frontiers | Frontiers in Applied Mathematics and Statistics

TYPE Original Research
PUBLISHED 07 August 2024
DOI 10.3389/fams.2024.1418589

OPEN ACCESS

EDITED BY
Artur Lemonte,
Federal University of Rio Grande do Norte, Brazil

REVIEWED BY
Ahmed Z. Afify,
Benha University, Egypt
Yuri A. Iriarte,
Universidad de Antofagasta, Chile

*CORRESPONDENCE
Cristian Carvajal-Muquillaza
✉ cristian.carvajal@upla.cl

RECEIVED 16 April 2024
ACCEPTED 16 July 2024
PUBLISHED 07 August 2024

CITATION
Carvajal-Muquillaza C, Manríquez R and Cabrera E (2024) θ-Weighted mixture distribution: the Weibull-Lomax case. *Front. Appl. Math. Stat.* 10:1418589. doi: 10.3389/fams.2024.1418589

θ-Weighted mixture distribution: the Weibull-Lomax case

Cristian Carvajal-Muquillaza*, Ronald Manríquez and Eduardo Cabrera

Laboratorio de Investigación Lab[e]saM, Departamento de Matemática, Física y Computación, Universidad de Playa Ancha, Valparaíso, Chile

Introduction: This article introduces a new family of weighted mixture distributions, referred to as θ-WM. The θ-WM family is generated by combining two distributions weighted by a parameter θ, offering notable flexibility to model a wide range of complex phenomena. A special case study of the θ-weighted mixture distribution of Weibull-Lomax (θ-WMWLx) is included, resulting from the combination of Weibull and Lomax distributions.

Methods: The research thoroughly examines the reliability and statistical properties of the θ-WMWLx distribution. Key aspects such as stochastic dominance, survival and hazard functions, mean residual life, and moments are addressed. The maximum likelihood method is used to estimate unknown parameters.

Results: The research findings show that the θ-WMWLx distribution provides a superior fit compared to competing distributions. The analyses are validated using three real datasets, demonstrating the effectiveness of the proposed distribution.

Discussion: The θ-WMWLx distribution stands out for its ability to model complex phenomena with high precision. Validation with real data confirms that the proposed distribution offers a better fit than existing distributions, highlighting its utility and applicability in various statistical analysis contexts.

KEYWORDS

mixture distribution, Weibull distribution, Lomax distribution, survival function, distribution family

1 Introduction

In scientific research, distribution models constitute conceptual frameworks to comprehend complex phenomena. These models span from the intricacies of climate change and pandemic spread to understanding the dynamics of economic indicators or the lifespan of cellular organisms. Granting these models, significant flexibility is essential to align them with the inherent complexities of the studied phenomena. The establishment of new parameters into distribution models gives them greater adaptability to diverse circumstances.

The study of generalized probability distributions begins with the famous study on Pearson's systems of continuous distributions, in which each probability density function satisfies a differential equation with four parameters on which the shape of the function depends (see Pearson [1]). In Amoroso [2], the generalization of the beta distribution to better fit certain income rates is discussed. Following Pearson's ideas, Burr [3] presents a system of continuous distributions that also satisfy a differential equation. Later, Johnson [4] proposed a system to generate distributions using the normalization transformation with four parameters: two of shape, one of scale, and one of location.

A pivotal contribution in this field emerged from the study of Marshall and Olkin [5], and their method defines a new survival function as follows:

$$\bar{G}(x|\alpha) = \frac{\alpha \bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}; \qquad (-\infty < x < \infty; 0 < \alpha < \infty), \quad (1)$$

where $\bar{\alpha} = 1 - \alpha$, and $\bar{F}$ represents the survival function of a random variable X.

From Equation (1), extensive research has led to new distribution families such as the Marshall-Olkin generalized exponential linear distribution proposed by Okasha and Kayid [6], the extended uniform distribution of Marshall-Olkin by Jose and Krishna [7], the Marshall-Olkin Topp-Leone half-logistic-G distribution by Sengweni et al. [8], the generalized Marshall-Olkin transmuted-G family in Handiquea et al. [9] that to extend the transmuted family proposed in Shaw and Buckley [10], and the Marshall-Olkin-Weibull-H family applied to COVID-19 data (see Afify et al. [11]). Based on incorporating parameters into a reference distribution, this approach has proven effective in fostering flexibility into new distribution families [12].

