frontiers | Frontiers in Applied Mathematics and Statistics

TYPE Original Research
PUBLISHED 14 June 2024
DOI 10.3389/fams.2024.1420671

On monoids of metric preserving functions

Viktoriia Bilet[1]* and Oleksiy Dovgoshey[1,2]

[1]Department of Theory of Functions, Institute of Applied Mathematics and Mechanics of NASU, Slovyansk, Ukraine, [2]Department of Mathematics and Statistics, University of Turku, Turku, Finland

OPEN ACCESS

EDITED BY
Kateryna Buryachenko,
Humboldt University of Berlin, Germany

REVIEWED BY
Ravindra K. Bisht,
National Defence Academy, India
Filip Turoboś,
Lodz University of Technology, Poland

*CORRESPONDENCE
Viktoriia Bilet
viktoriiabilet@gmail.com

RECEIVED 20 April 2024
ACCEPTED 20 May 2024
PUBLISHED 14 June 2024

CITATION
Bilet V and Dovgoshey O (2024) On monoids of metric preserving functions.
Front. Appl. Math. Stat. 10:1420671.
doi: 10.3389/fams.2024.1420671

Let X be a class of metric spaces and let P_X be the set of all $f\colon [0,\infty) \to [0,\infty)$ preserving X, i.e., $(Y, f\circ\rho) \in X$ whenever $(Y,\rho) \in X$. For arbitrary subset A of the set of all metric preserving functions, we show that the equality $P_X = A$ has a solution if A is a monoid with respect to the operation of function composition. In particular, for the set SI of all amenable subadditive increasing functions, there is a class X of metric spaces such that $P_X = SI$ holds.

2020 Mathematics Subject Classification: Primary 26A30, Secondary 54E35, 20M20

KEYWORDS

metric preserving function, monoid, subadditive function, ultrametric space, ultrametric preserving function

1 Introduction

The following is a particular case of the concept introduced by Jachymski and Turoboś [1].

Definition 1. Let $\mathbf{A}$ be a class of metric spaces. Let us denote by $\mathbf{P_A}$ the set of all functions $f : [0,\infty) \to [0,\infty)$ such that the implication

$$\big((X,d) \in \mathbf{A}\big) \Rightarrow \big((X, f\circ d) \in \mathbf{A}\big)$$

is valid for every metric space (X,d).

For mappings $F : X \to Y$ and $\Phi : Y \to Z$, we use the symbol $F \circ \Phi$ to denote the mapping

$$X \xrightarrow{F} Y \xrightarrow{\Phi} Z.$$

We also use the following notation:
$\mathbf{F}$, set of functions $f : [0,\infty) \to [0,\infty)$;
$\mathbf{F}_0$, set of functions $f \in \mathbf{F}$ with $f(0) = 0$;
$\mathbf{Am}$, set of functions $f \in \mathbf{F}_0$ with $f^{-1}(0) = \{0\}$;
$\mathbf{SI}$, set of subadditive increasing $f \in \mathbf{Am}$;
$\mathbf{M}$, class of metric spaces;
$\mathbf{U}$, class of ultrametric spaces;
$\mathbf{Dis}$, class of discrete metric spaces;
$\mathbf{M}_2$, class of two-points metric spaces;
$\mathbf{M}_1$, class of one-point metric spaces.
The main purpose of this article is to provide a solution to the following problems.

Problem 2. *Let* $\mathbf{A} \subseteq \mathbf{P_M}$. *Find conditions under which the equation*

$$\mathbf{P_X} = \mathbf{A} \tag{1}$$

has a solution $\mathbf{X} \subseteq \mathbf{M}$.

Problem 3. *Let* $\mathbf{A} \subseteq \mathbf{P_U}$. *Find conditions under which Equation (1) has a solution* $\mathbf{X} \subseteq \mathbf{U}$.

In addition, we find all solutions to Equation (1) for $\mathbf{A}$ equal to $\mathbf{F}$, $\mathbf{F}_0$, or $\mathbf{Am}$ and answer the following question.

Question 4. Is there a subclass $\mathbf{X}$ of the class $\mathbf{M}$ such that

$$\mathbf{P_X} = \mathbf{SI}?$$

This question was posed as a challenge in [2] in a different but equivalent form and it was the original motivation for our research.

The article is organized as follows. The next section contains some necessary definitions and facts from the theories of metric spaces and metric preserving functions.

Section 3 provides some definitions from the semigroup theory and describes solutions to Equation (1), for the cases when $\mathbf{A}$ is $\mathbf{F}$, $\mathbf{F}_0$, or $\mathbf{Am}$. In addition, we show that $\mathbf{P_X}$ is always a submonoid of $(\mathbf{F}, \circ)$. See Theorems 21, 23, 24, and Proposition 27, respectively.

Section 4 provides solutions to Problems 2 and 3, which are given, respectively, in Theorems 30 and 33. Theorem 32 gives a positive answer to Question 4.

