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# Some class of nonlinear partial differential equations in the ring of copolynomials over a commutative ring

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We study the copolynomials, i.e., $K$-linear mappings from the ring of polynomials $K[x]$ into the commutative ring $K$. With the help of the Cauchy–Stieltjes transform of a copolynomial, we introduce and examine a multiplication of copolynomials. We investigate the Cauchy problem related to the nonlinear partial differential equation $\frac{\partial u}{\partial t} = au^{m_0}\left(\frac{\partial u}{\partial x}\right)^{m_1}\left(\frac{\partial^2 u}{\partial x^2}\right)^{m_2}\left(\frac{\partial^3 u}{\partial x^3}\right)^{m_3}$, $m_0, m_1, m_2, m_3 \in \mathbb{N}_0$, $\sum_{j=0}^{3} m_j > 0$, $a \in K$ in the ring of copolynomials. To find a solution, we use the series of powers of the $\delta$-function. As examples, we consider the Cauchy problem with the Euler–Hopf equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$, for a Hamilton–Jacobi type equation $\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2$, and for the Harry Dym equation $\frac{\partial u}{\partial t} = u^3\frac{\partial^3 u}{\partial x^3}$.

KEYWORDS

**copolynomial, $\delta$-function, partial differential equation, Cauchy problem, Cauchy-Stieltjes transform, multiplication of copolynomials**

## 1 Introduction

The first, second, and third order equations play an important role in the theory of nonlinear partial differential equations. A significant portion of classical nonlinear differential equations is dedicated to these classes (see, for example, [1–5]). In this paper, we examine a purely algebraic approach to study the special Cauchy problem with the following evolution equation:

$$\frac{\partial u}{\partial t} = au^{m_0}\left(\frac{\partial u}{\partial x}\right)^{m_1}\left(\frac{\partial^2 u}{\partial x^2}\right)^{m_2}\left(\frac{\partial^3 u}{\partial x^3}\right)^{m_3} \tag{1.1}$$

$$u(0, x) = u_0\delta(x). \tag{1.2}$$

We study this Cauchy problem in the module $K[x]'$ of the $K$-linear functionals on the ring of polynomials $K[x]$, where $K$ is an arbitrary commutative integral domain with identity and $a, u_0 \in K$. We consider the module $K[x]'$ as an algebraic analog of space of distributions (see [6, 7]), where linear partial differential equations in the module $K[x]'$ were studied). In this paper, the elements of the module $K[x]'$ are called copolynomials (see Section 2). A copolynomial $\delta(x)$ is defined in the usual way: $(\delta, p) = p(0)$, $p \in K[x]$. A multiplication operation for copolynomials plays an important role for us. We define the product of copolynomials using the Cauchy–Stieltjes transform (see Section 3). We take note of several non-equivalent constructions of a multiplication that are considered in classical theories of distributions. For example, in the Colombeau theory [8, 9], the square of the $\delta$-function is well-defined, but in some other theories it is not defined (see, for example, Antosik et al. [10]; Section 12.5).

In Section 4, we prove the existence and uniqueness theorem for the Cauchy problem (1.1), (1.2), and establish a representation of the solution in the form of the series in powers of the $\delta$-function (Theorem 4.1). As examples, we consider the Cauchy problem for the Euler–Hopf equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$, for the Hamilton–Jacobi type equation $\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2$, and for the Harry Dym equation $\frac{\partial u}{\partial t} = u^3\frac{\partial^3 u}{\partial x^3}$. In some of these examples, an interesting connection between classical nonlinear partial differential equations and well-known integer sequences is discovered (see examples 4.1, 4.2, and 4.4, where the Euler–Hopf equation, the Hamilton–Jacobi equation, and the Harry Dym equation are studied, respectively). Note that we restrict our consideration of equations of type (1.1) to those of the order no higher than three for two reasons. First, the representation in the proof of Theorem 4.1 generally becomes more cumbersome. Second, we are unaware of any classical examples of nonlinear equations of type (1.1) of order higher than three (see [3, 5]).

Linear functionals in the space of polynomials were extensively studied from different points of views in algebra, combinatorics, and the theory of orthogonal polynomials (cf., for example, [11–13]). In a classical case of ($K = \mathbb{R}$ or $K = \mathbb{C}$), series with respect to derivatives of the $\delta$-function are intensively studied because of their applications to differential and functional-differential equations and the theory of orthogonal polynomials [13]. Formal power series solutions of nonlinear partial differential equations were examined in a number of studies (cf., for example, [14–16]).

## 2 Preliminary

Let $K$ be an arbitrary commutative integral domain with identity, and let $K[x]$ be a ring of polynomials with coefficients in $K$.

Definition 2.1. By a copolynomial over the ring $K$, we mean a $K$-linear functional defined on the ring $K[x]$, i.e., a homomorphism occurring from the module $K[x]$ to the ring $K$.

We denote the module of copolynomials over $K$ by $K[x]'$. Thus, $T \in K[x]'$ if and only if $T : K[x] \to K$ and $T$ has the property of $K$-linearity: $T(ap + bq) = aT(p) + bT(q)$ for all $p, q \in K[x]$ and $a, b \in K$. If $T \in K[x]'$ and $p \in K[x]$, are for the value of $T$ on $p$, we use the notation $(T, p)$. We also write the copolynomial $T \in K[x]'$ in the form $T(x)$, where $x$ is regarded as the argument of polynomials $p(x) \in K[x]$ and is subjected to the action of the $K$-linear mapping $T$. In this case, the result of action of $T$ upon $p$ can be represented in the form $(T(x), p(x))$.

