cases} \tag{83}$$

We can write inequality (82) as follows:

$$I_K(v) \geq I_K(u(t)) + (-u'(t) - A(u(t)) - B(t, u(t)) + f(t), v - u(t)) \quad \forall\, v \in V. \tag{84}$$

The functional I_K from V to $\mathbb{R}_\infty$ is proper, convex and lower semicontinuous. By the definition of the subdifferential $\partial I_K : V \to 2^{V'}$ inequality (84) is equivalent to inclusion

$$\partial I_K(u(t)) \ni -u'(t) - A(u(t)) - B(t, u(t)) + f(t),$$

i.e.,

$$u'(t) + A(u(t)) + \partial I_K(u(t)) + B(t, u(t)) \ni f(t). \tag{85}$$

We define

$$\Psi(v) := \int\limits_{\Omega} \left[p^{-1}(|\nabla v|^p + |v|^p) + 2^{-1} a |v|^2\right] dx, \quad v \in V, \tag{86}$$

and

$$\Phi(v) := \Psi(v) + I_K(v), \quad v \in V. \tag{87}$$

The functionals Ψ and Φ from V to $\mathbb{R}_\infty$ are proper, convex and lower semicontinuous. As easy to demonstrate, we have $\partial\Psi(v) = \{A(v)\} \subset V'$ for each $v \in V$, and

$$\partial\Phi(v) := A(v) + \partial I_K(v), \quad v \in V. \tag{88}$$

From the above [see, in particular, Equations (85) and (88)], it follows that the problem (79) can be written as such a subdifferential inclusion: *find a function $u \in L^p_{\text{loc}}(S; V)$ such that $u' \in L^2_{\text{loc}}(S; H)$ and, for a.e. $t \in S$, $u(t) \in D(\partial\Phi)$ and*

$$u'(t) + \partial\Phi(u(t)) + B(t, u(t)) \ni f(t) \quad \text{in} \quad H. \tag{89}$$

So problem (78) is a partial case of the problem $\mathbf{P}(\Phi, B, f)$. Based on this, let's illustrate the main results of this study (see Theorems 1, 2).

COROLLARY 5.1. *Let the following condition hold:*

$$\operatorname*{ess\,sup}_{t\in S} \|b(\cdot,\cdot,t)\|_{L^2(\Omega\times\Omega)} < \operatorname*{ess\,inf}_{x\in\Omega} a(x). \tag{90}$$

Then problem (78) has a unique solution. In addition, it belongs to the space $L^\infty_{\text{loc}}(S; W^{1,p}(\Omega)) \cap H^1_{\text{loc}}(S; L^2(\Omega))$ and for arbitrary $t_1, t_2 \in S$, $t_1 < t_2$, $\delta > 0$ satisfies the estimates:

$$\max_{t\in[t_1,t_2]} \int\limits_{\Omega} |u(x,t)|^2\, dx + \int\limits_{t_1}^{t_2}\int\limits_{\Omega} \left[|u(x,t)|^2 + |u(x,t)|^p + |\nabla u(x,t)|^p\right] dxdt \tag{91}$$

$$\leq C_{15}\Big[\delta^{-\frac{2}{q-2}} + \int\limits_{t_1-\delta}^{t_2}\int\limits_{\Omega} |f(x,t)|^2\, dxdt\Big], \tag{92}$$

$$\operatorname*{ess\,sup}_{t\in[t_1,t_2]} \int\limits_{\Omega} \left[|u(x,t)|^p + |\nabla u(x,t)|^p\right] dx + \int\limits_{t_1}^{t_2}\int\limits_{\Omega} |u_t(x,t)|^2\, dxdt$$

$$\leq C_{16}\Big[\max\{\delta^{-\frac{2}{q-2}}, \delta^{-\frac{q}{q-2}}\} + \int\limits_{t_1-2\delta}^{t_2}\int\limits_{\Omega} |f(x,t)|^2\, dxdt$$

$$+ \delta^{-1}\int\limits_{t_1-2\delta}^{t_1}\int\limits_{\Omega} |f(x,t)|^2\, dxdt\Big], \tag{93}$$

where C_{15}, C_{16} are positive constants depending on $\operatorname*{ess\,sup}_{t\in S} \|b(\cdot,\cdot,t)\|_{L^2(\Omega\times\Omega)}$, $\operatorname*{ess\,inf}_{x\in\Omega} a(x)$, *and p only.*

Proof. [Proof of Corollary 5.1] We need to demonstrate that functional Φ, defined in Equations (83)—(87), and family of operators $B(t,\cdot)$, $t \in S$, defined in Equation 80, satisfy the conditions of Theorems 1, 2.

