frontiers | Frontiers in Applied Mathematics and Statistics

TYPE Original Research
PUBLISHED 25 July 2024
DOI 10.3389/fams.2024.1450581

OPEN ACCESS

EDITED BY
Kateryna Buryachenko,
Humboldt University of Berlin, Germany

REVIEWED BY
Alexander Melnikov,
University of Alberta, Canada
Barbara Martinucci,
University of Salerno, Italy
Martynas Manstavičius,
Vilnius University, Lithuania

*CORRESPONDENCE
Vitaliy Golomoziy
✉ vitaliy.golomoziy@knu.ua

RECEIVED 17 June 2024
ACCEPTED 05 July 2024
PUBLISHED 25 July 2024

CITATION
Golomoziy V, Mishura Y and Kladívko K (2024) A discrete-time model that weakly converges to a continuous-time geometric Brownian motion with Markov switching drift rate. *Front. Appl. Math. Stat.* 10:1450581. doi: 10.3389/fams.2024.1450581

A discrete-time model that weakly converges to a continuous-time geometric Brownian motion with Markov switching drift rate

Vitaliy Golomoziy[1]*, Yuliya Mishura[1] and Kamil Kladívko[2]

[1]Faculty of Mechanics and Mathematics, Department of Probability, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine, [2]School of Business, Örebro University, Örebro, Sweden

This research is devoted to studying a geometric Brownian motion with drift switching driven by a 2 × 2 Markov chain. A discrete-time multiplicative approximation scheme was developed, and its convergence in Skorokhod topology to the continuous-time geometric Brownian motion with switching has been proved. Furthermore, in a financial market where the discounted asset price follows a geometric Brownian motion with drift switching, market incompleteness was established, and multiple equivalent martingale measures were constructed.

KEYWORDS

geometric Brownian motion, Markov switching, discrete-time multiplicative approximation, equivalent martingale measure, incomplete financial market

1 Introduction

In this article, we study a geometric Brownian motion with Markov switching in the drift coefficient. Assume that $(X_t)_{t\geq 0}$ follows a linear stochastic differential equation

$$dX_t = (\delta_0 Y_t + \delta_1(1 - Y_t))X_t dt + \sigma X_t dW_t,\ X_0 = x_0, \tag{1}$$

where $x_0 > 0$ is non-random, $\delta_0, \delta_1 \in \mathbb{R}$, $(W_t)_{t\geq 0}$ is a Brownian motion, and $(Y_t)_{t\geq 0}$ is an independent of W continuous-time Markov jump process with the values in the set $\{0, 1\}$, with the initial value $Y_0 = 0$ and with an infinitesimal matrix

$$\mathbb{A} = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}, \tag{2}$$

for some positive λ_0 and λ_1. Moreover, let the processes $(Y_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ be defined on a stochastic basis with filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathcal{F}_t = \sigma\{W_s, Y_s, 0 \leq s \leq t\}$. It is well known that the strong solution of Equation (1) can be represented as an exponent of the form

$$X_t = X_0 \exp\left(\int_0^t (\delta_0 Y_s + \delta_1(1 - Y_s))ds + \sigma W_t - \frac{\sigma^2}{2}t\right). \tag{3}$$

Drift-switching models have been applied in finance and economics for several decades. Early applications of drift switching in the context of time-series econometrics can be found in Quandt [1] or Quandt and Goldfeld [2]. Hamilton [3] used drift switching to model the business cycle, where the expected growth rates of a national product switch according to a Markov chain. In finance, geometric Brownian motion with a Markov chain-modulated drift rate has become popular for modeling asset price dynamics. For instance, Ang and Timmermann [4] and Sotomayor and Cdenillas [5] studied regime-switching models in finance, while Dai et al. [6] and Dai et al. [7] investigated optimal trend-following trading strategies for an asset price modeled by a stochastic differential (Equation 1). In this context, the switching drift rates correspond to bull and bear market conditions. Maheu et al. [8] focus on the identification and estimation aspects of such models. In a similar setting, Décamps et al. [9] and Klein [10] examine optimal investment timing in a risky project with a sunk cost. The study by Aingworth, Das and Motwani [11] was devoted to pricing equity options with Markov switching. Elliott et al. [12] also studied option pricing in models with Markov switching. Bae et al. [13] investigate the problem of asset allocation under regime switching and Ekström and Lu [14] study an optimal irreversible sale of an asset, while Ekström and Lindberg [15] analyze optimal closing strategies for momentum trades. Henderson et al. [16] study exercise patterns of American call executive stock options written on a stock whose drift parameter falls to a lower value at an exponentially distributed random time.

This study focuses on discretizing a geometric Brownian motion with a Markov switching drift rate, as described by Equation (1). Since explicit solutions to models with switching drift rates are rare, rigorous discretization and an understanding of its properties are essential for implementing numerical methods such as binomial and multinomial trees, PDE solvers, or Monte Carlo simulations for these models. Furthermore, in time-series econometrics, a discrete-time version of Equation (1) is typically used from the outset, albeit with only a vague connection to the continuous-time model. Our analysis rigorously connects the continuous- and discrete-time models and provides their convergence properties.

Note that a wide class of theorems on diffusion approximation of additive schemes were proved in the book of Liptser and Shiryaev [17] and generalized to multiplicative schemes in the book of Mishura and Ralchenko [18]. The present study is, in a context, a modification of the functional limit theorems obtained in Chapter 1 of the book [18]. However, to the best of our knowledge, multiplicative Markov switching schemes and their corresponding functional limit theorems have not been previously established.

In addition to the problem of the approximation (in the context of functional limit theorems) of a market with switching, we also investigated the question of the incompleteness of such a market. Intuitively, this incompleteness is obvious, since we have one risky asset with two independent sources of randomness. At the same time, it is easy to construct the so-called minimum martingale measure. It is more difficult to construct a class of equivalent martingale measures other than the minimal one. We managed to construct a fairly wide class of such measures, although it is obvious that all equivalent martingale measures are not exhausted by such a construction.

This study is organized as follows: In Sections 2 and 3, we develop a discretization for the switching component of the process (Equation 1) and prove the weak convergence of the respective probability measures, generated by the prelimit and limit processes, respectively. Section 4 is devoted to the weak convergence of the measures corresponding to the component responsible for volatility. Then, due to the independence of these processes and, consequently, of respective probability measures, we get the weak convergence of the products of these measures, or that is, of the sequence of probability measures generated by the prelimit sequence of probability measures, to the measure corresponding to the limit process. Note also the following: while prelimit and limit Markov processes (chains) are discontinuous, we can establish their weak convergence in Skorokhod topology. However, their integral sums and also the components that are responsible for the weak convergence to geometric Brownian motion converge even in the uniform topology. So, finally, our processes converge in the uniform topology. Finally, Section 5 is devoted to the construction of a wide class of equivalent martingale measures for the market, where Equation (1) represents the discounted price of a risky asset.

2 Discrete-time multiplicative approximation of the diffusion model with Markov switching

The main goal of this study is to construct a sequence of discrete-time versions of X, the geometric Brownian motion with Markov modulated drift given by Equation (1) and Equation (3), such that these discretized versions weakly converge in Skorokhod topology (in fact, convergence will be even in the uniform topology) to the process X on the fixed time interval $[0, T]$.

So, following this direction, we consider the limit process $(X_t)_{t\in[0,T]}$ on the fixed time interval $[0, T]$, where $T > 0$ is a maturity date, and create a series of discrete-time models numbered by $N \in \mathbb{N}$. Our Nth discrete-time market corresponds to the partition of the interval $[0, T]$ into N subintervals of the form $\left[\frac{(k-1)T}{N}, \frac{kT}{N}\right]$, $1 \le k \le N$. Let $X_0^{(N)} = x_0$, and $X_k^{(N)}$ be a strictly positive discounted price of the asset at a time $\frac{kT}{N}$ of Nth discrete-time market, $1 \le k \le N$.

Taking into account the multiplicative nature of the limit model, together with the assumption of independence of Y and W on $[0, T]$, we can assume that the ratio $\frac{X_k^{(N)}}{X_{k-1}^{(N)}}$, $1 \le k \le N$ can be represented as a product

$$\frac{X_k^{(N)}}{X_{k-1}^{(N)}} = \left(1 + R_k^{(1,N)}\right)\left(1 + R_k^{(2,N)}\right), \tag{4}$$

where random variables $R_k^{(i,N)}$, $i \in \{1, 2\}$, $1 \le k \le N$ are independent and $R_k^{(i,N)} > -1$ almost surely (a.s.) Taking logarithms in Equation (4), we can write

$$U_k^{(N)} = \log(X_k^{(N)}) = \log(X_0) + \sum_{j=1}^{k} \log\left(1 + R_j^{(1,N)}\right) + \sum_{j=1}^{k} \log\left(1 + R_j^{(2,N)}\right), \tag{5}$$

where $1 \leq k \leq N$ and $U_0^{(N)} = \log(x_0)$. We assume that the process $X^{(N)}$ is defined on the stochastic basis $(\Omega^{(N)}, \mathcal{F}^{(N)}, R_k^{(1,N)}(\mathcal{F}_t^{(N)})_{t\in[0,T]}, \mathbb{P}^{(N)})$, where filtration is generated by the respective random variables $R_k^{(i,N)}, i = 1, 2$ so that $X_k^{(N)}$ is $\mathcal{F}_{\frac{kT}{N}}^{(N)}$-measurable. In this model, random variables $R_k^{(1,N)}$ represent non-volatile net profit rates generated by the price process on the time intervals $\left[\frac{(k-1)T}{N}, \frac{kT}{N}\right]$, $1 \leq k \leq N$ in a model with switching. Recall that we consider $(Y_s)_{s\geq 0}$, which is the jump Markov process with values in the set $\{0, 1\}$ and an infinitesimal matrix (Equation 1). This process governs the switching in a continuous-time model. Recall also that state 0 generates income with intensity δ_0 and state 1 generates income with intensity δ_1. Once we consider a discrete-time model, we have to introduce a discrete-time switching process (note that in such a model, the switching of the interest rate may only occur at times $\frac{kT}{N}$). Let $\left(Y_k^{(N)}\right)_{k\geq 0}$ be a discrete-time, 2×2 Markov chain defined on the same probability space as $R^{(1,N)}$, $R^{(2,N)}$, and $U^{(N)}$ which is defined in Equation (5). It is independent of the processes $R^{(2,N)}$ and $U^{(2,N)}$. The chain takes values in the set $\{0, 1\}$ and has initial values $Y_0^{(N)} = 0$ and $Y_k^{(N)} = 0$, implying that the intensity of the interest on the kth interval of the Nth discrete-time market equals δ_0. Similarly, $Y_k^{(N)} = 1$ means that such intensity equals δ_1.