2 Preliminaries on metrics and metric preserving functions

Let X be non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a *metric* on the set X if for all $x, y, z \in X$ we have

(*i*) $d(x, y) \geqslant 0$ with equality if and only if $x = y$, the *positivity property*;
(*ii*) $d(x, y) = d(y, x)$, the *symmetry property*;
(*iii*) $d(x, y) \leqslant d(x, z) + d(z, y)$, the *triangle inequality*.

A metric space (X, d) is *ultrametric* if the *strong triangle inequality*

$$d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$$

holds for all $x, y, z \in X$.

Example 5. Let us denote $\mathbb{R}_0^+$ by the set $(0, \infty)$. Then the mapping $d^+ : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to [0, \infty)$,

$$d^+(p, q) := \begin{cases} 0 & \text{if} \quad p = q, \\ \max\{p, q\} & \text{otherwise.} \end{cases}$$

is the ultrametric on $\mathbb{R}_0^+$ introduced by Delhommé et al. [3].

Definition 6. Let (X, d) be a metric space. The metric d is *discrete* if there is $k \in (0, \infty)$ such that

$$d(x, y) = k$$

for all distinct $x, y \in X$.

In what follows we will say that a metric space (X, d) is discrete if d is a discrete metric on X. We will denote the class of all discrete metric space by $\mathbf{Dis}$. In addition, for given non-empty set X_1, we will denote by $\mathbf{Dis}_{X_1}$ the subclass of $\mathbf{Dis}$ consisting of all metric spaces (X_1, d) with discrete d.

Remark 7. All discrete topological spaces can be endowed with a metric which is discrete, but not every metric space with discrete topology is discrete in the sense of Definition 6.

Example 8. Let $\mathbf{M}_k$, for $k = 1, 2$, be the class of all metric spaces (X, d) satisfying the equality

$$\operatorname{card}(X) = k.$$

Then all metric spaces belonging to $\mathbf{M}_1 \cup \mathbf{M}_2$ are discrete.

Proposition 9. *The following statements are equivalent for each metric space* $(X, d) \in \mathbf{M}$.

(*i*) (X, d) *is discrete.*
(*ii*) *Every three-points subspace of* (X, d) *is discrete.*

Proof: The implication $(i) \Rightarrow (ii)$ is evidently valid.

Suppose that (ii) holds but $(X, d) \notin \mathbf{Dis}$. Then there are some different points $i, j, k, l \in X$ such that

$$d(i, j) \neq d(k, l). \tag{2}$$

We write $X_1 := \{i, j, k\}$ and $X_2 := \{j, k, l\}$. Then the spaces $(X_1, d|_{X_1 \times X_1})$ and $(X_2, d|_{X_2 \times X_2})$ are discrete subspaces of (X, d) by statement (ii). Consequently we have

$$d(i, j) = d(j, k) \tag{3}$$

and

$$d(j, k) = d(k, l), \tag{4}$$

by definition of the class $\mathbf{Dis}$. Now (3) and (4) give us

$$d(i, j) = d(k, l),$$

which contradicts (2).

Remark 10. The standard definition of discrete metric can be formulated as follows: "The metric on X is discrete if the distance from each point of X to every other point of X is one." (see, for example, Searcóid [4]).

Let $\mathbf{F}$ be the set of all functions $f : [0, \infty) \to [0, \infty)$.

Definition 11. A function $f \in \mathbf{F}$ is *metric preserving* (*ultrametric preserving*) if $f \in \mathbf{P_M}$ ($f \in \mathbf{P_U}$).

Remark 12. The concept of metric preserving functions can be traced back to Wilson [5]. Similar problems were considered by Blumenthal [6]. The theory of metric preserving functions was developed by Borsík, Doboš, Piotrowski, Vallin, and other mathematicians [7–19]. See also lectures by Doboš [20] and the introductory paper by Corazza [21]. The study of ultrametric preserving functions began by Pongsriiam and Termwuttipong [22] and was continued in [23, 24].

We will say that $f \in \mathbf{F}$ is *amenable* if

$$f^{-1}(0) = \{0\}$$

holds and the set of all amenable functions from $\mathbf{F}$ will be denoted by $\mathbf{Am}$. Let us denote the set of all functions $f \in \mathbf{F}$ satisfying the equality $f(0) = 0$ by $\mathbf{F}_0$. It follows directly from the definition that $\mathbf{Am} \subsetneq \mathbf{F}_0 \subsetneq \mathbf{F}$.

Moreover, a function $f \in \mathbf{F}$ is *increasing* if the implication

$$(x \leqslant y) \Rightarrow (f(x) \leqslant f(y))$$

is valid for all $x, y \in [0, \infty)$.

The following theorem was proved in [22].

Theorem 13. *A function $f \in \mathbf{F}$ is ultrametric preserving if and only if f is increasing and amenable.*

Remark 14. Theorem 13 was generalized in [25] to the special case of the so-called ultrametric distances. These distances were introduced by Priess-Crampe and Ribenboim [26] and were studies by different researchers [27–30].

Recall that a function $f \in \mathbf{F}$ is said to be *subadditive* if

$$f(x+y) \leqslant f(x) + f(y)$$

holds for all $x, y \in [0, \infty)$. Let us denote the set of all subadditive increasing functions $f \in \mathbf{Am}$ by $\mathbf{SI}$.