Another strategy for creating new distribution families involves functional transformations. For instance, Souza et al. [13] introduced the Sin-G family generated by the sine function. In addition, Shama et al. [14] proposed the Modified Generalized-G family, a flexible distribution built on power-exponential transformations. Their theoretical approach is as follows: let $\bar{G}$ be a survival function of an absolutely continuous distribution with support $(0, 1)$, and H be a decreasing continuous function such that $H(x) \in [0, 1]$ and $\lim_{x\to 0} H(x) = 1$. Then, the following function is a valid continuous distribution function:

$$F(x) = 1 - \bar{G}(x)H(x). \quad (2)$$

Shama et al. chose G as the survival function of the T2ExG family defined with a certain generic baseline distribution and H as a decreasing exponential function compound with a possible other baseline distribution.

There is extensive literature associated with the generalization and/or obtaining of families of distributions that contemplate different techniques and strategies (see, for instance, Nadarajah and Kotz [15], Eugene et al. [16], Cordeiro and de Castro [17], Mahdavi and Kundu [18], and Iriarte et al. [19]).

This study aims to construct a new distribution family called θ-Weighted Mixture distribution (θ-WM) based on the approach of Equation (2) by incorporating a parameter $\theta \in [0, 1]$ and considering H as a survival function and, thus, to expand de proposal of Shama et al. [14]. This general family is established as a flexible and adaptable framework for modeling diverse phenomena; it allows a gradual and flexible transition between G and H distributions through the weighting parameter θ, addressing a wider range of problems that require more accurate modeling. Specifically, we deal with a particular case within the θ-WM family, denoted by θ-WMWLx, composed of the Weibull and Lomax distributions. We study the fundamental reliability components associated with the θ-WMWLx distribution, along with its basic statistical properties such as survival and hazard functions, mean residual life, mean inactivity time, useful expansions, quantile function, Renyi entropy, moments, order statistics, and estimation methods. Furthermore, we present two real datasets to compare the behavior of the proposed distribution with four other models.

The θ-WM family has desirable properties, justified as follows: (i) The specific submodels of the θ-WM family, such as θ-WMEW and θ-WMWLx, can represent crucial hazard rate (hr) shapes, including increasing, decreasing, J-shape, and reversed J-shape; (ii) in addition, the densities of its submodels encompass reversed J-shaped, right-skewed, symmetric, left-skewed, and decreasing–increasing–decreasing patterns; (iii) the specific θ-WMWLx model provides a better fit compared to other generalized models using the same baseline distribution, as demonstrated in the case of the Lomax-exponential distribution (LED) and the exponentiated Kumaraswamy Inverse Weibull (EKIW) distribution.

The document is structured as follows: Section 2 introduces the θ-Weighted Mixture distribution family (θ-WM) based on Equation (2). In Section 3, we study the θ-WMWLx distribution as part of the θ-WM family. Section 4 encompasses a comprehensive reliability analysis, including a detailed exploration of the fundamental statistical properties of the θ-WMWLx distribution.

In Section 5, a simulation study is incorporated to evaluate the performance of maximum likelihood estimators (MLE), least squares estimators (LSE), and weighted least squares estimators (WLSE) across various sample sizes and parameter configurations. Finally, two real-world applications are presented to demonstrate the behavior of the proposed distribution compared to four other models, and the conclusion drawn from the study is summarized in Section 6.

2 The θ-Weighted Mixture distribution family (θ-WM)

In this section, we introduce the definition of the θ-Weighted Mixture distribution family (θ-WM), based on Equation (2). In addition, we show two distribution models from this proposed family as illustrative examples.

Definition 1. Let $G(x|\eta)$ and $\Psi(x|\xi)$ be two absolutely continuous baseline distribution functions, where $\bar{G}(x|\eta)$ and $\bar{\Psi}(x|\xi)$ represent their respective survival functions (SF). Then, for $\theta \in [0, 1]$, the corresponding cumulative distribution function (CDF) and probability density function (PDF) for the θ-WM distribution family are obtained as follows:

$$F_\theta(x|\eta,\xi) = 1 - \bar{G}(x|\eta)^{1-\theta}\,\bar{\Psi}(x|\xi)^{\theta}, \quad \eta,\xi > 0, x > 0, \quad (3)$$

and

$$f_\theta(x|\eta,\xi) = \bar{G}(x|\eta)^{-\theta}\,\bar{\Psi}(x|\xi)^{\theta} \left[(1-\theta)g(x|\eta) + \theta\bar{G}(x|\eta)\bar{\Psi}(x|\xi)^{-1}\psi(x|\xi)\right], \quad (4)$$

where g and ψ are the PDFs of the distributions G and Ψ, respectively.