The definition of the transition probabilities matrix for the process $Y^{(N)}$ follows from the requirement for occupation times of $Y^{(N)}$ to be close to those of $(Y_s)_{s\geq 0}$. This leads to the following definition of the transition probabilities of the chain $Y^{(N)}$ for $i \in \{0, 1\}$:

$$\begin{aligned}
&\mathbb{P}^{(N)}\left(Y_{k+1}^{(N)} = i | Y_k^{(N)} = i\right) \\
&= \mathbb{P}\left(Y_s = i, \frac{Tk}{N} \leq s \leq \frac{T(k+1)}{N} \mid Y_{\frac{Tk}{N}} = i\right) \\
&= \mathbb{P}(Y_s = i, 0 \leq s \leq T/N \mid Y_0 = i) = e^{-\frac{\lambda_i T}{N}},
\end{aligned}$$

where we used the Markov property of the process $(Y_s)_{s\geq 0}$. Such probabilities define a one-step transition probability matrix

$$\begin{pmatrix} e^{-\lambda_0 \frac{T}{N}} & 1 - e^{-\lambda_0 \frac{T}{N}} \\ 1 - e^{-\lambda_1 \frac{T}{N}} & e^{-\lambda_1 \frac{T}{N}} \end{pmatrix}. \tag{6}$$

Using the switching process $Y^{(N)}$, we can define random variables $R_k^{(1,N)}$, $0 \leq k \leq N$, as follows:

$$R_k^{(1,N)} = \frac{\delta_0 T}{N} Y_k^{(N)} + \frac{\delta_1 T}{N}(1 - Y_k^{(N)}). \tag{7}$$

Definition 7 has the following financial interpretation: Since $R_k^{(1,N)}$, $1 \leq k \leq N$ is a profit rate generated by the risky asset on the kth time interval, the accrual on this interval equals to

$$1 + R_k^{(1,N)} = \exp\left(\frac{\delta_0 T}{N}\right) Y_k^{(N)} + \exp\left(\frac{\delta_1 T}{N}\right)(1 - Y_k^{(N)}). \tag{8}$$

Equation (8) can be written as:

$$R_k^{(1,N)} = \left(\exp\left(\frac{\delta_0 T}{N}\right) - 1\right) Y_k^{(N)} + \left(\exp\left(\frac{\delta_1 T}{N}\right) - 1\right)(1 - Y_k^{(N)}). \tag{9}$$

Using the Taylor formula, we can write $R_k^{(1,N)}$ as follows:

$$R_k^{(1,N)} = \left(\frac{\delta_0 T}{N} + o\left(\frac{\delta_0 T}{N}\right)\right) Y_k^{(N)} + \left(\frac{\delta_1 T}{N} + o\left(\frac{\delta_1 T}{N}\right)\right)(1 - Y_k^{(N)}).$$

By neglecting asymptotically small terms $o\left(\frac{\delta_0 T}{N}\right)$ and $o\left(\frac{\delta_1 T}{N}\right)$, we arrive at the definition (Equation 8).

Now, we turn our attention to $R_k^{(2,N)}$. This random variable represents the pure volatility in the model. In our discrete-time markets, the sums $\sum_{j=1}^{k} \log(1 + R_j^{(2,N)})$, roughly speaking, will approximate the process $\sigma W_t - \frac{\sigma^2}{2} t$.

Now, as we defined discrete-time markets and prelimit processes $\left(U_k^{(N)}, 0 \leq k \leq N\right)$, we can give a mathematical formulation for the main goal of this study, which is the convergence of discrete-time markets to the market described by Equation (1). By "convergence of discrete-time markets," we mean weak convergence of probability measures associated with stochastic processes that drive such markets, or convergence of random processes in Skorokhod or uniform topology. It will be specified explicitly in any theorem.

Next, we define the logarithm of the limit price process by

$$U_t = \log(X_t) = \log(X_0) + \int_0^t (\delta_0 Y_s + \delta_1(1 - Y_s))ds + \sigma W_t - \frac{\sigma^2 t}{2},$$

$t \in [0, T]$. It is convenient to separate the components of U_t and $U_k^{(N)}$ as follows:

$$\begin{aligned}
U_t &= \log X_0 + U_t^{(1)} + U_t^{(2)}, \\
U_k^{(N)} &= \log X_0 + U_k^{(1,N)} + U_k^{(2,N)},
\end{aligned}$$

where

$$\begin{aligned}
U_t^{(1)} &= \int_0^t (\delta_0 Y_s + \delta_1(1 - Y_s))ds, \; U_t^{(2)} = \sigma W_t - \frac{\sigma^2 t}{2}, \\
U_k^{(i,N)} &= \sum_{j=1}^{k} \log\left(1 + R_j^{(i,N)}\right), \; i \in \{1, 2\}, 1 \leq k \leq N, U_0^{(i,N)} = 0.
\end{aligned}$$

Let us define for $t \in \left[\frac{(k-1)T}{N}, \frac{kT}{N}\right)$,

$$\begin{aligned}
U_t^{(N)} &= U_{k-1}^{(N)}, \; U_T^{(N)} = U_N^{(N)}, \\
U_t^{(i,N)} &= U_{k-1}^{(i,N)}, \; U_T^{(i,N)} = U_N^{(i,N)}, \\
R_t^{(N)} &= R_{k-1}^{(N)}, R_T^{(N)} = R_N^{(N)}, \\
Y_t^{(N)} &= Y_{k-1}^{(N)}, \; Y_T^{(N)} = Y_N^{(N)}, \; 1 \leq k \leq N,
\end{aligned} \tag{10}$$

$i \in \{1, 2\}$, $1 \leq k \leq N$. So, we consider step-wise discrete-time approximations of the limit process U. Thus, our goal is to prove the weak convergence of the sequence of stochastic processes $\left(U_t^{(N)}\right)_{t\in[0,T]}$ to the process $(U_t)_{t\in[0,T]}$. To this end, we will establish the convergence of $\left(Y_t^{(N)}\right)_{t\in[0,T]}$ to $(Y_t)_{t\in[0,T]}$ (Theorem 3), then the convergence of $\left(U_t^{(i,N)}\right)_{t\in[0,T]}$ to $\left(U_t^{(i)}\right)_{t\in[0,T]}$, $i \in \{1, 2\}$ (Theorems 4 and 5), and the desired result then follows because of the independence of probability measures respective to Markov chains and the components that converge to the geometric Brownian motion. Therefore, the respective products of the probability measures weakly converge to the product of probability measures corresponding to the limit Markov chain and the limit geometric Brownian motion, respectively.

3 Weak convergence of discrete-time Markov chains to the limit Markov process

In this section, we prove that the sequence of processes $\left(Y^{(N)}\right)_{t\in[0,T]}$ introduced in Section 2 converges in Skorokhod topology to the process $(Y_t)_{t\in[0,T]}$. As a consequence, we will obtain the convergence of the processes $\left(U^{(1,N)}\right)_{t\in[0,T]}$ to $\left(U^{(1)}\right)_{t\in[0,T]}$, however, even in the uniform topology.

Let N_t be the number of jumps of a process $(Y_s, s\geq 0)$ on a time interval $[0,t]$. Let us introduce the occupation times

$$\theta_0 = \inf\{t>0|Y_t\neq Y_0\}, \theta_n = \inf\{t>\theta_{n-1}: Y_t\neq Y_{\theta_{n-1}}\} - \theta_{n-1}, n\geq 1,$$

and jump times

$$\tau_k = \sum_{j=0}^{k}\theta_k, k\geq 0.$$

Recall that the Markov chain $(Y_k^{(N)}, k\geq 0)$, introduced in Section 2, has an initial value of $Y_0^{(N)} = 0$ and the transition probability matrix (Equation 6). For this chain, let us define the total number of jumps on the time interval $[0,T]$

$$\nu_N = \sum_{j=0}^{N-1}\left|Y_{j+1}^{(N)} - Y_j^{(N)}\right|,$$

occupation times

$$\theta_0^{(N)} = \inf\left\{k>0|Y_k^{(N)}\neq Y_0^{(N)}\right\},$$
$$\theta_n^{(N)} = \inf\left\{k>\theta_{n-1}^{(N)}: Y_k^{(N)}\neq Y_{\theta_{n-1}^{(N)}}^{(N)}\right\}, n\geq 1,$$

and jump times

$$\tau_k^{(N)} = \sum_{k=0}^{k}\theta_j^{(N)}, k\geq 0.$$

For a given $t\in[0,T]$ and integer N, define $k_{t,N}\in\{0,\dots,N\}$ in the following way: $k_{T,N} = N$, and for $t\in[0,T)$, we have $t\in\left[\frac{k_{t,N}T}{N}, \frac{(k_{t,N}+1)T}{N}\right)$. We will also use the notation $t^{(N)} = \frac{k_{t,N}T}{N}$.