In the next proposition, we restate the equivalence between statements (i) and (ii) of Corollary 36 [2].

Proposition 15. *The equality*

$$\mathbf{SI} = \mathbf{P_U} \cap \mathbf{P_M}$$

holds.

Remark 16. The metric preserving functions can be considered as a special case of metric products (= metric preserving functions of several variables). See, for example, [31–37]. An important special class of ultrametric preserving functions of two variables was first considered in 2009 [38].

3 Preliminaries on semigroups. Solutions to $\mathbf{F_X} = \mathbf{A}$ for $\mathbf{A} = \mathbf{F}$, $\mathbf{F}_0$, and $\mathbf{Am}$

Let us recall some basic concepts of semigroup theory, see, for example, “Fundamentals of Semigroup Theory” by Howie [39].

A *semigroup* is a pair $(S, *)$ consisting of a non-empty set S and an associative operation $* : S \times S \to S$, which is called the *multiplication* on S. A semigroup $S = (S, *)$ is a *monoid* if there is $e \in S$ such that

$$e * s = s * e = s$$

for every $s \in S$.

Definition 17. Let $(S, *)$ be a semigroup and $\varnothing \neq T \subseteq S$. Then T is a *subsemigroup* of S if $a, b \in T \Rightarrow a * b \in T$. If $(S, *)$ is a monoid with the identity e, then T is a *submonoid* of S if T is a subsemigroup of S and $e \in T$.

Example 18. The semigroups $(\mathbf{F}, \circ)$, $(\mathbf{Am}, \circ)$, $(\mathbf{P_M}, \circ)$, and $(\mathbf{P_U}, \circ)$ are monoids, and the identical mapping $\mathrm{id} : [0, \infty) \to [0, \infty)$, $\mathrm{id}(x) = x$ for every $x \in [0, \infty)$ is the identity of these monoids.

The following simple lemmas are well-known.

Lemma 19. *Let T be a submonoid of a monoid $(S, *)$ and let $V \subseteq T$. Then V is a submonoid of $(S, *)$ if and only if V is a submonoid of T.*

Lemma 20. *Let T_1 and T_2 be submonoids of a monoid $(S, *)$. Then the intersection $T_1 \cap T_2$ also is a submonoid of $(S, *)$.*

The next theorem describes all solutions to the equation $\mathbf{P_X} = \mathbf{F}$.

Theorem 21. *The following statements are equivalent for every $\mathbf{X} \subseteq \mathbf{M}$.*

- (i) *$\mathbf{X}$ is the empty subclass of $\mathbf{M}$.*
- (ii) *The equality*

$$\mathbf{P_X} = \mathbf{F} \tag{5}$$

holds.

Proof: $(i) \Rightarrow (ii)$. Let $\mathbf{X}$ be the empty subclass of $\mathbf{M}$. Definition 1 implies the inclusion $\mathbf{F} \supseteq \mathbf{P_X}$. Let us consider an arbitrary $f \in \mathbf{F}$. To prove equality (5), it is suffice to show that $f \in \mathbf{P_X}$. Since $\mathbf{X}$ is empty, the membership relation $(X, d) \in \mathbf{X}$ is false for every metric space (X, d). Consequently, the implication

$$((X, d) \in \mathbf{X}) \Rightarrow ((X, f \circ d) \in \mathbf{X})$$

is valid for every $(X, d) \in \mathbf{M}$. It implies $f \in \mathbf{P_X}$ by Definition 1. Equality (5) follows.

$(ii) \Rightarrow (i)$. Let (ii) hold. We must show that $\mathbf{X}$ is empty. Suppose contrary that there is a metric space $(X, d) \in \mathbf{X}$. Since, by definition, we have $X \neq \varnothing$, there is a point $x_0 \in X$. Consequently, $d(x_0, x_0) = 0$ holds. Let $c \in (0, \infty)$ and let $f : [0, \infty) \to [0, \infty)$ be a constant function,

$$f(t) = c$$

for every $t \in [0, \infty)$. In particular, we have

$$f(0) = c > 0. \tag{6}$$

Equality (5) implies that $f \circ d$ is a metric on X. Thus, we have

$$0 = f(d(x_0, x_0)) = f(0),$$

which contradicts (6). Statement (i) follows.

Remark 22. Theorem 21 becomes invalid if we allow the empty metric space to be considered. The equality

$$\mathbf{P_X} = \mathbf{F}$$

holds if the non-empty class $\mathbf{X}$ contains only the empty metric space.

Let us describe now all possible solutions to $\mathbf{P_X} = \mathbf{F}_0$.

Theorem 23. *The equality*

$$\mathbf{P_X} = \mathbf{F}_0 \tag{7}$$

holds if and only if $\mathbf{X}$ *is a non-empty subclass of* $\mathbf{M}_1$.