Let $p(x) = \sum_{n=0}^{m} a_n x^n \in K[x]$. For any $x \in K$, we consider the polynomial $p(x + h) \in K[h]$:

$$p(x+h) = \sum_{n=0}^{m} p_n(x) h^n,$$

where $p_n(x) \in K$. Since, in the case of a field with zero characteristic, $p_n(x) = \frac{p^{(n)}(x)}{n!}$, we also assume that by definition $\frac{p^{(n)}(x)}{n!} = p_n(x)$, $n = 0, ..., m$ is also true for any commutative ring $K$. For $n > m$, we assume that $\frac{p^{(n)}(x)}{n!} = 0$.

Definition 2.2. The derivative $T'$ of a copolynomial $T \in K[x]'$, as in the classical case, is given in the formula

$$(T', p) = -(T, p'), \quad p \in K[x].$$

By using this result, we arrive at the following expression for the $n$th order derivative:

$$(T^{(n)}, p) = (-1)^n (T, p^{(n)}), \quad p \in K[x].$$

Hence,

$$(T^{(n)}, p) = 0, \quad T \in K[x]', \quad p \in K[x],\ n > \deg p.$$

By virtue of the equality

$$\left(\frac{T^{(n)}}{n!}, p\right) = (-1)^n \left(T, \frac{p^{(n)}}{n!}\right), \quad p \in K[x] \qquad (2.1)$$

the copolynomials $\frac{T^{(n)}}{n!}$ are well defined for any $T \in K[x]'$ and $n \in \mathbb{N}$.

Example 2.1. The copolynomial $\delta$-function is given in the formula

$$(\delta, p) = p(0), \quad p \in K[x].$$

For the copolynomial $\delta$-function, we find its derivative of the $n$th order as follows:

$$(\delta^{(n)}, p) = (-1)^n (\delta, p^{(n)}) = (-1)^n p^{(n)}(0), \quad n \in \mathbb{N}.$$

Example 2.2. Let $K = \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue-integrable function such that

$$\int_{-\infty}^{\infty} |x^n f(x)| dx < +\infty, \quad n = 0, 1, 2, ... \qquad (2.2)$$

Then, $f$ generates the regular copolynomial $T_f$:

$$(T_f, p) = \int_{-\infty}^{\infty} p(x) f(x) dx, \quad p \in \mathbb{R}[x].$$

Note that, in this case, unlike the classical theory, all copolynomials are regular ([13], Theorem 7.3.4), although a nonzero function $f$ can generate the zero copolynomial {([17], Remark 1), ([18], Example 2.2)}. We present an example of a function that satisfies the property (2.2) and generates the $\delta$-function.

It is known that for any $\varepsilon > 0$ there exists an even function $\varphi_\varepsilon(x) \in C_0^\infty(\mathbb{R})$ such that $\varphi_\varepsilon(x) = 1$ for any $x \in (-\varepsilon; \varepsilon)$ [19]. Then, $\varphi_\varepsilon(0) = 1$ and $\varphi_\varepsilon^{(k)}(0) = 0$, and $k \in \mathbb{N}$. The inverse Fourier transform

$$f_\varepsilon(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_\varepsilon(\lambda) e^{i\lambda x} d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_\varepsilon(\lambda) \cos \lambda x d\lambda$$

is an element of the Schwarz space $S(\mathbb{R})$. Then, $\varphi_\varepsilon(\lambda)$ is the Fourier transform of $f_\varepsilon(x)$:

$$\varphi_\varepsilon(\lambda) = \int_{-\infty}^{\infty} f_\varepsilon(x)e^{-i\lambda x}dx$$

and

$$\int_{-\infty}^{\infty} f_\varepsilon(x)dx = \varphi_\varepsilon(0) = 1, \quad \int_{-\infty}^{\infty} x^k f_\varepsilon(x)dx = i^k\varphi_\varepsilon^{(k)}(0) = 0, \quad k \in \mathbb{N},$$

$$\int_{-\infty}^{\infty} p(x)f_\varepsilon(x)dx = p(0), \quad p \in K[x],$$

i.e., $f_\varepsilon(x)$ generates the copolynomial $\delta$-function for any $\varepsilon > 0$.

We now consider the issue of convergence in the space $K[x]'$. In the ring $K$, we consider the discrete topology. Further, in the module of copolynomials $K[x]'$, we consider the topology of pointwise convergence. The convergence of a sequence $\{T_n\}_{n=1}^{\infty}$ to $T$ in $K[x]'$ means that for every polynomial $p \in K[x]$, there exists a number $n_0 \in \mathbb{N}$ such that

$$(T_n, p) = (T, p), \quad n = n_0, n_0 + 1, n_0 + 2, \ldots.$$

By the definition of convergence in the module $K[x]'$, we arrive at the following statement [6].

Theorem 2.1. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of elements from $K$ and let $T \in K[x]'$. Then, the series $\sum_{n=0}^{\infty} a_n \frac{T^{(n)}}{n!}$ converges in $K[x]'$.

The following assertion [6] shows the possibility of an expansion of an arbitrary formal generalized function in a series in the system $\left\{\frac{\delta^{(n)}}{n!}\right\}_{n=0}^{\infty}$ {see also ([12], Proposition 2.3) in the case $K = \mathbb{C}$}.