Lemma 1. For all $k\geq 1$, the following inequality holds:

$$\mathbb{P}(N_T = k) \leq dC^k \exp\left(-\frac{|\lambda_1-\lambda_0|T}{2}k\right), \tag{11}$$

where

$$d = \max\left\{\left(\frac{\lambda_0}{\lambda_1}\right)^{\frac{1}{2}}, 1\right\}\max\left\{e^{-\lambda_0 T}, e^{-\frac{|3\lambda_1-\lambda_0|}{2}T}, e^{-\frac{|\lambda_0-\lambda_1|}{2}T}\right\}, \tag{12}$$

and

$$C = \frac{e^{|\lambda_1-\lambda_0|}-1}{|\lambda_1-\lambda_0|}(\lambda_0\lambda_1)^{\frac{1}{2}}. \tag{13}$$

Proof. We have the following relations:

$$\begin{aligned}
&\mathbb{P}(N_T = 2m)\\
&= \int_0^T\int_{t_0}^T\dots\int_{t_{2m-1}}^T \lambda_0 e^{-\lambda_0 t_0}\lambda_1 e^{-\lambda_1(t_1-t_0)}\lambda_0 e^{-\lambda_0(t_2-t_1)}\\
&\dots\lambda_1 e^{-\lambda_1(t_{2m-1}-t_{2m-2})}e^{-\lambda_0(T-t_{2m-1})}dt_0\dots dt_{2m-1}\\
&\leq \int_{[0,T]^{2m}}\lambda_0 e^{-\lambda_0 t_0}\lambda_1 e^{-\lambda_1(t_1-t_0)}\dots e^{-\lambda_0(T-t_{2m-1})}dt_0\dots dt_{2m-1}\\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T}\int_{[0,T]^{2m}}\\
&\quad\exp\left((\lambda_1-\lambda_0)(t_0-t_1+t_2\dots+t_{2m-1})\right)dt_0\dots dt_{2m-1}\\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T}\prod_{j=0}^{2m-1}\int_0^T e^{(-1)^j(\lambda_1-\lambda_0)t_j}dt_j\\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T}\prod_{j=0}^{2m-1}\frac{(-1)^j(e^{(-1)^j(\lambda_1-\lambda_0)T}-1)}{\lambda_1-\lambda_0}\\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T}\frac{(e^{(\lambda_1-\lambda_0)T}-1)^m(1-e^{-(\lambda_1-\lambda_0)T})^m}{(\lambda_1-\lambda_0)^{2m}}\\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T}\left(\frac{e^{(\lambda_1-\lambda_0)T}-1}{\lambda_1-\lambda_0}\right)^{2m}e^{-(\lambda_1-\lambda_0)Tm}.
\end{aligned}$$

In the case when $\lambda_1 > \lambda_0$, from these relations, we immediately get inequality (Equation 11) for $k = 2m$. In the case when $\lambda_0 > \lambda_1$, we can rewrite previous estimates as

$$\begin{aligned}
&\mathbb{P}(N_T = 2m)\\
&\leq (\lambda_0\lambda_1)^m e^{-\lambda_0 T}\left(\frac{e^{(\lambda_1-\lambda_0)T}-1}{\lambda_1-\lambda_0}\right)^{2m}e^{-(\lambda_1-\lambda_0)Tm}\\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T}\left(\frac{1-e^{(\lambda_1-\lambda_0)T}}{\lambda_0-\lambda_1}\right)^{2m}e^{-(\lambda_1-\lambda_0)Tm}\\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T}\left(\frac{e^{(\lambda_0-\lambda_1)T}-1}{\lambda_0-\lambda_1}\right)^{2m}e^{-(\lambda_1-\lambda_0)Tm+2(\lambda_0-\lambda_1)Tm}\\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T}\left(\frac{e^{(\lambda_0-\lambda_1)T}-1}{\lambda_0-\lambda_1}\right)^{2m}e^{-(\lambda_0-\lambda_1)Tm},
\end{aligned}$$

and also get the inequality (Equation 11).

Let us now switch to the case $k = 2m-1$. Following the same process as before, we obtain the inequality

$$\begin{aligned}
&\mathbb{P}(N_T = 2m-1)\\
&\quad\leq \lambda_0(\lambda_0\lambda_1)^{m-1}e^{-\lambda_1 T}\left(\frac{e^{(\lambda_1-\lambda_0)T}-1}{\lambda_1-\lambda_0}\right)^{2m-1}e^{-(\lambda_1-\lambda_0)Tm}.
\end{aligned}$$

If $\lambda_1 > \lambda_0$, then we can write

$$\begin{aligned}
\mathbb{P}(N_T = 2m-1) &\leq \left(\frac{\lambda_0}{\lambda_1}\right)^{\frac{1}{2}}C^{2m-1}\exp\left(-(\lambda_1-\lambda_0)Tm-\lambda_1 T\right)\\
&\leq C^{2m-1}\exp\left(-(\lambda_1-\lambda_0)T\frac{2m-1}{2}-\frac{3\lambda_1-\lambda_0}{2}T\right),
\end{aligned}$$

so that Equation (11) holds true in this case.

If $\lambda_0 > \lambda_1$, then

$$
\begin{aligned}
&\mathbb{P}(N_T = 2m-1) \\
&\leq \lambda_0(\lambda_0\lambda_1)^{m-1}e^{-\lambda_1 T}\left(\frac{e^{(\lambda_0-\lambda_1)T}-1}{\lambda_0-\lambda_1}\right)^{2m-1} e^{-(\lambda_0-\lambda_1)Tm-(\lambda_0-\lambda_1)T} \\
&= \left(\frac{\lambda_0}{\lambda_1}\right)^{\frac{1}{2}} C^{2m-1}e^{-(\lambda_0-\lambda_1)Tm} \\
&= \left(\frac{\lambda_0}{\lambda_1}\right)^{\frac{1}{2}} C^{2m-1}\exp\left(-(\lambda_0-\lambda_1)T\frac{2m-1}{2}-\frac{\lambda_0-\lambda_1}{2}T\right),
\end{aligned}
$$

and Equation (11) holds true. □

Corollary 1.

$$\mathbb{E}\exp\left(\frac{|\lambda_1-\lambda_0|T}{4}N_T\right) < \infty. \tag{14}$$

Proof. Let us define $\Lambda = |\lambda_1 - \lambda_0|$ and let constants C and d be as in Equations (13) and (12), respectively. From Lemma 1, we see that

$$\mathbb{P}(N_t = k) \leq dC^k e^{-\frac{\Lambda T}{2}k}, \tag{15}$$

Inequality (14) is a direct consequence of Inequality (15), indeed, we can put $\alpha := \frac{\Lambda T}{4} > 0$ and get that

$$\mathbb{E}e^{\alpha N_T} \leq d\sum_{k=0}^{\infty} C^k e^{-\frac{\Lambda T}{4}k} < \infty.$$

□

Theorem 1. Denote by $f_m(t_0,\ldots,t_m)$ a conditional density of $(\tau_0,\ldots,\tau_m)$ given $N_T = m$ and put

$$g_m(t_0,\ldots,t_m) = f_m(t_0,\ldots,t_m)\mathbb{P}(N_T = m).$$

Then, for any $\varepsilon > 0$, there exists an integer $N(m)$ such that for all $N \geq N(m)$ and all $0 \leq t_0 < \ldots < t_m \leq T$, we have

$$\left| g_m(t_0,\ldots,t_m) - \left(\frac{N}{T}\right)^{m+1}\hat{p}_N(k_{t_0,N},\ldots,k_{t_m,N})\right| < \varepsilon,$$

where $\hat{p}_N(k_0,\ldots,k_m) = \mathbb{P}^{(N)}\{\tau_0^{(N)} = k_0, \tau_1^{(N)} = k_1,\ldots,\tau_m^{(N)} = k_m, \tau_{m+1}^{(N)} > N\}$.

Proof. We prove the statement for even m (so that we will write $2m$ in the following theorem). The proof for the odd m is the same.

Let $\tilde{f}_{2m}(t_0,\ldots,t_{2m})$ be a conditional density of $(\theta_0,\ldots,\theta_{2m})$ given $N_T = 2m$ and $\tilde{g}_{2m}(t_0,\ldots,t_{2m}) = \tilde{f}_{2m}(t_0,\ldots,t_{2m})\mathbb{P}(N_T = 2m)$. Since $\{\theta_j, 0 \leq j \leq 2m\}$ are independent random variables with alternating exponential distributions, we can write

$$
\begin{aligned}
&\tilde{g}_{2m}(t_0,\ldots,t_{2m}) \\
&\quad= \lim_{h\to 0}\frac{1}{(2h)^{2m+1}}\mathbb{P}\left(|\theta_j - t_j| < h, \tau_{2m+1} > T,\ 0 \leq j \leq 2m\right) \\
&\quad= \mathbb{P}(\theta_{2m+1} > T - (t_0+\ldots+t_{2m})) \\
&\qquad \lim_{h\to 0}\prod_{j=0}^{2m}\left(\frac{1}{2h}\mathbb{P}(|\theta_j - t_j| < h)\right) \\
&\quad= \lambda_0 e^{-\lambda_0 t_0}\lambda_1 e^{-\lambda_1 t_1}\ldots\lambda_0 e^{-\lambda_0 t_{2m}}e^{-\lambda_1(T-(t_0+\ldots+t_{2m}))},
\end{aligned}
$$

for all $t_j \geq 0$, $0 \leq j \leq 2m$, such that $t_0 + \ldots + t_{2m} \leq T$. Recall that $\theta_0 = \tau_0$ and $\theta_j = \tau_j - \tau_{j-1}$, $1 \leq j \leq 2m$. So we have for all $0 \leq t_0 < \ldots < t_{2m} \leq T$

$$
\begin{aligned}
&g_{2m}(t_0,\ldots,t_{2m}) \\
&\quad= \tilde{g}_{2m}(t_0, t_1 - t_0,\ldots,t_{2m} - t_{2m-1}) \\
&\quad= \lambda_0 e^{-\lambda_0 t_0}\lambda_1 e^{-\lambda_1(t_1-t_0)}\ldots\lambda_0 e^{-\lambda_0(t_{2m}-t_{2m-1})}e^{-\lambda_1(T-t_{2m})} \\
&\quad= \lambda_0(\lambda_0\lambda_1)^m e^{-\lambda_1 T}\exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j t_j\right).
\end{aligned}
$$