Proof: Let $\mathbf{X} \subseteq \mathbf{M}_1$ be non-empty. Equality (7) holds if

$$\mathbf{P_X} \supseteq \mathbf{F}_0 \tag{8}$$

and

$$\mathbf{P_X} \subseteq \mathbf{F}_0. \tag{9}$$

Here, we prove the validity of (8). Let $f \in \mathbf{F}_0$ be arbitrary. Since every $(X, d) \in \mathbf{X}$ is a one-point metric space, we have $f \circ d = d$ for all $(X, d) \in \mathbf{X}$ by the positivity property of metric spaces, Inclusion (8) follows.

Here, we prove (9). The inclusion $\mathbf{P_X} \subseteq \mathbf{F}$ follows from Definition 1. Thus, if (9) does not hold, then there is $f_0 \in \mathbf{F}$ such that $f_0 \in \mathbf{P_X}$,

$$f_0(0) = k \quad \text{and} \quad k > 0. \tag{10}$$

Since $\mathbf{X}$ is non-empty, there is $(X_0, d_0) \in \mathbf{X}$. Let x_0 be a (unique) point of X_0. Since f_0 belongs to $\mathbf{P_X}$, the function $f_0 \circ d_0$ is a metric on X_0. Now, using (10), we obtain

$$f_0(d_0(x_0, x_0)) = f_0(0) = k > 0,$$

which contradicts the positivity property of metric spaces. Inclusion (9) follows.

Let (7) hold. We must show that $\mathbf{X}$ is a non-empty subclass of $\mathbf{M}_1$. If $\mathbf{X}$ is empty, then

$$\mathbf{P_X} = \mathbf{F} \tag{11}$$

holds by Theorem 21. Equality (11) contradicts equality (7). Hence, $\mathbf{X}$ is non-empty. To complete the proof, we must show that

$$\mathbf{X} \subseteq \mathbf{M}_1. \tag{12}$$

Let us consider the constant function $f_0 : [0, \infty) \to [0, \infty)$ such that

$$f_0(t) = 0, \tag{13}$$

for every $t \in [0, \infty)$. Then f_0 belongs to $\mathbf{F}_0$. Hence, for every $(X, d) \in \mathbf{X}$, the mapping $d_0 : = f_0 \circ d$ is a metric on X. Now (13) implies $d_0(x, y) = 0$ for all $x, y \in X$ and $(X, d) \in \mathbf{X}$. Hence, $\text{card}(X) = 1$ holds, because the metric space (X, d_0) is one-point by the positivity property. Inclusion (12) follows. The proof is completed.

The next theorem gives us all solutions to the equation $\mathbf{P_X} = \mathbf{Am}$.

Theorem 24. *The following statements are equivalent for every* $\mathbf{X} \subseteq \mathbf{M}$.

(*i*) *The inclusion*

$$\mathbf{X} \subseteq \mathbf{Dis} \tag{14}$$

holds, and there is $(Y, \rho) \in \mathbf{X}$ *with*

$$\text{card}(Y) \geqslant 2, \tag{15}$$

and we have

$$\mathbf{Dis}_{X_1} \subseteq \mathbf{X} \tag{16}$$

for every $(X_1, d_1) \in \mathbf{X}$.

(*ii*) *The equality*

$$\mathbf{P_X} = \mathbf{Am} \tag{17}$$

holds.

Proof: (*i*) ⇒ (*ii*). Let (*i*) hold. Equality (17) holds if

$$\mathbf{P_X} \supseteq \mathbf{Dis} \tag{18}$$

and

$$\mathbf{P_X} \subseteq \mathbf{Dis}. \tag{19}$$

Here, we prove (18). Inclusion (18) holds if we have

$$(X_1, f \circ d_1) \in \mathbf{X} \tag{20}$$

for all $f \in \mathbf{Am}$ and $(X_1, d_1) \in \mathbf{X}$. Relation (20) follows from Theorem 23 if $(X_1, d_1) \in \mathbf{M}_1$. To see it we only note that $\mathbf{Am} \subseteq \mathbf{F}_0$. Let us consider the case when

$$\text{card}(X_1) \geqslant 2.$$

Since (X_1, d_1) is discrete by (14), Definition 6 implies that there is $k_1 \in (0, \infty)$ satisfying

$$d_1(x, y) = k_1$$

for all distinct $x, y \in X_1$. Let $f \in \mathbf{Am}$ be arbitrary. Then $f(k_1)$ is strictly positive and

$$f(d_1(x, y)) = f(k_1)$$

holds for all distinct $x, y \in X_1$. Thus, $f \circ d_1$ is a discrete metric on X_1, i.e., we have

$$(X_1, f \circ d_1) \in \mathbf{Dis}_{X_1}. \tag{21}$$

Now, Equation (20) follows from Equations (16, 21).

Here, we prove (19). To prove, we must show that every $f \in \mathbf{P_X}$ is amenable.

Suppose contrary that f belongs to $\mathbf{P_X}$ but the equality

$$f(t_1) = 0 \tag{22}$$

holds with some $t_1 \in (0, \infty)$. By statement (*i*) we can find $(Y, \rho) \in \mathbf{X}$ such that (15) and

$$\rho(x, y) = t_1$$

hold for all distinct $x, y \in Y$. Now $f \in \mathbf{P_X}$ and $(Y, \rho) \in \mathbf{X}$ imply that $f \circ \rho$ is a metric on Y. Consequently, for all distinct $x, y \in Y$, we have

$$f(\rho(x,y)) = f(t_1) > 0,$$

which contradicts (22). The validity of (19) follows.