Lemma 2.1. Let $T \in K[x]'$. Then,

$$T = \sum_{n=0}^{\infty}(-1)^n(T, x^n)\frac{\delta^{(n)}}{n!}. \tag{2.3}$$

# 3 Multiplication of copolynomials

## 3.1 The Cauchy–Stieltjes transform

Let $K\left[\left[z, \frac{1}{z}\right]\right]$ be the module of formal Laurent series with coefficients in $K$. For $g \in K\left[\left[z, \frac{1}{z}\right]\right]$ and $g(z) = \sum_{k=-\infty}^{\infty} g_k z^k$, we naturally define the formal residue:

$$Res(g(z)) = g_{-1}.$$

Definition 3.1. Let $T \in K[x]'$. Consider the following formal Laurent series from the ring $\frac{1}{s}K[[\frac{1}{s}]]$:

$$C(T)(s) = \sum_{k=0}^{\infty} \frac{(T, x^k)}{s^{k+1}}.$$

The Laurent series $C(T)(s)$ will be called the *Cauchy–Stieltjes transform* of a copolynomial $T$.

We may write informally as follows: $C(T)(s) = \left(T, \frac{1}{s-x}\right)$. Obviously, that the mapping $C : K[x]' \to \frac{1}{s}K[[\frac{1}{s}]]$ is an isomorphism of $K$-modules.

Proposition 3.1. (The inversion formula). Let $T \in K[x]'$ and $p \in K[x]$. Then,

$$(T, p) = Res(C(T)(s)p(s)).$$

*Proof.* It is sufficient to consider the case $p(x) = x^n$ for some $n \in \mathbb{N}_0$. We have

$$C(T)(s)s^n = \sum_{k=0}^{\infty} \frac{(T, x^k)s^n}{s^{k+1}}.$$

Therefore, $Res(C(T)(s)s^n) = (T, x^n)$.

Example 3.1. For the copolynomial $\delta$-function, we have

$$C(\delta)(s) = \frac{1}{s}. \tag{3.1}$$

The following proposition shows that in some sense the differentiating commutes with the Cauchy–Stieltjes transform.

Proposition 3.2. For any $T \in K[x]'$, the equality

$$C\left(T^{(n)}\right) = C(T)^{(n)}, \quad n \in \mathbb{N}$$

holds valid.

*Proof.* It is sufficient to consider the case $n = 1$, so that

$$C\left(T'\right)(s) = \sum_{k=0}^{\infty} \frac{(T', x^k)}{s^{k+1}} =$$

$$= -\sum_{k=1}^{\infty} \frac{k(T, x^{k-1})}{s^{k+1}} = -\sum_{k=0}^{\infty} \frac{(k+1)(T, x^k)}{s^{k+2}} = C(T)'(s).$$

## 3.2 Multiplication of copolynomials and its properties

The Cauchy–Stieltjes transform and Proposition 3.2 allow to introduce the multiplication operation on the module of copolynomials such that this operation is consistent with the differentiation.

Definition 3.2. Let $T_1, T_2 \in K[x]'$, i.e., $T_1, T_2$ are copolynomials. Define their *product* by the following equality:

$$C(T_1T_2) = C(T_1)C(T_2), \tag{3.2}$$

i.e.,

$$T_1T_2 = C^{-1}\left(C(T_1)C(T_2)\right),$$

where $C : K[x]' \to \frac{1}{s}K[[\frac{1}{s}]]$ is a Cauchy–Stieltjes transform.

In the following lemma, the action of the product of copolynomials on monomials is expressed through the action of multipliers on monomials.

Lemma 3.1. Let $T_1, T_2 \in K[x]'$ and $n \in \mathbb{N}_0$. Then,

$$(T_1T_2, x^n) = \begin{cases} \sum_{k=0}^{n-1}(T_1, x^k)(T_2, x^{n-1-k}), & n \in \mathbb{N}, \\ 0, & n = 0. \end{cases} \tag{3.3}$$

*Proof.* By Equation 3.2, we have

$$C(T_1T_2)(s) = C(T_1)(s)C(T_2)(s) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(T_1, x^k)(T_2, x^j)}{s^{k+j+2}}$$

$$= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(T_1, x^k)(T_2, x^{n-1-k})\frac{1}{s^{n+1}}.$$

Applying the inversion formula to the both part of this equality (see Proposition 3.1), we obtain (3.3).

Remark 3.1. Definition 3.2 means that the module of copolynomials $K[x]'$ with the introduced product is a associative commutative ring, which isomorphic to the ring of formal Laurent series $\frac{1}{s}K[[\frac{1}{s}]]$ with a natural product operation. In particular, the ring of copolynomials is an integral domain and this is a ring without identity.

Example 3.2. Let $n = 1$. With the help of Proposition 3.2, we find the square of $\delta$-function:

$$C(\delta^2)(s) = (C(\delta))^2(s) = \frac{1}{s^2} = \left(\frac{-1}{s}\right)' = (-C(\delta))' = C(-\delta'),$$

i.e.,

$$\delta^2 = -\delta'.$$

Moreover, by Equations 2.1, 3.1, we have

$$C\left(\frac{\delta^{(n)}}{n!}\right)(s) = \sum_{k=0}^{\infty}\left(\frac{\delta^{(n)}}{n!}, x^k\right)\frac{1}{s^{k+1}} = \sum_{k=0}^{\infty}\left(\delta, \frac{1}{n!}\frac{d^n x^k}{dx^n}\right)\frac{(-1)^n}{s^{k+1}} = \frac{(-1)^n}{s^{n+1}} = (-1)^n(C(\delta))^{n+1},$$

so that

$$\frac{(-1)^n\delta^{(n)}}{n!} = \delta^{n+1}, \quad n = 0, 1, 2, ..., \tag{3.4}$$

and therefore,

$$(\delta^n)' = -n\delta^{n+1}, \quad n \in \mathbb{N}. \tag{3.5}$$

Hence, by Theorem 2.1 and (3.4), the series

$$\sum_{k=0}^{\infty}u_k\delta^{k+1} = \sum_{k=0}^{\infty}(-1)^k\frac{\delta^{(k)}}{k!}u_k$$

converges for any $u_k \in K$.