To simplify the further derivations, let us omit indices in $k_{t_i,N}$ and simply write k_i. Then we can rewrite $\hat{p}_N(k_0,\ldots,k_m)$ as

$$
\begin{aligned}
\hat{p}_N(k_0,\ldots,k_{2m}) &= e^{-\lambda_0\frac{k_0 T}{N}}\left(1-e^{\frac{-\lambda_0 T}{N}}\right)e^{-\lambda_1\frac{(k_1-k_0)T}{N}}\left(1-e^{\frac{-\lambda_1 T}{N}}\right)\times\ldots\times \\
&\times e^{-\lambda_0\frac{k_0 T}{N}}\left(1-e^{\frac{-\lambda_0 T}{N}}\right)e^{\frac{-\lambda_1(N-k_{2m})T}{N}} = \left(1-e^{\frac{-\lambda_0 T}{N}}\right)^{m+1}\left(1-e^{\frac{-\lambda_1 T}{N}}\right)^m\times \\
&\times\exp\left(-\frac{T}{N}\left(\lambda_0 k_0 + \lambda_1(k_1-k_0) + \lambda_0(k_2-k_1)\ldots\right.\right. \\
&\left.\left.+\lambda_0(k_{2m}-k_{2m-1}) + \lambda_1(N-k_{2m})\right)\right) \\
&= \left(1-e^{\frac{-\lambda_0 T}{N}}\right)^{m+1}\left(1-e^{\frac{-\lambda_1 T}{N}}\right)^m e^{-\lambda_1 T}\exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j\frac{k_j T}{N}\right) \\
&= \left(1-e^{\frac{-\lambda_0 T}{N}}\right)^{m+1}\left(1-e^{\frac{-\lambda_1 T}{N}}\right)^m e^{-\lambda_1 T}\exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j t_j^{(N)}\right).
\end{aligned}
$$

Furthermore, the following limit holds:

$$
\begin{aligned}
&\left(\frac{T}{N}\right)^{-(2m+1)}\left(1-e^{-\lambda_0\frac{T}{N}}\right)^{m+1}\left(1-e^{-\lambda_1\frac{T}{N}}\right)^m \\
&= \left(\frac{1-e^{-\lambda_0\frac{T}{N}}}{(T/N)}\right)^{m+1}\left(\frac{1-e^{-\lambda_1\frac{T}{N}}}{(T/N)}\right)^m \\
&\to \lambda_0^{m+1}\lambda_1^m,
\end{aligned}
$$

as $N \to \infty$. For any $\varepsilon_1 > 0$, we can now find an integer $N(m,\varepsilon_1)$ such that for all $N \geq N(m,\varepsilon_1)$, we have

$$\left|\left(\frac{T}{N}\right)^{-2m-1}\left(1-e^{\frac{-\lambda_0 T}{N}}\right)^{m+1}\left(1-e^{\frac{-\lambda_1 T}{N}}\right)^m - \lambda_0^{m+1}\lambda_1^m\right| < \varepsilon_1.$$

Put

$$\varepsilon_2 = \lambda_0^{m+1}\lambda_1^m\varepsilon_1,\text{ and } B = \exp(|\lambda_1-\lambda_0|T).$$

Note that

$$\exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{n}(-1)^j s_j\right) \leq B,$$

for all integer $n > 0$ and all $0 \le s_0 < s_1 < \ldots < s_n \le T$. We can now write

$$\frac{e^{\lambda_1 T}}{\lambda_0^{m+1}\lambda_1^m}\left|g_{2m}(t_0,\ldots,t_{2m}) - \left(\frac{N}{T}\right)^{2m+1}\hat{p}_N(k_{t_0,N},\ldots,k_{t_{2m},N})\right| \le$$
$$\le \left|\exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j t_j\right) - \exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j t_j^{(N)}\right)\right| +$$
$$+ \varepsilon_2 \exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j t_j^{(N)}\right) =$$
$$= B\left(\left|\exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j\left(t_j - t_j^{(N)}\right)\right) - 1\right| + \varepsilon_2\right)$$
$$\le B\left(\left|B^{\frac{2m+1}{N}} - 1\right| + \varepsilon_2\right).$$

Clearly, we can now choose an integer $N_0 = N(m,\varepsilon)$ such that for all $N \ge N_0$,

$$\left|g_{2m}(t_0,\ldots,t_{2m}) - \left(\frac{N}{T}\right)^{2m+1}\hat{p}_N(k_{t_0,N},\ldots,k_{t_{2m},N})\right| < \varepsilon.$$

The theorem is proved. □

Theorem 2. Let $0 \le t_0 < t_1 < \ldots < t_n \le T$ be fixed. Then, for any $\varepsilon > 0$ there exists an integer $N(n,\varepsilon)$ such that for all $N \ge N(n,\varepsilon)$, we have

$$\left|\mathbb{P}\left(Y_{t_i} = x_i, 0 \le i \le n\right) - \mathbb{P}^{(N)}\left(Y^{(N)}_{t_i^{(N)}} = x_i, 0 \le i \le n\right)\right| < \varepsilon, \tag{16}$$

where $x_i \in \{0,1\}$, $0 \le i \le m$.

Proof. This result follows from Theorem 1 and Lemma 1. Indeed, for every fixed $\varepsilon > 0$, we can find an integer m such that $\mathbb{P}(N_T > m) + \mathbb{P}^{(N)}(\nu_N > m) < \varepsilon/2$ for all $N > 0$, so that (Equation 16) is reduced to

$$\begin{aligned}&\left|\mathbb{P}\left(Y_{t_i} = x_i, 0 \le i \le n, N_t \le m\right)\right. \\ &\left.-\mathbb{P}^{(N)}\left(Y^{(N)}_{t_k^{(N)}} = x_i, 0 \le i \le n, \nu_N \le m\right)\right| < \varepsilon/2.\end{aligned} \tag{17}$$

Let us introduce the random variables r_j of the form

$$r_j = \inf\{k \ge 0 : \tau_k \le t_j < \tau_{k+1}\}. \tag{18}$$

In fact, r_j is the index number of the occupation interval that covers the fixed point t_j. Note that r_j is defined on the same probability space as $(Y_s)_{s\ge 0}$, and for $\omega \in \{N_t \le m\}$, each $r_k(\omega)$ takes value in the set $\{0,1,\ldots,m\}$. Put $A_n^m = \{(r_0(\omega),\ldots,r_n(\omega)),\ \omega \in \Omega\} \subset \mathbb{R}^{n+1}$. It is clear that A_n^m is a finite set, and

$$|A_n^m| \le m^{n+1}.$$

Then using formula (18), we get an equality

$$\mathbb{P}\left(Y_{t_i} = x_i, 0 \le i \le n, N_t \le m\right)$$
$$= \sum_{(r_0,\ldots,r_n)\in A_n^m} \mathbb{P}\left(\tau_{r_j} \le t_j < \tau_{r_j+1}, 0 \le j \le n, N_t \le m\right).$$

By Theorem 1, we can find an integer $N(\varepsilon,n)$ such that for all $N \ge N(\varepsilon,n)$

$$\begin{aligned}&\left|\mathbb{P}\left(\tau_{r_j} \le t_j < \tau_{r_j+1}, 0 \le j \le n, N_t \le m\right)\right. \\ &\left.-\mathbb{P}^{(N)}\left(\tau^{(N)}_{r_j} \le t_j < \tau^{(N)}_{r_j+1}, 0 \le j \le n, N_t \le m\right)\right| \\ &\le \frac{\varepsilon}{2m^{n+1}},\end{aligned}$$

which proves Equation 17 and hence the statement of the theorem follows. □

Theorem 3. Processes $\left(Y_t^{(N)}\right)_{t\in[0,T]}$ converge to $(Y_t)_{t\in[0,T]}$, $N \to \infty$ in Skorokhod topology.

Proof. In Theorem 2, we already proved the convergence of finite-dimensional distributions. Therefore, by Theorem 4, Section VI.5 from Gikhman and Skokohod [19], we have to verify that for all $\varepsilon > 0$

$$\lim_{h\to 0}\limsup_{N\to\infty}\sup_{x\in\{0,1\},0\le s-t\le h}\mathbb{P}^{(N)}\left(\left|Y_s^{(N)} - Y_t^{(N)}\right| > \varepsilon \,\middle|\, Y_t^{(N)} = x\right) = 0. \tag{19}$$

Let us examine the probability

$$\mathbb{P}^{(N)}\left(\left|Y_{t+h}^{(N)} - Y_t^{(N)}\right| > \varepsilon \,\middle|\, Y_t^{(N)} = 0\right).$$

Since the chain $Y^{(N)}$ takes values in the set $\{0,1\}$, the condition $\left|Y_{t_h}^{(N)} - Y_t^{(N)}\right| > \varepsilon$ means that $Y_{t+h}^{(N)} = 1 - Y_t^{(N)}$. Thus, we can write

$$\begin{aligned}\mathbb{P}^{(N)}\left(\left|Y_{t+h}^{(N)} - Y_t^{(N)}\right| > \varepsilon \,\middle|\, Y_t^{(N)} = 0\right) &= \mathbb{P}^{(N)}\left(Y_{t+h}^{(N)} = 1|\ Y_t^{(N)} = 0\right) \\ &= \mathbb{P}^{(N)}\left(Y_h^{(N)} = 1|\ Y_0^{(N)} = 0\right),\end{aligned}$$

where the last equality follows from homogeneity.
Similarly,

$$\begin{aligned}\mathbb{P}^{(N)}\left(|Y_{t+h}^{(N)} - Y_t^{(N)}| > \varepsilon \mid Y_t^{(N)} = 1\right) &= \mathbb{P}^{(N)}\left(Y_{t+h}^{(N)} = 0|\ Y_t^{(N)} = 1\right) \\ &= \mathbb{P}^{(N)}\left(Y_h^{(N)} = 0|\ Y_0^{(N)} = 1\right).\end{aligned}$$