Remark 1. From the definition 1, we have:

1. The parameter θ determines the weighting between the survival functions $\bar{G}$ and $\bar{\Psi}$ in the formation of the combined distribution.
2. If $\theta = 0$, then $F_0(x|\eta,\xi) = G(x|\eta)$.

3. If $\theta = 1$, then $F_1(x|\eta, \xi) = \Psi(x|\xi)$.
4. If $G(x|\Phi) = \Psi(x|\Phi)$, then $F_\theta(x|\Phi) = G(x|\Phi)$.

2.1 Some models based on the θ-WM distribution family

In this subsection, two models are shown using Equations (3, 4) to determine both the cumulative distribution function and probability density function of the θ-WM distribution family for each of these models.

Figures 1–3 show the HRFs of the submodels in the θ-WM family. The flexibility of these graphs can be observed, displaying increasing shapes, J-shapes, unimodal, and asymmetric forms. Similarly, the densities of these submodels provide great flexibility in their shapes, which can exhibit left-skewed, bimodal, right-skewed, unimodal, symmetrical, and J-shaped forms, as illustrated in Figures 1–3.

2.1.1 The θ-WMEW distribution

We define the θ-WMEW distribution by taking the exponential and Weibull distributions as baseline distributions in the θ-WM distribution family.

Suppose X_1 is a random variable that follows an exponential distribution with parameter λ. Then, the CDF and PDF of X_1 are defined by $G(x|\lambda) = 1 - e^{-\lambda x}$ and $g(x|\lambda) = \lambda e^{-\lambda x}$, respectively, and the random variable X_2 follows a Weibull distribution with shape parameter α and scale parameter β. The CDF and PDF of X_2 are expressed as $\Psi(x|\alpha, \beta) = 1 - e^{-(\beta x)^\alpha}$ and $\psi(x|\alpha, \beta) = \alpha\beta(\beta x)^{\alpha-1} e^{-(\beta x)^\alpha}$, respectively. Consequently, the CDF and PDF of the θ-WMEW distribution are given by

$$F_\theta(x|\theta, \lambda, \alpha, \beta) = 1 - \left[e^{-\lambda x}\right]^{1-\theta} \left[e^{-(\beta x)^\alpha}\right]^\theta,$$

and

$$f_\theta(x|\theta, \lambda, \alpha, \beta) = e^{\lambda x(\theta-1)-\theta(\beta x)^\alpha} \left[(1-\theta)\lambda + \theta\alpha\beta(\beta x)^{\alpha-1}\right].$$

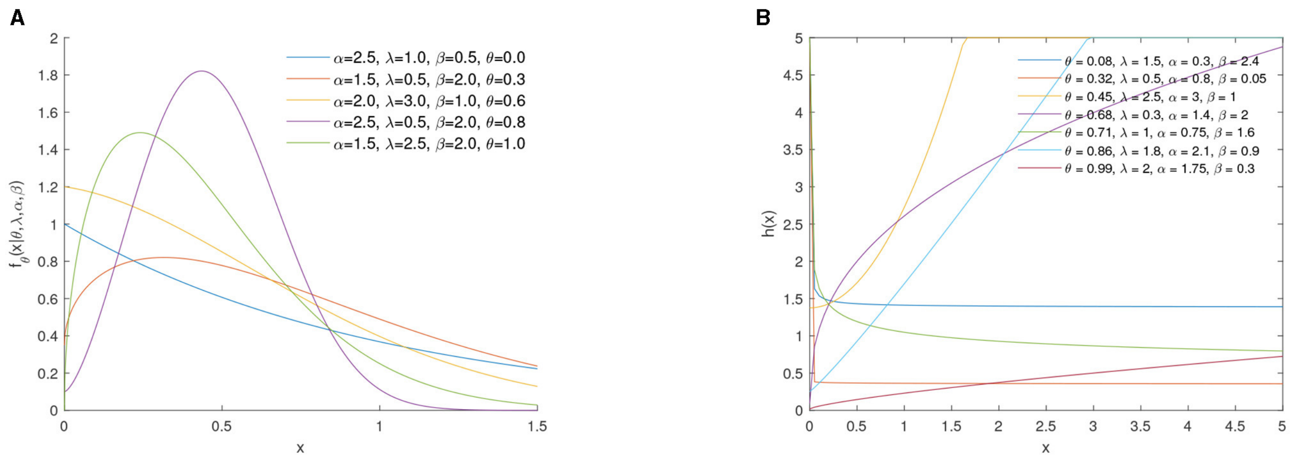

FIGURE 1
(A) PDF of θ-WMEW. **(B)** HRF of θ-WMEW.