$(ii) \Rightarrow (i)$. Let $\mathbf{X}$ satisfy equality (17). Since $\mathbf{Am} \neq \mathbf{F}$ holds, the class $\mathbf{X}$ is non-empty by Theorem 21. Moreover, using Theorem 23, we see that $\mathbf{X}$ contains a metric space (X, d) with $\text{card}(X) \geqslant 2$, because $\mathbf{Am} \neq \mathbf{F}_0$.

If the inequality

$$\text{card}(Y) \leqslant 2$$

holds for every $(Y, \rho) \in \mathbf{X}$, then all metric spaces belonging to $\mathbf{X}$ are discrete (see Example 8). Using the definitions of **Dis** and **Am**, it is easy to prove that for each $(X_1, d_1) \in \mathbf{Dis}$ and every $(X_1, d) \in \mathbf{Dis}_{X_1}$ there exists $f \in \mathbf{Am}$ such that $d = f \circ d_1$. Hence to complete the proof, it is suffice to show that every $(X, d) \in \mathbf{X}$ is discrete when

$$\text{card}(X) \geqslant 3. \tag{23}$$

Let us consider arbitrary $(X, d) \in \mathbf{X}$ satisfying (23). Suppose that $(X, d) \notin \mathbf{Dis}$. Then by Proposition 9 there are distinct $a, b, c, \in X$ such that

$$d(b,c) \notin \{d(a,b), d(c,a)\}. \tag{24}$$

Let c_1 and c_2 be points of $(0, \infty)$ such that

$$c_2 > 2c_1. \tag{25}$$

Now we can define $f \in \mathbf{Am}$ as

$$f(t) := \begin{cases} 0 & \text{if} \quad t = 0, \\ c_2 & \text{if} \quad t = d(b,c), \\ c_1 & \text{otherwise}. \end{cases} \tag{26}$$

Equality (17) implies that $f \circ d$ is a metric on X. Consequently, we have

$$f(d(b,c)) \leqslant f(d(b,a)) + f(d(b,c)) \tag{27}$$

by triangle inequality. Now using Equations (24, 26) we can rewrite Equation (27) as

$$c_2 \leqslant c_1 + c_1,$$

which contradicts Equation (25). It implies $(X, d) \in \mathbf{Dis}$. The proof is completed.

Corollary 25. *The equalities*

$$\mathbf{P_{Dis}} = \mathbf{P_{M_2}} = \mathbf{Am}$$

hold.

Remark 26. The equality

$$\mathbf{P_{M_2}} = \mathbf{Am}$$

is known, see, for example, Remark 1.2 in the article [13]. This article also contains "constructive" characterizations of the smallest bilateral ideal and the largest subgroup of the monoid $\mathbf{P_M}$.

Proposition 27. *Let $\mathbf{X}$ be a subclass of $\mathbf{M}$. Then $\mathbf{P_X}$ is a submonoid of $(\mathbf{F}, \circ)$.*

Proof: It follows directly from Definition 1 that

$$\mathbf{P_X} \subseteq \mathbf{F}$$

holds and that the identity mapping $\text{id}: [0, \infty) \to [0, \infty)$ belongs to $\mathbf{P_X}$. Hence, by Lemma 19, it is suffice to prove

$$f \circ g \in \mathbf{P_X} \tag{28}$$

for all $f, g \in \mathbf{P_X}$.

Let us consider arbitrary $f \in \mathbf{P_X}$ and $g \in \mathbf{P_X}$. Then, using Definition 1, we see that $(X, g \circ d)$ belongs to $\mathbf{X}$ for every $(X, d) \in \mathbf{X}$. Consequently,

$$(X, f \circ (g \circ d)) \in \mathbf{X} \tag{29}$$

holds. Since the composition of functions is always associative, the equality

$$(f \circ g) \circ d = f \circ (g \circ d) \tag{30}$$

holds for every $(X, d) \in \mathbf{X}$. Now Equation (28) follows from Equations (29, 30).

The above proposition implies the following corollary.

Corollary 28. *If the equation*

$$\mathbf{P_X} = \mathbf{A}$$

has a solution, then $\mathbf{A}$ is a submonoid of $\mathbf{F}$.

The following example shows that the converse of Corollary 28 is, generally speaking, false.