To evaluate the latter probabilities, we will need a general form of n-step transition probability for a 2×2 Markov chain, which has the form

$$\mathbb{P}^{(N)}\left(Y_m^{(N)} = 1|\ Y_0^{(N)} = 0\right) = \pi_1^{(N)} - \pi_1^{(N)}(a^{(N)} - 1)^m,$$

$$\mathbb{P}^{(N)}\left(Y_m^{(N)} = 0|\ Y_0^{(N)} = 1\right) = \pi_0^{(N)} - \pi_0^{(N)}(a^{(N)} - 1)^m,$$

where $a^{(N)} = e^{-\lambda_0\frac{T}{N}} + e^{-\lambda_1\frac{T}{N}}$ (note that $a^{(N)} \in (0,2)$), and

$$\pi^{(N)} = (\pi_0^{(N)}, \pi_1^{(N)}) = \left(\frac{1 - e^{-\lambda_1\frac{T}{N}}}{2 - a^{(N)}}, \frac{1 - e^{-\lambda_0\frac{T}{N}}}{2 - a^{(N)}}\right)$$

is an invariant distribution for the chain $Y^{(N)}$ (see Appendix, Equation A1). For a fixed $h \in [0,T]$, recall the notation

$$h^{(N)} = \left\lfloor\frac{hN}{T}\right\rfloor.$$

Now, we can rewrite the left-hand side of Equation (19) as

$$\begin{aligned}
&\lim_{h\to 0}\limsup_{N\to\infty}\sup_{x\in\{0,1\},0\le s-t\le h}\mathbb{P}^{(N)}\left(|Y_s^{(N)}-Y_t^{(N)}|>\varepsilon \mid Y_t^{(N)}=x\right)\le\\
&\le \lim_{h\to 0}\limsup_{N\to\infty}\sup_{x\in\{0,1\},0\le s\le h}\mathbb{P}^{(N)}\left(Y_s^{(N)}=1-x|\ Y_0^{(N)}=x\right)\\
&\le \lim_{h\to 0}\limsup_{N\to\infty}\sup_{x\in\{0,1\},0\le s\le h}\left(\pi_x^{(N)}-\pi_x^{(N)}(a^{(N)}-1)^{s^{(N)}}\right)\\
&= \lim_{h\to 0}\limsup_{N\to\infty}\max_{x\in\{0,1\}}\left(\pi_x^{(N)}-\pi_x^{(N)}(a^{(N)}-1)^{h^{(N)}}\right)\\
&\le \lim_{h\to 0}\limsup_{N\to\infty}\left(1-(a^{(N)}-1)^{\frac{hN}{T}}\right)\\
&= \lim_{h\to 0}(1-e^{-(\lambda_0+\lambda_1)h})=0.
\end{aligned}$$

□

Theorem 4. Processes $\left(U_t^{(1,N)}\right)_{t\in[0,T]}$ converge to $\left(U_t^{(1)}\right)_{t\in[0,T]}$ in the uniform topology.

Proof. Using the Taylor formula for logarithm, we get the following representation: for $x>0$,

$$\log(1+x)=x+\rho(x)x,$$

where $|\rho(x)|\le h(N)$ when $x\in\left(0,\frac{C}{N}\right)$ for some constant C, and $h(N)\to 0$, $N\to\infty$. For any fixed $t\in\left[\frac{(k-1)T}{N},\frac{kT}{N}\right)$ we have

$$\begin{aligned}
U_t^{(1,N)}&=\sum_{j=0}^{k-1}\log\left(1+R_j^{(1,N)}\right)=\sum_{j=0}^{k-1}\left(R_j^{(1,N)}+\rho\left(R_j^{(1,N)}\right)R_j^{(1,N)}\right)\\
&=\sum_{j=0}^{k-1}\left(\frac{\delta_0 T}{N}Y_j^{(N)}+\frac{\delta_1 T}{N}\left(1-Y_j^{(N)}\right)\right)+\sum_{j=0}^{k-1}\rho\left(R_j^{(1,N)}\right)R_j^{(1,N)}\\
&=\int_0^{\frac{(k-1)T}{N}}\left(\delta_0 Y_s^{(N)}+\delta_1\left(1-Y_s^{(N)}\right)\right)ds+\sum_{j=0}^{k-1}\rho\left(R_j^{(1,N)}\right)R_j^{(1,N)}\\
&=\int_0^t\left(\delta_0 Y_s^{(N)}+\delta_1\left(1-Y_s^{(N)}\right)\right)ds\\
&-\int_{\frac{(k-1)T}{N}}^t\left(\delta_0 Y_s^{(N)}+\delta_1\left(1-Y_s^{(N)}\right)\right)ds\\
&+\sum_{j=0}^{k-1}\rho\left(R_j^{(1,N)}\right)R_j^{(1,N)}.
\end{aligned}$$

Next, we have a.s.

$$\left|\sum_{j=0}^{k-1}\rho\left(R_j^{(1,N)}\right)R_j^{(1,N)}\right|\le h(N)T\max\{\delta_0,\delta_1\}\to 0,\ N\to\infty,$$

$$\left|\int_{\frac{(k-1)T}{N}}^t\left(\delta_0 Y_s^{(N)}+\delta_1\left(1-Y_s^{(N)}\right)\right)ds\right|\le\frac{\max\{\delta_0,\delta_1\}T}{N}\to 0,\ N\to\infty.$$

Using Theorem 3, we may conclude that for each fixed $t\in[0,T]$,

$$\int_0^t\left(\delta_0 Y_s^{(N)}+\delta_1\left(1-Y_s^{(N)}\right)\right)ds\to^d\int_0^t\left(\delta_0 Y_s+\delta_1\left(1-Y_s\right)\right)ds,\ N\to\infty.$$

We can now use the Slutsky theorem, and conclude that

$$U_t^{(1,N)}\to^d U_t^{(1)},$$

where by $\to^d$ we denote a weak convergence in distribution. Let us now consider a linear combination of the form

$$\sum_{j=0}^m\alpha_j U_{t_j}^{(1,N)},\ \alpha_j\in\mathbb{R},\ m\ge 0,\ 0\le t_0<\ldots<t_m\le T.$$

Using the properties of the Riemann integral and Slutsky theorem we can apply similar reasoning to conclude that

$$\sum_{j=0}^m\alpha_j U_{t_j}^{(1,N)}\to^d\sum_{j=0}^m\alpha_j U_{t_j}^{(1)},\ N\to\infty,$$

which implies weak convergence of finite-dimensional distributions of the process $\left(R_t^{(1,N)}\right)_{t\in[0,T]}$ to that of $\left(U_t^{(1)}\right)_{t\in[0,T]}$.

Let us consider the modulus of continuity of the sequences of the processes $\left(\int_0^t\left(\delta_0 Y_s^{(N)}+\delta_1(1-Y_s^{(N)})\right)ds\right)_{t\in[0,T]}$. Obviously, for all $0\le u<t\le T$

$$\left|\int_u^t\left(\delta_0 Y_s^{(N)}+\delta_1(1-Y_s^{(N)})\right)ds\right|\le(t-u)\max\{\delta_0,\delta_1\}.$$

The latter inequality implies that the family of processes $\left(\int_0^t\left(\delta_0 Y_s^{(N)}+\delta_1(1-Y_s^{(N)})\right)ds\right)_{t\in[0,T]}$ is tight in the uniform topology. The statement of the theorem follows from this fact, together with the convergence of finite-dimensional distributions. □

4 Weak convergence to a geometric Brownian motion with Markov switching drift rate in the multiplicative scheme of series

Conditions of weak convergence of the sequence of processes

$$U^{(2,N)}:=\{U_t^{(2,N)},t\in[0,T]\},\ N\ge 1,$$

created in Equation 10, to the process $U_2(t)=\sigma W_t-\frac{\sigma^2}{2}t$, are classical. They can be deduced from the respective results contained in the books [20] and [18]. However, for the reader's convenience, we describe them briefly, basing them on the Skorokhod theorem about weak convergence of sums of independent random variables to the continuous process with independent increments (see, e.g., Theorem 1, pages 452–453 from Gikhman and Skokohod [21]). So, we consider the scheme of series of the form $U_t^{(2,N)}=0,\ t\in[0,\frac{T}{N})$, $U_T^{(2,N)}=\sum_{i=1}^N\log\left(1+R_i^{(2,N)}\right)$, and

$$U_t^{(2,N)}=\sum_{i=1}^{\left[\frac{Nt}{T}\right]}\log\left(1+R_i^{(2,N)}\right),\ t\in\left[\frac{T}{N},T\right).$$

We can simplify these records by putting $\sum_{i=1}^0$ and

$$U_t^{(2,N)}=\sum_{i=1}^{\left[\frac{Nt}{T}\right]}\log\left(1+R_i^{(2,N)}\right),\ t\in[0,T].$$

Assume that there exist two real-valued sequences $\{\alpha_N, \beta_N, N \geq 1\}$ such that $-1 < \alpha_N < R_i^{(2,N)} < \beta_N$ with probability 1 and $\alpha_n, \beta_N \to 0$ as $N \to \infty$. Then

$$U_t^{(2,N)} = \sum_{i=1}^{[\frac{Nt}{T}]} \left(R_i^{(2,N)} - \frac{1}{2} \left(R_i^{(2,N)} \right)^2 \right) + \Delta_N(t),$$

where $|\Delta_N(t)| \leq \Delta(\alpha_N, \beta_N) \sum_{i=1}^{N} \left(R_i^{(2,N)} \right)^2$, and real-valued positive sequence $\Delta(\alpha_N, \beta_N) \to 0$ as $N \to \infty$. Recall that we already assumed that $\left(R_i^{(2,N)}, 1 \leq i \leq N \right)$ are mutually independent.