Remark 3.2. By Lemma 2.1 and (3.4) for any copolynomial $T \in K[x]'$, the expansion in powers of the $\delta$-function holds:

$$T = \sum_{k=0}^{\infty}(T, x^k)\delta^{k+1}.$$

Remark 3.3. The equalities (3.1) and (3.4) show that in a certain sense $\delta(x)$ and $\frac{1}{s}$ are related (see also [1], p. 79).

# 4 Main results and examples

## 4.1 Formal power series over the ring of copolynomials

The ring of formal power series in the form $u(t,x) = \sum_{k=0}^{\infty}u_k(x)t^k$ with coefficients $u_k(x) \in K[x]'$ will be denoted by $K[x]'[[t]]$. In this subsection, we remind several notations from Gefter and Piven' [6].

The partial derivative with respect to $t$ of the series $u(t,x) \in K[x]'[[t]]$ is defined by the formula

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{\infty}ku_k(x)t^{k-1}.$$

The partial derivative $\frac{\partial u}{\partial x}$ of the series $u(t,x) \in K[x]'[[t]]$ is defined as follows:

$$\frac{\partial u}{\partial x} = \sum_{k=0}^{\infty}u_k'(x)t^k.$$

By $(u(t,x), p(x))$, we denote the action of $u(t,x) \in K[x]'[[t]]$ on $p(x) \in K[x]$, which is defined coefficient-wise.

$$(u(t,x), p(x)) = \sum_{k=0}^{\infty}(u_k(x), p(x))t^k.$$

Thus, $(u(t,x), p(x)) \in K[[t]]$.

## 4.2 Existence and uniqueness theorem

Let $a, u_0 \in K$ and let $m_j \in \mathbb{N}_0$ $(j = 0,1,2,3)$, $\sum_{j=0}^{3}m_j > 0$. Consider the Cauchy problem (1.1), (1.2) in the ring $K[x]'[[t]]$. We prove the following existence and uniqueness theorem for this Cauchy problem.

Theorem 4.1. Let $K \supset \mathbb{Q}$. Then, the Cauchy problem (1.1), (1.2) has a unique solution in $K[x]'[[t]]$. This solution is in the form

$$u(t,x) = \sum_{k=0}^{\infty}u_k\delta^{nk+1}t^k, \tag{4.1}$$

where $u_k \in K$ and $n = \sum_{j=0}^{3}(j+1)m_j - 1$. Moreover, for every $t \in K$, this series converges in the topology of $K[x]'$.

*Proof.* We will find the solution of the Cauchy problem (1.1), (1.2) in the form (4.1). Differentiating (4.1) on $x$ and $t$ and taking into account (3.5), we have

$$\frac{\partial u}{\partial t} = \sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{nk+n+1}t^k, \tag{4.2}$$

$$\frac{\partial u}{\partial x} = -\sum_{k=0}^{\infty}(nk+1)u_k\delta^{nk+2}t^k,$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{k=0}^{\infty}(nk+1)(nk+2)u_k\delta^{nk+3}t^k,$$

$$\frac{\partial^3 u}{\partial x^3} = -\sum_{k=0}^{\infty}(nk+1)(nk+2)(nk+3)u_k\delta^{nk+4}t^k.$$

Then,

$$u^{m_0} = \sum_{\tau_0=0}^{\infty}\sum_{|\alpha|=\tau_0} u_{\alpha_1}\cdots u_{\alpha_{m_0}}\delta^{n\tau_0+m_0}t^{\tau_0},$$

$$\left(\frac{\partial u}{\partial x}\right)^{m_1} = (-1)^{m_1}\sum_{\tau_1=0}^{\infty}\sum_{|\beta|=\tau_1} (n\beta_1+1)\cdots(n\beta_{m_1}+1)u_{\beta_1}\cdots u_{\beta_{m_1}}\delta^{n\tau_1+2m_1}t^{\tau_1},$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^{m_2} = \sum_{\tau_2=0}^{\infty}\sum_{|\gamma|=\tau_2}(n\gamma_1+1)\cdots(n\gamma_{m_2}+1)(n\gamma_1+2)\cdots(n\gamma_{m_2}+2)u_{\gamma_1}\cdots u_{\gamma_{m_2}}\delta^{n\tau_2+3m_2}t^{\tau_2},$$

$$\left(\frac{\partial^3 u}{\partial x^3}\right)^{m_3} = (-1)^{m_3}\sum_{\tau_3=0}^{\infty}\sum_{|\sigma|=\tau_3}(n\sigma_1+1)\cdots(n\sigma_{m_3}+1)(n\sigma_1+2)\cdots(n\sigma_{m_3}+2)\cdot(n\sigma_1+3)\cdots(n\sigma_{m_3}+3)u_{\sigma_1}\cdots u_{\sigma_{m_3}}\delta^{n\tau_3+4m_3}t^{\tau_3},$$