Example 29. Let us define $\mathbf{A}_1 \subseteq \mathbf{F}$ as

$$\mathbf{A}_1 = \{f_1, \text{id}\},$$

where $f_1 \in \mathbf{F}$ is defined such that

$$f_1(t) := \begin{cases} 1 & \text{if} \quad t = 0, \\ 0 & \text{if} \quad t = 1, \\ t & \text{otherwise} \end{cases} \tag{31}$$

and id is the identical mapping of $[0, \infty)$. The equalities $f_1 \circ f_1 = \text{id}$, $f_1 \circ \text{id} = f_1 = \text{id} \circ f_1$ show that $\mathbf{A}_1$ is a submonoid of $(\mathbf{F}, \circ)$. Suppose that there is $\mathbf{X}_1 \subseteq \mathbf{M}$ satisfying the equality

$$\mathbf{P}_{\mathbf{X}_1} = \mathbf{A}_1. \tag{32}$$

Then using Theorem 21, we see that $\mathbf{X}_1$ is non-empty because $\mathbf{A}_1 \neq \mathbf{F}$ holds. Let (X_1, d_1) be an arbitrary metric space from $\mathbf{A}_1$. Since X_1 is non-empty, we can find $x_1 \in X_1$. Then (32) implies that $f_1 \circ d_1$ is metric on X_1. Now using (31), we obtain

$$f_1(d_1(x_1, x_1)) = f_1(0) = 1,$$

which contradicts the positivity property of metrics.

4 Submonoids of monoids $\mathbf{P_M}$ and $\mathbf{P_U}$

The following theorem provides a solution to Problem 2.

Theorem 30. *Let* $\mathbf{A}$ *be a non-empty subset of the set* $\mathbf{P_M}$ *of all metric preserving functions. Then the following statements are equivalent.*

(*i*) *The equation*

$$\mathbf{P_X} = \mathbf{A} \tag{33}$$

has a solution $\mathbf{X} \subseteq \mathbf{M}$.

(*ii*) $\mathbf{A}$ *is a submonoid of* $(\mathbf{F}, \circ)$.

(*iii*) $\mathbf{A}$ *is a submonoid of* $(\mathbf{P_M}, \circ)$.

Proof: (*i*) $\Rightarrow$ (*ii*). Suppose that there is $\mathbf{X} \subseteq \mathbf{M}$ such that (33) holds. Then $\mathbf{A}$ is a submonoid of $(\mathbf{F}, \circ)$ by Proposition 27.

(*ii*) $\Rightarrow$ (*iii*). Let $\mathbf{A}$ be a submonoid of $(\mathbf{F}, \circ)$. By Proposition 27, the monoid $(\mathbf{P_M}, \circ)$ also is a submonoid of $(\mathbf{F}, \circ)$. Then using the inclusion $\mathbf{A} \subseteq \mathbf{P_M}$, we obtain that $\mathbf{A}$ is a submonoid of $(\mathbf{P_M}, \circ)$ by Lemma 19.

(*iii*) $\Rightarrow$ (*i*). Let $\mathbf{A}$ be a submonoid of $(\mathbf{P_M}, \circ)$. We must prove that (33) has a solution $\mathbf{X} \subseteq \mathbf{M}$.

Let (X, d) be a metric space such that

$$\{d(x,y) : x, y \in X\} = [0, \infty). \tag{34}$$

$$\mathbf{X} := \{(X, f \circ d) : f \in \mathbf{A}\}. \tag{35}$$

Thus, a metric space (Y, ρ) belongs to $\mathbf{X}$ if and only if $Y = X$ and there is $f \in \mathbf{A}$ such that $\rho = f \circ d$.

We claim that Equation (33) holds if $\mathbf{X}$ is defined by Equality (35). To prove it, we note that Equation (33) holds if

$$\mathbf{A} \subseteq \mathbf{P_X} \tag{36}$$

and

$$\mathbf{A} \supseteq \mathbf{P_X}. \tag{37}$$

Here, we prove Inclusion (36). This inclusion holds if for every $f \in \mathbf{A}$ and each $(Y, \rho) \in \mathbf{X}$ we have $(Y, f \circ \rho) \in \mathbf{X}$. Let us consider arbitrary $(Y, \rho) \in \mathbf{X}$ and $f \in \mathbf{A}$. Then, using Equation (35), we can find $g \in \mathbf{A}$ such that

$$X = Y \quad \text{and} \quad \rho = g \circ d. \tag{38}$$

Since $\mathbf{A}$ is a monoid, the membership relations $f \in \mathbf{A}$ and $g \in \mathbf{A}$ imply $g \circ f \in \mathbf{A}$. Hence, we have

$$(X, g \circ f \circ d) \in \mathbf{X} \tag{39}$$

by Equality (35). Now $(Y, f \circ \rho) \in \mathbf{X}$ follows from Equations (38, 39).

Here, we prove Inclusion (37). Let g_1 belongs to $\mathbf{P_X}$ and let (X, d) be the same as in (35). Then $(X, g_1 \circ d)$ belongs to $\mathbf{X}$ and, using (35), we can find $f_1 \in \mathbf{A}$ such that

$$(X, g_1 \circ d) = (X, f_1 \circ d). \tag{40}$$

Equality (40) implies

$$g_1(d(x,y)) = f_1(d(x,y)), \tag{41}$$

for all $x, y \in X$. Consequently, $g_1(t) = f_1(t)$ holds for every $t \in [0, \infty)$ by Equation (34, 41). Thus, we have $g_1 = f_1$. That implies $g_1 \in \mathbf{A}$. Inclusion (37) follows. The proof is completed.

Remark 31. A reviewer of the article noted that condition (34) can be neatly expressed in terms of center distances which stems from article [40].