Theorem 5. Assume that the following conditions hold:

(i) $\mathbb{E} R_i^{(2,N)} = 0, 1 \leq i \leq N, N \geq 1$.

(ii) For any $t \in [0, T]$

$$\sum_{i=1}^{t^{(N)}} \mathbb{E} \left[R_i^{(2,N)} \right]^2 \to \sigma^2 t.$$

Then the sequence $\mathbb{P}_T^{(2,N)}$ of measures corresponding to processes $\{U_t^{(2,N)}, t \in [0, T]\}$ weakly converges to the measure $\mathbb{P}_T^{(2)}$ corresponding to process $\{\sigma W_t - \frac{\sigma^2}{2} t, t \in [0, T]\}$.

Proof. Conditions (i) and (ii) mentioned in Theorem 5, together with Theorem 5.53 from Föllmer et al. [20], imply that for any $0 \leq s < t \leq T$, the distribution of the increment $U_t^{(2,N)} - U_s^{(2,N)}$ weakly converges to $\sigma(W_t - W_s) - \frac{\sigma^2}{2}(t - s)$. Moreover, these conditions, together with restrictions on the values of $R_i^{(2,N)}$, support Lindeberg's condition in Theorem 1, pages 452–453 from Gikhman and Skokohod [21], whence the proof follows. □

5 Incompletenesses of the market with switching

This section explores the incompleteness of the continuous-time market with drift Markov switching, as described by Equation 1. Although this topic is not directly related to the convergence problem studied in the previous sections, it is of interest to the financial applications of the model.

In this section, we assume that $(X_t)_{t \geq 0}$ represents the discounted asset price in an arbitrage-free market, which consists of this risky asset and a risk-free asset. Since the risky asset price involves two independent sources of randomness, the financial market is incomplete. To demonstrate the incompleteness explicitly, let us construct a MMM and separately a class of martingale measures especially related to the Markov process. First, fix the interval $[0, T]$ and attempt to construct an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$, whose Radon-Nikodym derivative restricted to the interval $[0, T]$ has the form

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}_T} = \exp \left(\sigma \int_0^T \varphi(u) dW_u - \frac{\sigma^2}{2} \int_0^T \varphi^2(u) du \right), \tag{20}$$

where $\varphi(u)$ is a $\mathcal{F}_u$-adapted stochastic process satisfying condition $\mathbb{E} \left(\frac{d\mathbb{Q}_T}{d\mathbb{P}_T} \right) = 1$ (in this case $\mathbb{Q}_T$ is indeed a probability measure). Moreover, recall the notion of the MMM from Föllmer and Schweizer [22]:

Definition 1. (Föllmer and Schweizer [22]) Let the discounted asset price in a financial market be given by the real-valued semimartingale of the form

$$S = S_0 + M + A,$$

where $S_0 > 0$ is a constant, M is a local $\mathbb{P}$-martingale, A is a process of locally bounded variation, $\mathbb{P}$ is the initial probability measure, and $M_0 = A_0 = 0$. The minimal martingale measure (MMM) for S is an equivalent probability measure $\hat{\mathbb{P}}$ that is characterized by the properties that it transforms S into a local martingale and preserves the martingale property for any local $\mathbb{P}$-martingale that is strongly orthogonal to M.

According to Föllmer and Schweizer [22], assume additionally that M is a $\mathbb{P}$-square-integrable martingale, and A has a form

$$A_t = \int_0^t \lambda_s d\langle M \rangle_s, t \in [0, T],$$

where $\int_0^T \lambda_s^2 d\langle M \rangle_s < \infty$ a.s., $\langle M \rangle$ is the quadratic characteristics of M (see, e.g., Liptser and Shiryayev [17] for detail). Moreover, if

$$dS_t = S_t(\rho_t dt + \sigma_t dW_t),$$

and σ is a strictly positive adapted process on $[0, T]$, then $\lambda_s = \rho_s \sigma_s^{-2}$, $s \in [0, T]$, and

$$\int_0^t \lambda_s^2 \langle M \rangle_s = \int_0^t \rho_s^2 \sigma_s^{-2} ds, \ t \in [0, T].$$

If the MMM $\hat{\mathbb{P}}$ exists, then its Radon-Nikodym derivative restricted to the interval $[0, T]$ is given by the stochastic exponent of the form

$$\frac{d\hat{\mathbb{P}}_T}{d\mathbb{P}_T} = \mathcal{E} \left(- \int \lambda dM \right) = \exp \left\{ - \int_0^T \lambda_s dM_s - \frac{1}{2} \int_0^T \lambda_s^2 d\langle M \rangle_s \right\} \times$$
$$\times \prod_{0 \leq s \leq T} (1 - \lambda \Delta M_s) \exp \left(\lambda \Delta M_s - \frac{1}{2} \lambda^2 (\Delta M)_s^2 \right).$$

Lemma 2. The equivalent martingale measure for the market is described by Equation (1), which has the form Equation (20), is unique, and the function φ equals

$$\varphi(u) = - \left(\frac{\delta_0}{\sigma^2} Y_u + \frac{\delta_1}{\sigma^2} (1 - Y_u) \right), u \in [0, T]. \tag{21}$$

and $\mathbb{Q} = \hat{\mathbb{P}}$ is a MMM in this market.

Proof. For all $t \in [0, T]$ the following equality holds

$$\mathbb{E}_{\mathbb{Q}}[X_t - X_s | \mathcal{F}_s] = \frac{\mathbb{E} \left[\frac{d\mathbb{Q}_T}{d\mathbb{P}_T} (X_t - X_s) \,\middle|\, \mathcal{F}_s \right]}{\mathbb{E} \left[\frac{d\mathbb{Q}_T}{d\mathbb{P}_T} \,\middle|\, \mathcal{F}_s \right]}. \tag{22}$$

Assume that

$$\mathbb{E} \exp \left(\sigma \int_0^T \varphi(u) dW_u - \frac{\sigma^2}{2} \int_0^T \varphi^2(u) du \right) = 1. \tag{23}$$

Then the process $\exp\left(\sigma \int_0^t \varphi(u)dW_u - \frac{\sigma^2}{2}\int_0^t \varphi^2(u)du\right)$, $t \in [0,T]$ is a martingale, in particular,

$$\mathbb{E}\left(\exp\left(\sigma \int_0^t \varphi(u)dW_u - \frac{\sigma^2}{2}\int_0^t \varphi^2(u)du\right)\middle|\mathcal{F}_s\right)$$
$$= \exp\left(\sigma \int_0^s \varphi(u)dW_u - \sigma^2\int_0^s \varphi^2(u)du\right),$$

therefore,

$$\mathbb{E}\left[\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}(X_t - X_s)\middle|\mathcal{F}_s\right] = X_s\mathbb{E}\left[\exp\left(\sigma\int_0^t \varphi(u)dW_u - \frac{\sigma^2}{2}\int_0^t \varphi^2(u)du\right)\right.$$
$$\left.\times\left(\exp\left(\int_s^t (\delta_0 Y_u + \delta_1(1-Y_u))du + \sigma(W_t - W_s) - \frac{\sigma^2}{2}(t-s)\right) - 1\right)\middle|\mathcal{F}_s\right] = 0 \tag{24}$$

if and only if

$$\mathbb{E}\left[\exp\left(\int_s^t(\sigma\varphi(u)\right.\right.$$
$$\left.\left.+\sigma)dW_u + \int_s^t(\delta_0 Y_u + \delta_1(1-Y_u) - \frac{\sigma^2}{2}\varphi^2(u) - \frac{\sigma^2}{2})du\right)\middle|\mathcal{F}_s\right] = 1,$$

which in turn is true if and only if

$$\delta_0 Y_u + \delta_1(1-Y_u) - \frac{\sigma^2}{2}\varphi^2(u) - \frac{\sigma^2}{2} = -\frac{\sigma^2}{2}(\varphi(u)+1)^2,$$

whence $\varphi(u)$ satisfies equality (Equation 21). According to Föllmer and Schweizer [22], measure $\mathbb{Q}$ is a MMM for this market.

Indeed, in our case, $M_t = \sigma\int_0^t X_s dW_s$ and $A_t = \int_0^t(\delta_0 Y_s + \delta_1(1-Y_s))X_s ds$. Obviously, M is a continuous square-integrable martingale, $\lambda_t = \sigma^{-2}(\delta_0 Y_t + \delta_1(1-Y_t))$, and for MMM

$$\frac{d\hat{\mathbb{P}}_T}{d\mathbb{P}_T} = \mathcal{E}\left(-\int_0^T \lambda dM\right)$$
$$= \exp\left\{-\sigma^{-1}\int_0^T(\delta_0 Y_s + \delta_1(1-Y_s))dW_s\right.$$
$$\left.-\frac{\sigma^{-2}}{2}\int_0^T(\delta_0 Y_s + \delta_1(1-Y_s))^2 ds\right\},$$

therefore, $\hat{\mathbb{P}}_T = \mathbb{Q}_T$ from (Equation 20) with Equations 21-24 in hand. Moreover, equality (Equation 23) holds. So, the lemma is proved. □

Nevertheless, there can be other equivalent martingale measures. To construct a wide class of equivalent martingale measures, let us consider the following objects: First, we shall use the standard definition of the Feller process (see e.g., Chung [23], p. 50) and the following definition of the left quasi-continuous process, taken from Chung [23] and Liptser and Shiryayev [17].

Definition 2. Let us have a stochastic basis with filtration and an adapted process $U = \{U_t, t \geq 0\}$. Process U is left quasi-continuous, if for any stopping time τ and any sequence of stopping times $\tau_n \uparrow \tau$, $U_\tau = \lim_{\tau_n\uparrow\tau} U_{\tau_n}$ P-a.s. on the set $\{\tau < \infty\}$.

Now we summarize the following facts from Liptser and Shiryayev [17] and Gushchin [24], simplifying them for our situation (in general, these properties can be formulated in a local version, but our processes under consideration are integrable). We consider càdlàg processes, which have a.s. continuous trajectories from the right and with left limits at all points.