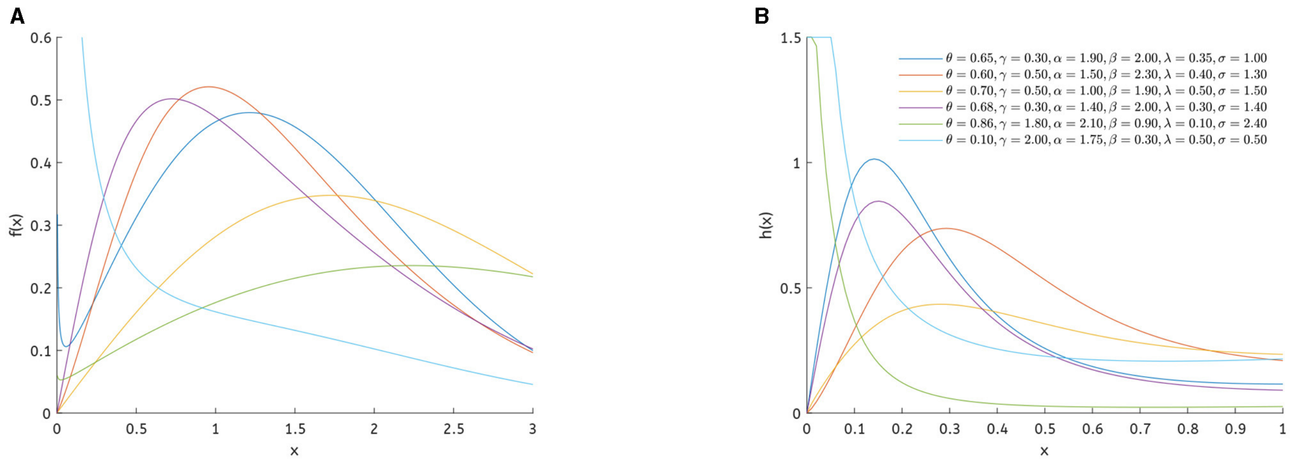

FIGURE 2
(A) PDF of θ-WMPLxR. **(B)** HRF of θ-WMPLxR.

where θ represents a parameter that weights the characteristics of the exponential and Weibull distributions. Figure 1 shows the graph of PDF and hazard rate function HRF for different parameters.

2.1.2 The θ-WMPLxR distribution

The θ-WMPLxR distribution corresponds to the composition of the Marshall-Olkin power Lomax and Rayleigh distributions. Then, the CDF and PDF of the θ-WMPLxR distribution are expressed as follows:

$$F_\theta(x|\Theta) = 1 - \left[1 - \left(\frac{1-\lambda^\alpha(\lambda+x^\beta)^{-\alpha}}{1-(1-\gamma)(\lambda^\alpha(\lambda+x^\beta)^{-\alpha})}\right)\right]^{1-\theta} \left[e^{-\frac{x^2}{2\sigma^2}}\right]^\theta,$$

and

$$f_\theta(x|\Theta) = \left[1 - \left(\frac{1-\lambda^\alpha(\lambda+x^\beta)^{-\alpha}}{1-(1-\gamma)(\lambda^\alpha(\lambda+x^\beta)^{-\alpha})}\right)\right]^{-\theta} \left[e^{-\frac{x^2}{2\sigma^2}}\right]^\theta \times \left[(1-\theta)\frac{\gamma\alpha\beta\lambda^\alpha x^{\beta-1}(\lambda+x^\beta)^{-\alpha-1}}{\left[1-(1-\gamma)\lambda^\alpha(\lambda+x^\beta)^{-\alpha}\right]^2} + \frac{\theta x}{\sigma^2}\left[1-\left(\frac{1-\lambda^\alpha(\lambda+x^\beta)^{-\alpha}}{1-(1-\gamma)(\lambda^\alpha(\lambda+x^\beta)^{-\alpha})}\right)\right]\right].$$

Figure 2 shows the graph of the PDF and HRF of the θ-WMPLxR distribution for different parameter values.

3 θ-Weighted Mixture Weibull-Lomax distribution (θ-WMWLx)

Within the framework of the family of θ-weighted mixture distributions (θ-WM) emerges the θ-weighted mixture Weibull-Lomax distribution (θ-WMWLx). Based on the strategic fusion of the Weibull and Lomax distributions, this theoretical expansion provides a powerful and adaptable tool for modeling phenomena with diverse and complementary properties.