where $\alpha,\beta,\gamma,\sigma$ are multi-indexes, $\alpha = (\alpha_1,...,\alpha_{m_0}), \beta = (\beta_1,...,\beta_{m_1}), \gamma = (\gamma_1,...,\gamma_{m_2}), \sigma = (\sigma_1,...,\sigma_{m_3})$. Therefore,

$$au^{m_0}\left(\frac{\partial u}{\partial x}\right)^{m_1}\left(\frac{\partial^2 u}{\partial x^2}\right)^{m_2}\left(\frac{\partial^3 u}{\partial x^3}\right)^{m_3} = (-1)^{m_1+m_3}a\sum_{k=0}^{\infty}\sum_{|\tau|=k}\sum_{|\alpha|=\tau_0} u_{\alpha_1}\cdots u_{\alpha_{m_0}}\cdot \sum_{|\beta|=\tau_1}(n\beta_1+1)\cdots(n\beta_{m_1}+1)u_{\beta_1}\cdots u_{\beta_{m_1}}\cdot \sum_{|\gamma|=\tau_2}(n\gamma_1+1)\cdots(n\gamma_{m_2}+1)(n\gamma_1+2)\cdots(n\gamma_{m_2}+2)u_{\gamma_1}\cdots u_{\gamma_{m_2}}\cdot \sum_{|\sigma|=\tau_3}(n\sigma_1+1)\cdots(n\sigma_{m_3}+1)(n\sigma_1+2)\cdots(n\sigma_{m_3}+2)(n\sigma_1+3)\cdots(n\sigma_{m_3}+3)u_{\sigma_1}\cdots u_{\sigma_{m_3}}\delta^{nk+n+1}t^k, \tag{4.3}$$

where $\tau = (\tau_0,\tau_1,\tau_2,\tau_3)$. Equating coefficients at $\delta^{nk+n+1}t^k$ in right-hand sides of (4.2) and (4.3), we obtain

$$(k+1)u_{k+1} = (-1)^{m_1+m_3}a\sum_{|\tau|=k}\sum_{|\alpha|=\tau_0} u_{\alpha_1}\cdots u_{\alpha_{m_0}}\cdot \sum_{|\beta|=\tau_1}(n\beta_1+1)\cdots(n\beta_{m_1}+1)u_{\beta_1}\cdots u_{\beta_{m_1}}\cdot \sum_{|\gamma|=\tau_2}(n\gamma_1+1)\cdots(n\gamma_{m_2}+1)(n\gamma_1+2)\cdots(n\gamma_{m_2}+2)u_{\gamma_1}\cdots u_{\gamma_{m_2}}\cdot \sum_{|\sigma|=\tau_3}(n\sigma_1+1)\cdots(n\sigma_{m_3}+1)(n\sigma_1+2)\cdots(n\sigma_{m_3}+2)(n\sigma_1+3)\cdots(n\sigma_{m_3}+3)u_{\sigma_1}\cdots u_{\sigma_{m_3}}.$$

Since $K\supset\mathbb{Q}$, we obtain that for any $k\in\mathbb{N}_0$ the element $u_{k+1}$ is uniquely expressed through $u_0,...,u_k$. Now, if $t\in K$, then by Equation 3.4

$$u(t,x) = \sum_{k=0}^{\infty}u_k\delta^{nk+1}t^k = \sum_{k=0}^{\infty}(-1)^{nk}\frac{\delta^{(nk)}}{(nk)!}u_kt^k$$

so that the convergence of the series (4.1) follows from Theorem 2.1. Now, we prove the uniqueness of the solution of the Cauchy problem (1.1), (1.2) in the ring $K[x]'[[t]]$. We will find a solution of the Cauchy problem (1.1), (1.2) in the form

$$u(t,x) = \sum_{k=0}^{\infty}v_k(x)t^k,$$

where $v_k(x)\in K[x]'$. Then, by the initial condition (1.2), we have $v_0(x) = u_0\delta(x)$. Substitute $u(t,x)$ into Equation 1.1 and equate coefficients of $t^k$. Then, there exist polynomials $p_k\in K[z_1,...,z_{4(k+1)}]$ $(k=0,1,2,...)$ such that

$$(k+1)v_{k+1}(x) = p_k\left(v_0(x),\frac{\partial v_0}{\partial x},\frac{\partial^2 v_0}{\partial x^2},\frac{\partial^3 v_0}{\partial x^3},..., v_k(x),\frac{\partial v_k}{\partial x},\frac{\partial^2 v_k}{\partial x^2},\frac{\partial^3 v_k}{\partial x^3}\right).$$

Since the ring $K$ contains the field of rational numbers, from this we uniquely find $u_k(x)$, $k\in\mathbb{N}$:

$$v_k(x) = k^{-1}p_{k-1}\left(v_0(x),\frac{\partial v_0}{\partial x},\frac{\partial^2 v_0}{\partial x^2},\frac{\partial^3 v_0}{\partial x^3},..., v_{k-1}(x),\frac{\partial v_{k-1}}{\partial x},\frac{\partial^2 v_{k-1}}{\partial x^2},\frac{\partial^3 v_{k-1}}{\partial x^3}\right).$$

The proof is complete.

## 4.3 Examples

We consider some examples of classical equations that illustrate Theorem 4.1. In what follows, we suppose that $K$ is of characteristic 0 ([20], Section 1.43). We denote by $F$ the quotient field of $K$. Obviously, $K\supset\mathbb{Z}$ and $F\supset\mathbb{Q}$.