(i) For any adapted process A of integrable variation, there exists a predictable process A^π of integrable variation (dual predictable projection, or compensator of A) such that the process $M = A - A^\pi$ is a martingale.

(ii) If process A is left quasi-continuous, then process A^π is continuous.

(iii) The left quasi-continuity of the adapted process A of integrable variation is equivalent to any of the following properties:

(a) for any predictable stopping moment τ $\Delta_\tau A \mathbb{1}_{\tau<\infty} = 0$, where $\Delta_t A = A_t - A_{t-}$, the jump at point t, which is correctly defined for càdlàg processes.

(b) for any bounded stopping moment τ and for any sequence of non-decreasing stopping times $\tau_n \uparrow \tau$

$$\mathbb{E}A_{\tau_n} \to \mathbb{E}A_\tau,\ n \to \infty.$$

Now we are in a position to construct a wide class of equivalent martingale measures for our market with Markov switching, but we decide to operate only with the Markov process Y. It should be noted that Y has bounded variation $|Y|$ on $[0,T]$ with finite moments of any order (variation $|Y|$ on $[0,t]$ is simply a number of jumps N_t, which, according to Corollary 1, has a finite exponential moment). Therefore, Y is a process of integrable variation and admits a dual predictable projection Y^π of integrable variation.

Lemma 3. Process Y is left quasi-continuous.

Proof. The desired property follows directly from Theorem 4 (Section 2.4, page 70) in Chung [23], once we establish that Y is a Feller process.

Recall that time-homogenous Markov process has values in some compact space E is called Feller if the following two conditions hold true:

(i) for all $f \in C(E)$

$$\lim_{t\downarrow 0}\int_E P_t(\cdot, dy)f(y) = f(\cdot),$$

(ii) for every fixed t and $f \in C(E)$

$$\int_E P_t(\cdot, dy)f(y) \in C(E),$$

where $C(E)$ is a space of all functions continuous on E and $P_t(x, A)$ is a transition probability on the time interval $[0,t]$.

In our case, $E = \{0,1\}$, so every finite function on E is continuous, and (ii) follows immediately.

Since the matrix $\mathbb{A}$ defined in Equation (2) is a generator of the process Y, we have by definition

$$\mathbb{A}f(\cdot) = \lim_{t\downarrow 0}\frac{\int_E P_t(\cdot, dy)f(y) - f(\cdot)}{t},$$

for all continuous functions f on $\{0,1\}$, which implies (i). □

Now, according to Gushchin [24], any left quasi-continuous process of integrable variation has a continuous integrable dual predictable projection (compensator). Therefore, we can consider the dual predictable projection Y^{π} of Y, which is a continuous process of integrable variation, and let $M_t = Y_t - Y_t^{\pi}$. Then M is a martingale. Therefore, according to Liptser and Shiryayev [17], M admits a decomposition $M = M^c + M^d$, where M^c is a continuous local martingale, and M^d is a purely discontinuous local martingale where pure discontinuity means that common quadratic variation $[M^c, M^d]$ is a zero process.

Lemma 4. M is a purely discontinuous martingale with a finite a.s. number of jumps on any fixed interval $[0, T]$.

Proof. Pure discontinuity immediately follows from the fact that both the purely jump process Y and the continuous compensator of Y^{π} have zero common quadratic variations $[Y, B]$ and $[Y^{\pi}, B]$ with any continuous process B. The lemma is proven. □

Therefore, if we create a stochastic exponent $\mathcal{E}(M)$, it will have the form

$$\mathcal{E}_t(M) = \prod_{0 \le u \le t} (1 + \Delta M_u) \exp\{-\Delta M_u\} = \prod_{0 \le u \le t} (1 + \Delta Y_u) \exp\{-\Delta Y_u\},$$

where $\Delta(\cdot)_s$ stands for the jump of the respective process at point s, and these jumps are correctly defined for càdlàg processes. However, the problem with this stochastic exponent is that the jumps of M can equal -1. To avoid this difficulty, let us consider any strictly positive continuous process $\psi_t, 0 \le t \le T$ adapted to $\sigma_{0,t}(Y)$ such that $\psi(t) \le \left(\frac{|\lambda_1 - \lambda_0| T}{4} \wedge \frac{1}{2}\right)$, consider stochastic integral $M_t^{(\psi)} = \int_0^t \psi_s dM_s$, which is in fact a sum of a finite number of terms, and construct stochastic exponent $\mathcal{E}_t\left(M^{(\psi)}\right)$. Introduce the following notations: $\mathcal{E}_{s,t}\left(M^{(\psi)}\right) = \mathcal{E}_t\left(M^{(\psi)}\right)\left(\mathcal{E}_s\left(M^{(\psi)}\right)\right)^{-1}$, $0 < s \le t$, and

$$M_t^{(\varphi)} = \sigma \int_0^t \varphi(u) dW_u, \ \langle M^{(\varphi)} \rangle_t = \sigma^2 \int_0^t \varphi^2(u) du,$$

where φ is defined in Equation (21),

$$\begin{aligned} \mathcal{E}_t\left(M^{(\varphi)}\right) &= \exp\left\{M_t^{(\varphi)} - \tfrac{1}{2}\langle M^{(\varphi)} \rangle_t\right\}, \ \mathcal{E}_{s,t}\left(M^{(\varphi)}\right) \\ &= \mathcal{E}_t\left(M^{(\varphi)}\right)\left(\mathcal{E}_s\left(M^{(\varphi)}\right)\right)^{-1}, \ 0 < s \le t. \end{aligned}$$

Theorem 6. Probability measures $\mathbb{Q}^{\varphi,\psi}$, for which its Radon-Nikodym derivative restricted on the interval $[0, T]$ has the form

$$\frac{d\mathbb{Q}_T^{\varphi,\psi}}{d\mathbb{P}_T} = \mathcal{E}_T\left(M^{(\psi)}\right) \mathcal{E}_T\left(M^{(\varphi)}\right),$$

is a probability equivalent martingale measure for the market defined by Equation (1).

Proof. First, notice that for any $s > 0$

$$\begin{aligned} &\left(1 + \Delta M_s^{(\psi)}\right) \exp\{-\Delta M_s^{(\psi)}\} \\ &\le \left(1 + \tfrac{|\lambda_1 - \lambda_0| T}{4}\right) \mathbb{1}_{\Delta Y_s = 1} + e^{\frac{|\lambda_1 - \lambda_0| T}{4}} \mathbb{1}_{\Delta Y_s = -1}, \end{aligned}$$

therefore, according to Corollary 1, $\mathcal{E}_t\left(M^{(\psi)}\right)$ does not exceed $\exp\left(\frac{|\lambda_1 - \lambda_0| T}{4} N_T\right)$, and so, it is integrable. It means that being a local martingale and stochastic exponent, and also being an integrable, $\mathcal{E}_t\left(M^{(\psi)}\right)$, $t \in [0, T]$ is a martingale. In particular, $\mathbb{E}\mathcal{E}_T\left(M^{(\psi)}\right) = 1$ and $\mathcal{E}_T\left(M^{(\psi)}\right)$ define a probability measure $\mathbb{P}^{(\psi)}$ on $(\Omega, \mathcal{F})$, equivalent to measure $\mathbb{P}$. Now, for any $0 \le s \le t \le T$, introduce the σ-fields $\sigma_{s,t}(Y) = \sigma\{Y_u, s \le u \le t\}$ generated by the process Y on the respective intervals. Then

$$\mathbb{E}\left(\mathcal{E}_T\left(M^{(\psi)}\right) \mathcal{E}_T\left(M^{(\varphi)}\right)\right) = \mathbb{E}\left(\mathcal{E}_T\left(M^{(\psi)}\right) \mathbb{E}\left(\mathcal{E}_T\left(M^{(\varphi)}\right) \Big| \sigma_{0,T}(Y)\right)\right). \tag{25}$$

Denote $x = x_t$, $t \in [0, T]$ some bounded, measurable, and non-random function. Then, taking into account the independence of W and Y, we can write that

$$\begin{aligned} &\mathbb{E}\left(\mathcal{E}_T\left(M^{(\varphi)}\right) \Big| \sigma_{0,T}(Y)\right) \\ &= \left(\mathbb{E} \exp\left(\int_0^T x_t dW_t - \frac{1}{2}\int_0^T x_t^2 dt\right) \Bigg|_{\varphi_t = x_t, \, t \in [0,T]}\right) = 1. \end{aligned} \tag{26}$$

Therefore,

$$\mathbb{E}\left(\mathcal{E}_T\left(M^{(\psi)}\right) \mathcal{E}_T\left(M^{(\varphi)}\right)\right) = \mathbb{E}\left(\mathcal{E}_T\left(M^{(\psi)}\right)\right) = 1,$$

whence $\mathbb{Q}^{\varphi,\psi}$ is a probability measure and $\frac{d\mathbb{Q}_t^{\varphi,\psi}}{d\mathbb{P}_t}$, $t \in [0, T]$ is a martingale. Now we shall use the independence of W and Y again in order to prove that $\mathbb{Q}$ is an equivalent martingale measure. Indeed, similarly to the proof of Lemma 2,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}^{\varphi,\psi}}\left[X_t - X_s | \mathcal{F}_s\right] = \frac{\mathbb{E}\left[\frac{d\mathbb{Q}_T^{\varphi,\psi}}{d\mathbb{P}_T}(X_t - X_s) \Big| \mathcal{F}_s\right]}{\mathbb{E}\left[\frac{d\mathbb{Q}_T^{\varphi,\psi}}{d\mathbb{P}_T} \Big| \mathcal{F}_s\right]} \\ &= X_s \mathbb{E}\Big[\mathcal{E}_{s,t}\left(M^{(\varphi)}\right) \mathcal{E}_{s,t}\left(M^{(\psi)}\right) \times \\ &\times \left(\exp\left\{\int_s^t (\delta_0 Y_u + \delta_1(1 - Y_u)) du + \sigma(W_t - W_s) - \frac{\sigma^2}{2}(t - s)\right\} - 1\right) \Big| \mathcal{F}_s\Big] \\ &= X_s \mathbb{E}\Bigg[\mathcal{E}_{s,t}\left(M^{(\psi)}\right)\left(\exp\left\{\int_s^t \sigma(\varphi_u + 1) dW_u - \frac{\sigma^2}{2}\int_s^t (\varphi_u + 1)^2 du\right\}\right. \\ &\left. - \exp\left\{\sigma \int_s^t \varphi(u) dW_u - \frac{1}{2}\sigma^2 \int_s^t \varphi^2(u) du\right\}\right) \Bigg| \mathcal{F}_s\Bigg] =: G(s,t). \end{aligned}$$