Let us turn now to Question 4. Proposition 15 and Lemma 20 provide the following result.

Theorem 32. *There is* $\mathbf{X} \subseteq \mathbf{M}$ *such that*

$$\mathbf{P_X} = \mathbf{SI}. \tag{42}$$

Proof: By Proposition 27, the monoids $(\mathbf{P_M}, \circ)$ and $(\mathbf{P_U}, \circ)$ are submonoids of $(\mathbf{F}, \circ)$. The equality

$$\mathbf{SI} = \mathbf{P_M} \cap \mathbf{P_U} \tag{43}$$

holds by Proposition 15. Using Equality (43) and Lemma 20 with $T_1 = \mathbf{P_M}$, $T_2 = \mathbf{P_U}$, and $\mathbf{S} = \mathbf{F}$, we see that $\mathbf{SI}$ also is a submonoid of $\mathbf{F}$. Consequently, Theorem 30 with $\mathbf{A} = \mathbf{SI}$ implies that there is $\mathbf{X} \subseteq \mathbf{M}$ such that (42) holds.

The next theorem is an ultrametric analog of Theorem 30 and it gives us a solution to Problem 3.

Theorem 33. *Let* $\mathbf{A}$ *be a non-empty subset of the set* $\mathbf{P_U}$ *of all ultrametric preserving functions. Then the following statements are equivalent.*

(*i*) *The equation* $\mathbf{P_X} = \mathbf{A}$ *has a solution* $\mathbf{X} \subseteq \mathbf{U}$.

(*ii*) $\mathbf{A}$ *is a submonoid of* $(\mathbf{F}, \circ)$.

(*iii*) $\mathbf{A}$ *is a submonoid of* $(\mathbf{P_U}, \circ)$.

A proof of Theorem 33 can be obtained by a simple modification of the proof of Theorem 30. We only note that the ultrametric space defined in Example 5 satisfies equality (34) with $X = \mathbb{R}_0^+$ and $d = d^+$.

5 Two conjectures

Conjecture 34. *The equation*

$$\mathbf{P_X} = \mathbf{A}$$

has a solution $\mathbf{X} \subseteq \mathbf{M}$ *for every submonoid* $\mathbf{A}$ *of the monoid* $\mathbf{Am}$.

Example 29 shows that we cannot replace $\mathbf{Am}$ with $\mathbf{F}$ in Conjecture 34, but we hope that the following is valid.

Conjecture 35. *For every submonoid* $\mathbf{A}$ *of the monoid* $\mathbf{F}$*, there exists* $\mathbf{X} \subseteq \mathbf{M}$ *such that* $\mathbf{P_X}$ *and* $\mathbf{A}$ *are isomorphic submonoids.*

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

VB: Writing – original draft, Writing – review & editing. OD: Methodology, Project administration, Supervision, Validation, Writing – original draft, Writing – review & editing.

Funding

The author(s) declare financial support was received for the research, authorship, and/or publication of this article. VB was partially supported by a grant from the Simons Foundation (Award 1160640, Presidential Discretionary-Ukraine Support Grants, VB). OD was supported by grant 359772 of the Academy of Finland.

Acknowledgments

We express our deep gratitude toward the reviewers for the careful study of our paper. Their elaborate, constructive criticism allowed us to improve this paper and suggested further directions of the research.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

References

1. Jachymski J, Turoboś F. On functions preserving regular semimetrics and quasimetrics satisfying the relaxed polygonal inequality. *Rev R Acad Cienc Exactas Fís Nat, Ser A Mat*. (2020) 114:159. doi: 10.1007/s13398-020-00891-7

2. Dovgoshey O. Strongly ultrametric preserving functions. *Topol Appl*. (2024) 351:108931. doi: 10.1016/j.topol.2024.108931

3. Delhommé C, Laflamme C, Pouzet M, Sauer N. Indivisible ultrametric spaces. *Topol Appl*. (2008) 155:1462–78. doi: 10.1016/j.topol.2008.03.012

4. Searcóid MO. *Metric Spaces*. Berlin: Springer (2006).

5. Wilson WA. On certain types of continuous transformations of metric spaces. *Am J Mathem*. (1935) 57:62–8. doi: 10.2307/2372019

6. Blumenthal LM. Remarks concerning the Euclidean four-point property. *Ergeb Math Kolloq Wien*. (1936) 7:7–10.

7. Borsík J, Doboš J. Functions the composition with a metric of which is a metric. *Math Slovaca*. (1981) 31:3–12.

8. Borsík J, Doboš J. On metric preserving functions. *Real Anal Exch*. (1988) 13:285–93. doi: 10.2307/44151879

9. Doboš J. On modification of the Euclidean metric on reals. *Tatra Mt Math Publ*. (1996) 8:51–4.

10. Doboš J, Piotrowski Z. Some remarks on metric-preserving functions. *Real Anal Exch*. (1994). 19:317–20. doi: 10.2307/44153846

11. Doboš J, Piotrowski Z. A note on metric preserving functions. *Int J Math Math Sci*. (1996) 19:199–200. doi: 10.1155/S0161171296000282

12. Doboš J, Piotrowski Z. When distance means money. *Internat J Math Ed Sci Tech*. (1997) 28:513–8. doi: 10.1080/0020739970280405

13. Dovgoshey O, Martio O. Functions transferring metrics to metrics. *Beitráge zur Algebra und Geometrie*. (2013) 54:237–61. doi: 10.1007/s13366-011-0061-7

14. Vallin RW. Metric preserving functions and differentiation. *Real Anal Exch*. (1997) 22:86. doi: 10.2307/44152729

15. Vallin RW. On metric preserving functions and infinite derivatives. *Acta Math Univ Comen, New Ser*. (1998) 67:373–6.