Example 4.1. Let $u_0 \in K$. In $K[x]'[[t]]$, consider the following Cauchy problem for the Euler–Hopf equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{4.4}$$

$$u(0,x) = u_0\delta(x). \tag{4.5}$$

By Theorem 4.1, the Cauchy problem (4.4), (4.5) has a unique solution in $F[x]'[[t]]$ and this solution can be represented in the form (4.1) of $n = 2$:

$$u(t,x) = \sum_{k=0}^{\infty} u_k\delta^{2k+1}t^k, \tag{4.6}$$

where $u_k \in F$. Substituting (4.6) into (4.4), we obtain (see Proof of Theorem 4.1):

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{2k+3}t^k = \sum_{k=0}^{\infty}\sum_{j=0}^{k}(2j+1)u_ju_{k-j}\delta^{2k+3}t^k. \tag{4.7}$$

Equating coefficients at $\delta^{2k+3}t^k$ in (4.7), we have

$$(k+1)u_{k+1} = \sum_{j=0}^{k}(2j+1)u_ju_{k-j}, \quad k \in \mathbb{N}_0. \tag{4.8}$$

Since

$$\sum_{j=0}^{k}(2j+1)u_ju_{k-j} = (k+1)\sum_{j=0}^{k}u_ju_{k-j},$$

the equality (4.8) implies

$$(k+1)u_{k+1} = (k+1)\sum_{j=0}^{k}u_ju_{k-j}, \quad k \in \mathbb{N}_0. \tag{4.9}$$

Since $K$ is of characteristic 0, the equality (4.9) is reduced to the following recurrence equation:

$$u_{k+1} = \sum_{j=0}^{k}u_ju_{k-j}, \quad k \in \mathbb{N}_0. \tag{4.10}$$

If $u_0 = 1$, then the solution of (4.10) is $u_k = C_k$, where $C_k = (k+1)^{-1}\binom{2k}{k}$ $(k \in \mathbb{N}_0)$ is the sequence of the Catalan numbers ([21], Section 7.5). Generally, the solution of (4.10) is in the form $u_k = C_ku_0^{k+1}$ $(k \in \mathbb{N}_0)$, so that

$$u(t,x) = \sum_{k=0}^{\infty}C_k\delta^{2k+1}u_0^{k+1}t^k = \sum_{k=0}^{\infty}C_k\frac{\delta^{(2k)}(x)}{(2k)!}u_0^{k+1}t^k \tag{4.11}$$

(see Equation 3.4). Since $u(t,x) \in K[x]'[[t]]$, it is a unique solution of the Cauchy problem (4.4), (4.5) in the ring $K[x]'[[t]]$.

Remark 4.1. Note that for any $t \in K$, the series (4.11) converges in the topology of $K[x]'$. The Cauchy–Stieltjes transform of (4.11) is the following Laurent series $\sum_{k=0}^{\infty}\frac{C_ku_0^{k+1}t^k}{x^{2k+1}}$. If $K = \mathbb{R}$, then this series is an expansion of the function $w(t,x) = \frac{x-\sqrt{x^2-4u_0t}}{2t}$ in the domain $D = \{(t,x) \in \mathbb{R}^2 : x > 0, x^2 - 4u_0t > 0\}$. The function $w(t,x)$ is a classical solution of the Euler–Hopf equation (4.4) in the domain $D$.

Example 4.2. Let $u_0 \in K$. In $K[x]'[[t]]$, consider the following Cauchy problem for a Hamilton–Jacobi type equation ([5], Section 24.1.6):

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2, \tag{4.12}$$

$$u(0,x) = u_0\delta(x). \tag{4.13}$$

By Theorem 4.1, the Cauchy problem (4.12), (4.13) has a unique solution in $F[x]'[[t]]$ and this solution can be represented in the form (4.1) for $n = 3$:

$$u(t,x) = \sum_{k=0}^{\infty}u_k\delta^{3k+1}t^k, \tag{4.14}$$

where $u_k \in F$. Substituting (4.14) into (4.4), we obtain (see Proof of Theorem 4.1):

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{3k+4}t^k = \sum_{k=0}^{\infty}\sum_{j=0}^{k}(3j+1)(3(k-j)+1)u_ju_{k-j}\delta^{3k+4}t^k. \tag{4.15}$$

Equating coefficients at $\delta^{3k+4}t^k$ in Equation 4.15, we have

$$(k+1)u_{k+1} = \sum_{j=0}^{k}(3j+1)(3(k-j)+1)u_ju_{k-j}, \quad k \in \mathbb{N}_0. \tag{4.16}$$

We prove that $y_k = \frac{2^kC_k^{(3)}}{k+1}$ is a solution of the recurrence Equation 4.16 with the initial condition $u_0 = 1$, where $C_k^{(3)} = (3k+1)^{-1}\binom{3k+1}{k} = \frac{(3k)!}{k!(2k+1)!}$ $(k \in \mathbb{N}_0)$ are the Fuss–Catalan numbers {[21], Section 7.5, Formula (7.67)}.