Consider the σ-field

$$\mathcal{H}_s^t = \mathcal{F}_s \vee \sigma_{s,t}(Y),$$

the smallest σ-field containing $\mathcal{F}_s$ and $\sigma_{s,t}(Y)$. Then $\mathcal{E}_{s,t}\left(M^{(\psi)}\right)$ is $\mathcal{H}_s^t$-measurable, and, similarly to Equations (25, 26),

$$\begin{aligned} G(s,t) = X_s \mathbb{E}\Bigg[\mathcal{E}_{s,t}\left(M^{(\psi)}\right) \mathbb{E}\Bigg(&\exp\left\{\int_0^t \sigma(\varphi_u + 1) dW_u - \frac{\sigma^2}{2}\int_0^t (\varphi_u + 1)^2 du\right\} \\ &- \exp\left\{\sigma \int_s^t \varphi(u) dW_u - \frac{1}{2}\sigma^2 \int_s^t \varphi^2(u) du\right\}\Bigg) \Bigg| \mathcal{H}_s^t\Bigg) \Bigg| \mathcal{F}_s\Bigg], \end{aligned}$$

where

$$
\begin{aligned}
&\mathbb{E}\Bigg(\exp\left\{ \int_0^t \sigma(\varphi_u+1)dW_u - \frac{\sigma^2}{2}\int_0^t (\varphi_u+1)^2 du \right\} \\
&\quad - \exp\left\{ \sigma \int_s^t \varphi(u)dW_u - \frac{\sigma^2}{2}\int_s^t \varphi^2(u)du \right\} \Bigg| \mathcal{H}_s^t \Bigg) \\
&= \mathbb{E}\Bigg(\exp\left\{ \int_s^t \sigma(x_u+1)dW_u - \frac{\sigma^2}{2}\int_s^t (x_u+1)^2 du \right\} \\
&- \exp\left\{ \sigma \int_s^t x(u)dW_u - \frac{\sigma^2}{2}\int_s^t x^2(u)du \right\} \Bigg) \Bigg|_{\varphi_t = x_t,\, t\in[0,T]} = 1 - 1 = 0.
\end{aligned}
$$

It means that $\mathbb{Q}^{\varphi,\psi}$ is an equivalent martingale measure for the market defined by Equation (1), and the theorem is proved. □

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

Author contributions

VG: Conceptualization, Formal analysis, Investigation, Writing – original draft. YM: Conceptualization, Formal analysis, Investigation, Methodology, Project administration, Supervision, Validation, Writing – review & editing. KK: Conceptualization, Methodology, Writing – review & editing.

Funding

The author(s) declare financial support was received for the research, authorship, and/or publication of this article. YM was supported by the Swedish Foundation for Strategic Research (Grant No. UKR24-0004), the by Japan Science and Technology Agency CREST JPMJCR2115, and ToppForsk (Project No. 274410) of the Research Council of Norway with the title STORM: Stochastics for Time-Space Risk Models.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

All claims expressed in this article are solely those of the authors and do not necessarily represent those of their affiliated organizations, or those of the publisher, the editors and the reviewers. Any product that may be evaluated in this article, or claim that may be made by its manufacturer, is not guaranteed or endorsed by the publisher.

References

1. Quandt RE. The estimation of the parameters of a linear regression system obeying two separate regimes. *J Am Stat Assoc.* (1958) 53:873–80. doi: 10.1080/01621459.1958.10501484

2. Quandt RE, Goldfeld SM. A Markov model for switching regressions. *J Econom.* (1973) 1:3–15. doi: 10.1016/0304-4076(73)90002-X

3. Hamilton JD. A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica.* (1989) 57:357–84. doi: 10.2307/1912559

4. Ang A, Timmermann A. Regime changes and financial markets. *Annu. Rev. Financ. Econ.* (2012) 04:313–37. doi: 10.1146/annurev-financial-110311-101808

5. Sotomayor LR, Cdenillas A. Explicit solutions of consumption-investment problems in financial markets with regime switching. *Math. Finance.* (2009) 19:251–79. doi: 10.1111/j.1467-9965.2009.00366.x

6. Dai M, Zhang Q, Zhu QJ. Trend following trading under a regime switching model. *SIAM J Financ Math.* (2010) 1:780–810. doi: 10.1137/090770552

7. Dai M, Yang Z, Zhang Q, Zhu QJ. Optimal trend following trading rules. *Math Oper Res.* (2016) 41:626–42. doi: 10.1287/moor.2015.0743

8. Maheu JM, McCurdy TH, Song Y. Components of bull and bear markets: bull corrections and bear rallies. *J Bus Econ Stat.* (2012) 30:391–403. doi: 10.1080/07350015.2012.680412

9. Décamps JP, Mariotti T, Villeneuve S. Investment timing under incomplete information. *Math Oper Res.* (2005) 30:472–500. doi: 10.1287/moor.1040.0132

10. Klein M. Comment on "Investment timing under incomplete information". *Math Oper Res.* (2009) 34:249–54. doi: 10.1287/moor.1080.0349

11. Aingworth DD, Das SR, Motwani R. A simple approach for pricing equity options with Markov switching state variables. *Quant Finance.* (2006) 6:95–105. doi: 10.1080/14697680500511215

12. Elliott RJ, Siu TK, Chan L. Option pricing for Garch models with Markov switching. *Int J Theor Appl Finance.* (2006) 09:825–41. doi: 10.1142/S0219024906003846

13. Bae GI, Kim WC, Mulvey JM. Dynamic asset allocation for varied financial markets under regime switching framework. *Eur J Oper Res.* (2014) 234:450–8. doi: 10.1016/j.ejor.2013.03.032

14. Ekström E, Lu B. Optimal selling of an asset under incomplete information. *Int J Stoch Anal.* (2011) 17:543590. doi: 10.1155/2011/543590

15. Ekström E, Lindberg C. Optimal closing of a momentum trade. *J Appl Probab.* (2013) 2:374–87. doi: 10.1239/jap/1371648947

16. Henderson V. Kladívko K, Monoyios M, Reisinger C. Executive stock option exercise with full and partial information on a drift change point. *SIAM J Financ Math.* (2020) 11:1007–62. doi: 10.1137/18M1222909

17. Liptser R, Shiryayev A. *Theory of Martingales (Mathematics and its Applications, 49).* London: Kluwer Academic Publisher. (1989). doi: 10.1007/978-94-009-2438-3

18. Mishura Y, Ralchenko K. *Discrete-Time Approximations and Limit Theorems In Applications to Financial Markets, Vol 2.* Berlin: De Gruyter. (2021). doi: 10.1515/9783110654240

19. Gikhman II, Skokohod AV. *The Theory of Stochastic Processes I.* Cham: Springer (2004). doi: 10.1007/978-3-642-61921-2

20. Föllmer H, Schied A. *Stochastic Finance: An Introduction in Discrete Time.* Berlin: De Gruyter (2011). doi: 10.1515/9783110218053

21. Gikhman II, Skokohod AV. *Introduction to the Theory of Random Processes.* Philadelphia, PA: WB Saunders Company (1969).

22. Föllmer H, Schweizer M. Minimal martingale measure. *Encycl Quant Finance.* (2010) 3:1200–4. doi: 10.1002/9780470061602.eqf04015

23. Chung KL. *Lectures from Markov Processes to Brownian Motion.* Berlin: Springer Science (1982). doi: 10.1007/978-1-4757-1776-1

24. Gushchin A. *Stochastic Calculus for Quantitative Finance.* Amsterdam: Elsevier (2015).

10.3389/fams.2024.1450581

Appendix

In this appendix, we present, for the reader's convenience, a direct formula for the n-step transition probability of a 2×2 discrete-time Markov chain. Consider a transition probability matrix of the form

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix},$$

for some $\alpha, \beta \in (0, 1)$. Transition probability P admits a unique invariant probability measure

$$\pi = (\pi_0, \pi_1) = \left(\frac{1-\beta}{2-\alpha-\beta}, \frac{1-\alpha}{2-\alpha-\beta} \right).$$

Let us find an eigendecomposition of P. Clearly, 1 is an eigenvalue, and the corresponding eigenvector is $(1, 1)$. The second eigenvalue is $\lambda = \alpha + \beta - 1$, and the corresponding eigenvector is $v = (1-\alpha, \beta-1)$. Thus, we have a decomposition

$$P^n = \begin{pmatrix} 1 & 1-\alpha \\ 1 & \beta-1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (\alpha+\beta-1)^n \end{pmatrix} \begin{pmatrix} \frac{1-\beta}{2-\alpha-\beta} & \frac{1-\alpha}{2-\alpha-\beta} \\ \frac{1}{2-\alpha-\beta} & -\frac{1}{2-\alpha-\beta} \end{pmatrix} \tag{A1}$$

$$= \begin{pmatrix} \pi_0 + \pi_1(\alpha+\beta-1)^n & \pi_1 - \pi_1(\alpha+\beta-1)^n \\ \pi_0 - \pi_0(\alpha+\beta-1)^n & \pi_1 + \pi_0(\alpha+\beta-1)^n \end{pmatrix}.$$