Writing the functional Ψ defined in Equation (86) in the form

$$\Psi(v) = p^{-1}\|v\|^p_{W^{1,p}(\Omega)} + 2^{-1}\int\limits_{\Omega} a|v|^2\, dx, \quad v \in W^{1,p}(\Omega), \tag{94}$$

we obtain that the functional Ψ is proper and $\text{dom}(\Psi) = W^{1,p}(\Omega)$.

Note that for arbitrary $r \geq 2$, function $F_r(\xi) = |\xi|^r$, $\xi \in \mathbb{R}^n$, is convex. Indeed, for all $\alpha \in [0,1]$, we have

$$F_r(\alpha\xi + (1-\alpha)\eta) = |\alpha\xi + (1-\alpha)\eta|^r \leq (\alpha|\xi| + (1-\alpha)|\eta|)^r$$

$$\leq \alpha|\xi|^r+(1-\alpha)|\eta|^r=\alpha F_r(\xi)+(1-\alpha)F_r(\eta),\quad \xi,\eta\in\mathbb{R}^n. \quad (95)$$

Here we used the convex function $g_r(s)=s^r$, $s\in[0,+\infty)$, since $g_r''(s)=r(r-1)s^{r-2}>0$ for all $s\in(0,+\infty)$.

From Equation (95), with $r=p$ and $r=2$, it is easy to see that functional Ψ is convex, hence functional Φ satisfies the condition $(\mathcal{A}_1)$.

Let $v_k \underset{k\to\infty}{\longrightarrow} v$ in $W^{1,p}(\Omega)$. Then $\|v_k\|_{W^{1,p}(\Omega)} \underset{k\to\infty}{\longrightarrow} \|v\|_{W^{1,p}(\Omega)}$ and $v_k \underset{k\to\infty}{\longrightarrow} v$ in $L^2(\Omega)$. From this, it follows:

$$\|v_k\|^p_{W^{1,p}(\Omega)} \underset{k\to\infty}{\longrightarrow} \|v\|^p_{W^{1,p}(\Omega)}, \quad (96)$$

$$\Big|\int_\Omega a|v_k|^2\,dx-\int_\Omega a|v|^2\,dx\Big| \leq \int_\Omega a|v_k^2-v^2|\,dx=\int_\Omega a|v_k+v|\,|v_k-v|\,dx$$

$$\leq \operatorname{ess\,sup} a\cdot(\|v_k\|_{L^2(\Omega)}+\|v\|_{L^2(\Omega)})\cdot\|v_k-v\|_{L^2(\Omega)} \underset{k\to\infty}{\longrightarrow} 0. \quad (97)$$

From Equations (94), (96), and (97), it follows that the functional Ψ is lower semicontinuous, hence functional Φ satisfies the condition $(\mathcal{A}_2)$.

Since $a>0$ a.e. on Ω, then [see Equation (94)]

$$\Psi(v)\geq p^{-1}\|v\|^p_{W^{1,p}(\Omega)},\quad v\in W^{1,p}(\Omega).$$

Hence, given that $I_K(v)\geq 0$, $v\in V$, condition $(\mathcal{A}_3)$ holds with $K_2:=p^{-1}$.

It is easy to show that

$$\partial\Psi(v)=\{A(v)\}\subset(W^{1,p}(\Omega))'\quad\forall v\in W^{1,p}(\Omega),$$

where $A(\cdot)$ is defined in Equation (79).

Then for any $v_1,v_2\in W^{1,p}(\Omega)$ we have

$$(A(v_1)-A(v_2),v_1-v_2)=\int_\Omega[(|\nabla v_1|^{p-2}\nabla v_1-|\nabla v_2|^{p-2}\nabla v_2)$$

$$(\nabla v_1-\nabla v_2)$$

$$+(|v_1|^{p-2}v_1-|v_2|^{p-2}v_2)(v_1-v_2)+a|v_1-v_2|^2]\,dx. \quad (98)$$

Since the function $F_p(\xi)=|\xi|^r$, $\xi\in\mathbb{R}^n$, is convex, from the convexity criterion we have

$$(\nabla F_p(\xi)-\nabla F_p(\eta))(\xi-\eta)\geq 0,\quad \xi,\eta\in\mathbb{R}^n. \quad (99)$$

Since $\nabla F_p(\xi)=p|\xi|^{p-2}\xi$, $\xi\in\mathbb{R}^n$, then from Equation (99) it follows:

$$\int_\Omega[(|\nabla v_1|^{p-2}\nabla v_1-|\nabla v_2|^{p-2}\nabla v_2)(\nabla v_1-\nabla v_2)\,dx\geq 0. \quad (100)$$