Consider the following combinatorial identity that was proved in Gould [22]:

$$\frac{4}{3k+4}\binom{3k+4}{k} = \sum_{j=0}^{k}\frac{2}{3j+2}\binom{3j+2}{j}\frac{2}{3(k-j)+2}\binom{3(k-j)+2}{k-j}, \quad k \in \mathbb{N}_0. \tag{4.17}$$

Since

$$\frac{2}{3j+2}\binom{3j+2}{j} = \frac{2(3j+1)!}{j!(2j+1)!(2j+2)} = \frac{1}{j+1}\binom{3j+1}{j}, \quad j \in \mathbb{N}_0,$$

the equality (4.17) can be written in the form

$$\frac{4}{3k+4}\binom{3k+4}{k}=\sum_{j=0}^{k}\frac{1}{j+1}\binom{3j+1}{j}\frac{1}{k-j+1}\binom{3(k-j)+1}{k-j},\quad k\in\mathbb{N}_0. \tag{4.18}$$

Since

$$\frac{4}{3k+4}\binom{3k+4}{k}=\frac{4(3k+4)!}{k!(2k+4)!(3k+4)}=$$
$$=\frac{2(3k+4)!(k+1)}{(2k+3)!k!(k+1)(k+2)(3k+4)}$$
$$=\frac{2(k+1)}{(k+2)(3k+4)}\binom{3k+4}{2k+3}=$$
$$=\frac{2(k+1)}{(k+2)(3k+4)}\binom{3(k+1)+1}{k+1}=\frac{2(k+1)}{k+2}C^{(3)}_{k+1}=\frac{(k+1)y_{k+1}}{2^k},$$

after the multiplication (4.18) by $2^k$, we have

$$(k+1)y_{k+1}=\sum_{j=0}^{k}\frac{2^j}{j+1}\binom{3j+1}{j}\frac{2^{k-j}}{k-j+1}\binom{3(k-j)+1}{k-j}=$$
$$=\sum_{j=0}^{k}(3j+1)(3(k-j)+1)y_jy_{k-j},\quad k\in\mathbb{N}_0,$$

i.e., $y_k$ satisfy (4.16). Since $y_k=\frac{2^k(3k)!}{(k+1)!(2k+1)!}$ is the number of inequivalent rooted maps of some vertices {[23], p.409, Section 5 and Formula (5.7)}, we have $y_k\in\mathbb{Z}$ (see also the integer sequence A000309 in Sloane [24]). Therefore, if $u_0=1$, then $u_k=y_k\in\mathbb{Z}$.

Now, we consider an arbitrary $u_0\in K$. Multiplying the equality

$$(k+1)y_{k+1}=\sum_{j=0}^{k}(3j+1)(3(k-j)+1)y_jy_{k-j},\quad k\in\mathbb{N}_0$$

by $u_0^{k+2}$, we obtain

$$u_0^{k+2}y_{k+1}=\frac{1}{k+1}\sum_{j=0}^{k}(3j+1)(3(k-j)+1)u_0^{j+1}y_ju_0^{k-j+1}y_{k-j},\quad k\in\mathbb{N}_0.$$

Therefore, for any $u_0\in K$, the sequence $u_k=u_0^{k+1}y_k\in K$ satisfies Equation 4.16. Hence, Equation 4.14 defines the unique solution to the Cauchy problem (4.12), (4.13) in $K[x]'[[t]]$.

Example 4.3. Let $b,u_0\in K$. Consider the following Cauchy problem for the heat equation in $K[x]'[[t]]$

$$\frac{\partial u}{\partial t}=b\frac{\partial^2 u}{\partial x^2}, \tag{4.19}$$

$$u(0,x)=u_0\delta(x). \tag{4.20}$$

By Theorem 4.1, the Cauchy problem (4.19), (4.20) has a unique solution in $F[x]'[[t]]$ and this solution can be represented in the form (4.1) for $n=2$:

$$u(t,x)=\sum_{k=0}^{\infty}u_k\delta^{2k+1}t^k, \tag{4.21}$$

where $u_k\in F$. Substituting (4.21) into (4.19), we obtain (see Proof of Theorem 4.1):

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{2k+3}t^k=b\sum_{k=0}^{\infty}(2k+1)(2k+2)u_k\delta^{2k+3}t^k. \tag{4.22}$$

Equating coefficients at $\delta^{3k+4}t^k$ in Equation 4.22, we have

$$(k+1)u_{k+1}=b(2k+1)(2k+2)u_k,\quad k\in\mathbb{N}_0$$

Since $K$ is of characteristic 0, this implies the following difference equation

$$u_{k+1}=2b(2k+1)u_k,\quad k\in\mathbb{N}_0,$$

which, for any given $u_0\in K$, has the unique solution $u_k=(2b)^k(2k-1)!!u_0$, $k\in\mathbb{N}_0$, where $(-1)!!=1$. Therefore, the unique solution of the Cauchy problem (4.19, 4.20) is in the form

$$u(t,x)=\sum_{k=0}^{\infty}(2b)^k(2k-1)!!u_0\delta^{2k+1}t^k=\sum_{k=0}^{\infty}b^ku_0\frac{\delta^{(2k)}(x)}{k!}t^k \tag{4.23}$$

(see also Equation 3.4). Since $u(t,x)\in K[x]'[[t]]$, it is a unique solution of the Cauchy problem (4.19, 4.20) in the ring $K[x]'[[t]]$.