16. Vallin RW. A subset of metric preserving functions. *Int J Math Math Sci*. (1998) 21:409–10. doi: 10.1155/S0161171298000568

17. Vallin RW. Continuity and differentiability aspects of metric preserving functions. *Real Anal Exch*. (1999/2000) 25:849–68. doi: 10.2307/44154040

18. Pongsriiam P, Termwuttipong I. On metric-preserving functions and fixed point theorems. *Fixed Point Theory Appl*. (2014) 2014:14. doi: 10.1186/1687-1812-2014-179

19. Vallin RW. On preserving $(\mathbb{R}, \text{Eucl.})$ and almost periodic functions. *Tatra Mt Math Publ*. (2002) 24:1–6.

20. Doboš J. *Metric Preserving Functions*. Košice, Slovakia: Štroffek (1998).

21. Corazza P. Introduction to metric-preserving functions. *Amer Math Monthly*. (1999) 104:309–23. doi: 10.1080/00029890.1999.12005048

22. Pongsriiam P, Termwuttipong I. Remarks on ultrametrics and metric-preserving functions. *Abstr Appl Anal*. (2014) 2014:1–9. doi: 10.1155/2014/163258

23. Dovgoshey O. On ultrametric-preserving functions. *Math Slovaca*. (2020) 70:173–82. doi: 10.1515/ms-2017-0342

24. Vallin RW, Dovgoshey OA. P-adic metric preserving functions and their analogues. *Math Slovaca*. (2021) 71:391–408. doi: 10.1515/ms-2017-0476

25. Dovgoshey O. Combinatorial properties of ultrametrics and generalized ultrametrics. *Bull Belg Math Soc Simon Stevin*. (2020) 27:379–417. doi: 10.36045/bbms/1599616821

26. Priess-Crampe S, Ribenboim P. Fixed points, combs and generalized power series. *Abh Math Sem Univ Hamburg*. (1993) 63:227–44. doi: 10.1007/BF02941344

27. Priess-Crampe S, Ribenboim P. Generalized ultrametric spaces I. *Abh Math Sem Univ Hamburg*. (1996) 66:55–73. doi: 10.1007/BF02940794

28. Priess-Crampe S, Ribenboim P. Generalized ultrametric spaces II. *Abh Math Sem Univ Hamburg*. (1997) 67:19–31. doi: 10.1007/BF02940817

29. Ribenboim P. The new theory of ultrametric spaces. *Periodica Math Hung*. (1996) 32:103–11. doi: 10.1007/BF01879736

30. Ribenboim P. The immersion of ultrametric spaces into Hahn Spaces. *J Algebra*. (2009) 323:1482–93. doi: 10.1016/j.jalgebra.2009.11.032

31. Borsík J, Doboš J. On a product of metric spaces. *Math Slovaca*. (1981) 31:193–205.

32. Bernig A, Foertsch T, Schroeder V. Non standard metric products. *Beitráge zur Algebra und Geometrie*. (2003) 44:499–510.

33. Dovgoshey O, Petrov E, Kozub G. Metric products and continuation of isotone functions. *Math Slovaca*. (2014) 64:187–208. doi: 10.2478/s12175-013-0195-1

34. Foertsch T, Schroeder V. Minkowski-versus Euclidean rank for products of metric spaces. *Adv Geom*. (2002) 2:123–31. doi: 10.1515/advg.2002.002

35. Herburt I. Moszyńska M. On metric products. *Colloq Math*. (1991) 62:121–33. doi: 10.4064/cm-62-1-121-133

36. Kazukawa D. Construction of product spaces. *Anal Geom Metr Spaces*. (2021) 9:186–218. doi: 10.1515/agms-2020-0129

37. Lichman M, Nowakowski P, Turobo F. On Functions Preserving Products of Certain Classes of Semimetric Spaces. *Tatra Mount Mathem Publicat*. (2021) 78:175–98. doi: 10.2478/tmmp-2021-0013

38. Dovgoshey O, Martio O. Products of metric spaces, covering numbers, packing numbers and characterizations of ultrametric spaces. *Rev Roumaine Math Pures Appl*. (2009) 54:423–39.

39. Howie JM. *Fundamentals of Semigroup Theory*. Oxford: Clarendan Press. (1995).

40. Bielas W, Plewik S, Walczyska M. On the center of distances. *Eur J Mathem*. (2018) 4:687–98. doi: 10.1007/s40879-017-0199-4