By Bokalo [9], for arbitrary $s_1,s_2\in\mathbb{R}$, the inequality

$$(|s_1|^{p-2}s_1-|s_2|^{p-2}s_2)(s_1-s_2)\geq 2^{2-p}|s_1-s_2|^p$$

holds. Hence, for all $v_1,v_2\in L_p(\Omega)$, we have

$$\int_\Omega(|v_1|^{p-2}v_1-|v_2|^{p-2}v_2)(v_1-v_2)\,dx\geq 2^{2-p}\int_\Omega|v_1-v_2|^p\,dx. \quad (101)$$

Using Hölder's inequality (see Proposition 2.3) with $r=p/2$, we have this chain of inequalities:

$$\int_\Omega|v_1-v_2|^2\,dx\leq\Big(\int_\Omega 1^{r'}\,dx\Big)^{\frac{1}{r'}}\Big(\int_\Omega|v_1-v_2|^p\,dx\Big)^{\frac{1}{r}}=$$

$$(\mathrm{mes}_n\Omega)^{\frac{p-2}{p}}\Big(\int_\Omega|v_1-v_2|^p\,dx\Big)^{\frac{2}{p}}.$$

From this, we obtain

$$\int_\Omega|v_1-v_2|^p\,dx\geq(\mathrm{mes}_n\Omega)^{\frac{2-p}{2}}\Big(\int_\Omega|v_1-v_2|^2\,dx\Big)^{\frac{p}{2}}$$
$$=(\mathrm{mes}_n\Omega)^{\frac{2-p}{2}}\|v_1-v_2\|^p_{L^2(\Omega)}. \quad (102)$$

From Equations (101), (102) it follows:

$$\int_\Omega(|v_1|^{p-2}v_1-|v_2|^{p-2}v_2)(v_1-v_2)\,dx$$
$$\geq 2^{2-p}(\mathrm{mes}_n\Omega)^{\frac{2-p}{2}}\|v_1-v_2\|^p_{L^2(\Omega)}. \quad (103)$$

Also, we have

$$\int_\Omega a|v_1-v_2|^2\,dx\geq(\operatorname*{ess\,inf}_\Omega a)\int_\Omega|v_1-v_2|^2\,dx. \quad (104)$$

Hence, from Equation (98), using Equations (100), (103), and (104), we have

$$(A(v_1)-A(v_2),v_1-v_2)\geq K_2\|v_1-v_2\|^2_{L^2(\Omega)}+K_3\|v_1-v_2\|^p_{L^2(\Omega)},$$
$$v_1,v_2\in W^{1,p}(\Omega), \quad (105)$$

where $K_2:=\operatorname*{ess\,inf}_\Omega a$, $K_3:=2^{2-p}(\mathrm{mes}_n\Omega)^{\frac{2-p}{2}}$.

From Equation (94) and the monotonicity of $I_K(\cdot)$ it follows condition $(\mathcal{A}_4)$ with $q=p$.

Let us prove that condition $(\mathcal{B})$ holds. Since Equation (80), we have for almost all $t\in S$ and for all $v_1,v_2\in L^2(\Omega)$:

$$\|B(t,v_1)(\cdot)-B(t,v_2)(\cdot)\|_{L^2(\Omega)}=\Big\|\int_\Omega b(\cdot,y,t)(v_1(y)-v_2(y))\,dy\Big\|_{L^2(\Omega)}$$

$$\leq\int_\Omega|v_1(y)-v_2(y)|\cdot\|b(\cdot,y,t)\|_{L^2(\Omega)}\,dy\leq\|b(\cdot,\cdot,t)\|_{L^2(\Omega\times\Omega)}\cdot$$

$$\|v_1-v_2\|_{L^2(\Omega)}\leq L\|v_1-v_2\|_{L^2(\Omega)},$$

where $L:=\operatorname*{ess\,sup}_{t\in S}\|b(\cdot,\cdot,t)\|_{L^2(\Omega\times\Omega)}$, i.e., condition $(\mathcal{B})$ holds.

From the above, it follows that in this case, condition (8) has form (90). Estimates (91) and (93) are derived directly from estimates (9) and (10).

6 Conclusion

We investigated the problem without initial conditions for some strictly nonlinear functional-differential variational inequalities in the form of sub-differential inclusions with functionals. The conditions for the existence of a unique solution to this problem in the absence of restrictions on the solution's behavior and the growth of input data when the time variable is directed to $-\infty$ have been obtained. There are also estimates of the solution to the researched problem provided.

The results obtained here can be used to study mathematical models in many fields of science, such as ecology, economics, physics, cybernetics, etc.

In the future, it would be worthwhile to obtain similar results for functional-differential variational inequalities that do not have the form of subdifferential inclusions with functionals.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

Author contributions

MB: Conceptualization, Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing. IS: Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing. TB: Investigation, Validation, Writing – original draft, Writing – review & editing.