Now let $K=\mathbb{R}$, $b>0$ and $t>0$. Taking into account the equality (3.14) [6] from Equation 4.23, we arrive

$$\left(\sum_{k=0}^{\infty}(2b)^k(2k-1)!!\delta^{2k+1}t^k,x^j\right)=\frac{1}{\sqrt{4\pi bt}}\int_{-\infty}^{\infty}x^je^{-\frac{x^2}{4bt}}dx,\quad j\in\mathbb{N}_0,$$

i.e.,

$$\sum_{k=0}^{\infty}(2b)^k(2k-1)!!\delta^{2k+1}t^k=\frac{1}{\sqrt{4\pi bt}}e^{-\frac{x^2}{4bt}}\text{ in }\mathbb{R}[x]'.$$

Example 4.4. Let $K\supset\mathbb{Q}$ and $u_0\in K$. Consider the following Cauchy problem for the Harry Dym equation in the ring $K[x]'[[t]]$ ([5], Section 13.1.4)

$$\frac{\partial u}{\partial t}=u^3\frac{\partial^3 u}{\partial x^3} \tag{4.24}$$

$$u(0,x)=u_0\delta(x). \tag{4.25}$$

By Theorem 4.1, the Cauchy problem (4.12, 4.13) has a unique solution in $K[x]'[[t]]$ and this solution can be represented in the form (4.1) for $n=6$:

$$u(t,x)=\sum_{k=0}^{\infty}u_k\delta^{6k+1}t^k, \tag{4.26}$$

where $u_k \in K$. As in the proof of Theorem 4.1, we have

$$\frac{\partial u}{\partial t} = \sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{6k+7}t^k, \quad (4.27)$$

$$\frac{\partial^3 u}{\partial x^3} = -\sum_{k=0}^{\infty}(6k+1)(6k+2)(6k+3)u_k\delta^{6k+4}t^k, \quad (4.28)$$

$$u^3 = \sum_{k=0}^{\infty}\sum_{|\alpha|=k} u_{\alpha_1}u_{\alpha_2}u_{\alpha_3}\delta^{6k+3}t^k, \quad (4.29)$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Substituting (4.27–4.29) into (4.24), we obtain

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{6k+7}t^k = -\sum_{k=0}^{\infty}\sum_{|\tau|=k}(6\tau_4+1)(6\tau_4+2)(6\tau_4+3)u_{\tau_1}u_{\tau_2}u_{\tau_3}u_{\tau_4}\delta^{6k+7}t^k, \quad (4.30)$$

where $\tau = (\tau_1, \tau_2, \tau_3, \tau_4)$. Equating coefficients at $\delta^{6k+7}t^k$ in the right-hand side of (4.30), we obtain

$$u_{k+1} = -(k+1)^{-1}\sum_{|\tau|=k}(6\tau_4+1)(6\tau_4+2)(6\tau_4+3)u_{\tau_1}u_{\tau_2}u_{\tau_3}u_{\tau_4}.$$

Computer experiments demonstrate that the first 200 terms of the sequence $u_k$ are integers. Although this sequence is not found in the online encyclopedia of integer sequences [24], we formulate the conjecture that $u_k \in \mathbb{Z}$ for all $k \in \mathbb{N}_0$.

The following example shows that the condition $K \supset \mathbb{Q}$ is essential for the assertion of Theorem 4.1.

Example 4.5. Let $K \supset \mathbb{Q}$. Consider the following Cauchy problem in $K[x]'[[t]]$:

$$\frac{\partial u}{\partial t} = u\left(\frac{\partial u}{\partial x}\right)^2, \quad (4.31)$$

$$u(0,x) = \delta(x). \quad (4.32)$$

By Theorem 4.1, the Cauchy problem (4.31, 4.32) has a unique solution in $K[x]'[[t]]$ and this solution can be represented in the form (4.1) for $n = 4$:

$$u(t,x) = \sum_{k=0}^{\infty}u_k\delta^{4k+1}t^k, \quad (4.33)$$

where $u_0 = 1$. Substituting (4.33) into (4.31), we obtain

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{4k+5}t^k = \sum_{k=0}^{\infty}\sum_{|\tau|=k}(4\tau_1+1)(4\tau_2+1)u_{\tau_1}u_{\tau_2}u_{\tau_3}\delta^{4k+5}t^k, \quad (4.34)$$

where $\tau = (\tau_1, \tau_2, \tau_3)$.

Equating coefficients at $\delta^{4k+5}t^k$ in the right-hand side of Equation 4.34, we obtain

$$u_{k+1} = (k+1)^{-1}\sum_{|\tau|=k}(4\tau_1+1)(4\tau_2+1)u_{\tau_1}u_{\tau_2}u_{\tau_3}, \quad k \in \mathbb{N}_0.$$

This implies that $u_1 = 1$ and $u_2 = \frac{11}{2} \notin \mathbb{Z}$. Therefore, the Cauchy problem (4.31), (4.32) in $\mathbb{Z}[x]'[[t]]$ has no solutions.

## 5 Conclusion

We investigated the Cauchy problem of the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = au^{m_0}\left(\frac{\partial u}{\partial x}\right)^{m_1}\left(\frac{\partial^2 u}{\partial x^2}\right)^{m_2}\left(\frac{\partial^3 u}{\partial x^3}\right)^{m_3},$$

$$m_0, m_1, m_2, m_3 \in \mathbb{N}_0, \quad \sum_{j=0}^{3}m_j > 0, \quad a \in K$$

in the ring of copolynomials. We have found a solution to this Cauchy problem, as the series in powers of the $\delta$-function. We considered the Cauchy problem for the Euler–Hopf equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$, for a Hamilton–Jacobi type equation $\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2$ and for the Harry Dym equation $\frac{\partial u}{\partial t} = u^3\frac{\partial^3 u}{\partial x^3}$. In the first two examples, an interesting connection between classical nonlinear partial differential equations and well-known integer sequences is revealed. The conjecture were formulated that all the coefficients of an expanding in powers of the $\delta$-function of the solution of the Cauchy problem for the Harry Dym equation are integers.

## Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

## Author contributions

AP: Writing – original draft, Writing – review & editing. SG: Writing – original draft, Writing – review & editing.

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## Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

## Publisher's note



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