Funding

The author(s) declare that no financial support was received for the research, authorship, and/or publication of this study.

Acknowledgments

The authors thank all the reviewers for their constructive suggestions.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

All claims expressed in this article are solely those of the authors and do not necessarily represent those of their affiliated organizations, or those of the publisher, the editors and the reviewers. Any product that may be evaluated in this article, or claim that may be made by its manufacturer, is not guaranteed or endorsed by the publisher.

References

1. Lions J-L. *Quelques méthodes de résolution des problémes aux limites non linéaires*. Paris: Dunod Gauthier-Villars. (1969).

2. Pankov A. *Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations*. Dordrecht: Kluwer. (1990). doi: 10.1007/978-94-011-9682-6

3. Tikhonov A, Samarskii A. *Equations of Mathematical Physics*. M: Nauka. (1972).

4. Showalter R. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. Providence: Amer Math Soc. (1997).

5. Tychonoff A. Théorémes d'unicité pour l'équation de la chaleur. *Mat Sb*. (1935) 42:199–216.

6. Oleinik O, Iosifjan G. Analog of Saint-Venant's principle and uniqueness of solutions of the boundary problems in unbounded domain for parabolic equations. *Usp Mat Nauk*. (1976) 31:142–66. doi: 10.1070/RM1976v031n06ABEH001583

7. Showalter R. Singular nonlinear evolution equations. *Rocky Mountain J Math*. (1980) 10:499–507. doi: 10.1216/RMJ-1980-10-3-499

8. Ivasishen SD. Parabolic boundary problems without initial conditions. *Ukr Mat Zh*. (1982) 34:547–552. doi: 10.1007/BF01086129

9. Bokalo NM. Problem without initial conditions for some classes of nonlinear parabolic equations. *J Soviet Mathem*. (1990) 51:2291–322. doi: 10.1007/BF01094990

10. Pukach PYa. On problem without initial conditions for some nonlinear degenarated parabolic system. *Ukrainian Mathem J*. (1994) 46:484-487. doi: 10.1007/BF01060422

11. Bokalo M. Well-posedness of problems without initial conditions for nonlinear parabolic variational inequalities. *Nonlin Boundary Problem*. (1998) 8:58–63.

12. Buhrii O. Some parabolic variational inequalities without initial conditions. *Visnyk Lviv Univ Series Mech Mathem*. (1998) 49:113–21.

13. Bokalo M, Dmytryshyn Yu. Problems without initial conditions for degenerate implicit evolution equations. *Electr J Differ Equat*. (2008) 2008:1–16.

14. Bokalo M, Lorenzi A. Linear evolution first-order problems without initial conditions. *Milan J Mathematics*. (2009) 77:437–94. doi: 10.1007/s00032-009-0107-6

15. Bokalo M. Almost periodic solutions of anisotropic elliptic-parabolic equations with variable exponents of nonlinearity. *Electr J Differ Equat*. (2014) 2014:1–13. doi: 10.1007/s10958-015-2290-z

16. Bokalo MM, Tsebenko AM. Optimal control for systems governed by parabolic equations without initial conditions with controls in the coefficients. *Electr J Differ Equat*. (2017) 2017:1–22. doi: 10.15330/cmp.8.1.21-37

17. Bokalo M, Skira I. Fourier problem for weakly nonlinear evolution inclusions with functionals. *J Optimiz Differ Equat Applic*. (2019) 27:1-20. doi: 10.15421/141901

18. Bokalo MM, Ilnytska OV. Problems without initial conditions for nonlinear evolution inclusions with variable time-delay. *J Nonl Evol Equat Applic*. (2020) 2019:59–79.

19. Bokalo Mykola, Ilnytska Olga, Bokalo Taras. Nonlinear parabolic variational inequalities with variable time-delay in time unbounded domains. *J Optimiz Differ Equat Applic*. (2023) 31:67-88. doi: 10.15421/142311

20. Gajewski H, Gröger K, Zacharias K. *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*. Berlin:Akademie-Verlag (1974). doi: 10.1515/9783112717899

21. Aubin J-P. Un theoreme de compacite. *Comptes rendus hebdomadaires des seances de l'academie des sci*. (1963) 256:5042–4.

22. Rockafellar R. On the maximal monotonicity of subdifferential mappings. *Pacific J Math*. (1970) 33:209–16. doi: 10.2140/pjm.1970.33.209

23. Brézis H. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert Amsterdam*. London: North-Holland Publishing Comp. (1973).

24. Brezis H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York, Dordrecht, Heidelberg, London: Springer. (2011). doi: 10.1007/978-0-387-70914-7

25. Yoshida K. *Functional Analysis*. Berlin, Heidelberg: Springer-Verlag. (1995).