The specific choice of the Weibull and Lomax distributions is not arbitrary; rather, it is grounded in their inherent characteristics that make them exceptionally suitable for modeling different contexts and phenomena. The Weibull distribution is recognized for its versatility in modeling product lifetimes, reliability phenomena, and time to failure, while the Lomax distribution, known as the two-parameter Pareto distribution, stands out for its ability to describe phenomena with heavy tails and long-tailed distributions. For more details, refer to the studies Tahir et al. [20], Afify et al. [21], Hassan and Abd-Allah [22], Ijaz et al. [23], and Alzaghal et al. [24].

The main distinction of the θ-WMWLx lies in its ability to encapsulate both the shape of the Weibull distribution, with its flexibility in modeling different behaviors, and the heavy-tailed characteristics of the Lomax distribution. This fusion provides a powerful and adaptable statistical framework to describe a wide range of phenomena, from data exhibiting reliability trends and lifetimes to those showing extreme behaviors and long tails.

Let $\bar{G}(x|\lambda,k)$ and $\bar{\Psi}(x|\alpha,\beta)$ be the survival functions of the Weibull and Lomax distributions, respectively, that is $\bar{G}(x|\lambda,k) = e^{-(\lambda x)^k}$ and $\bar{\Psi}(x|\alpha,\beta) = (1+\beta x)^{-\alpha}$. Then, according to Definition 1, and for $0 \le \theta \le 1$, the CDF and PDF for θ-WMWLx are defined by

$$F_\theta(x|\theta,\lambda,k,\alpha,\beta) = 1 - e^{-(1-\theta)(\lambda x)^k}(1+\beta x)^{-\theta\alpha}, \quad \lambda,k,\alpha,\beta > 0, x > 0, \tag{5}$$

and

$$f_\theta(x|\theta,\lambda,k,\alpha,\beta) = e^{(\theta-1)(\lambda x)^k}(1+\beta x)^{-\theta\alpha} \left[(1-\theta)\lambda k(\lambda x)^{k-1} + \theta\alpha\beta(1+\beta x)^{-1}\right]. \tag{6}$$

respectively.

Figure 3 shows the graph of the PDF and HRF of the θ-WMWLx distribution for a combination of λ, k, β, α, and θ parameters.

It is evident that the densities of the θ-WMWLx distribution (Equation 6) exhibit diverse shapes, ranging from symmetric to asymmetric, with skewness, inverted J-shaped, and unimodal distributions.

These graphs illustrate how the PDF may show various behaviors for θ and k values. For instance, when $\theta = 0$ and $0 < k \le 1$, the PDF decreases. However, if $\theta = 0$ and $k > 1$, the PDF increases when $X < 1/k((k-1)/k)^{1/k}$ and decreases when $X < 1/k((k-1)/k)^{1/k}$. Finally, for $\theta = 1$, the PDF always exhibits a decreasing trend.

Proposition 1 shows a stochastic order connection between $\bar{G}(x|\lambda,k)$ and $F_\theta(x|\theta,\lambda,k,\alpha,\beta)$.

Proposition 1. If $F_\theta(x|\theta,\lambda,k,\alpha,\beta)$ is defined as in Equation (5) and $F_\theta^{\circledast}(x|\theta,\lambda,k) = 1 - \left[\bar{G}(x|\lambda,k)\right]^{1-\theta}$, then $F_\theta(x|\theta,\lambda,k,\alpha,\beta)$ exhibits first-order stochastic dominance over $F_\theta^{\circledast}(x|\theta,\lambda,k)$, that is,

$$F_\theta(x|\theta,\lambda,k,\alpha,\beta) \succeq_1 F_\theta^{\circledast}(x|\theta,\lambda,k).$$

Proof. To prove first-order stochastic dominance, we need to prove that

$$F_\theta(x|\theta,\lambda,k,\alpha,\beta) \ge F_\theta^{\circledast}(x|\theta,\lambda,k), \forall x > 0.$$

Subtracting the CDF functions yields

$$\begin{aligned} &F_\theta(x|\theta,\lambda,k,\alpha,\beta) - F_\theta^{\circledast}(x|\theta,\lambda,k) \\ &= \left(1 - e^{-(1-\theta)(\lambda x)^k}(1+\beta x)^{-\theta\alpha}\right) - \left(1 - e^{-(1-\theta)(\lambda x)^k}\right) \\ &= -e^{-(1-\theta)(\lambda x)^k}(1+\beta x)^{-\theta\alpha} + e^{-(1-\theta)(\lambda x)^k} \\ &= e^{-(1-\theta)(\lambda x)^k}\left(1 - (1+\beta x)^{-\theta\alpha}\right). \end{aligned}$$

We know that $e^{-(1-\theta)(\lambda x)^k} > 0$ for all x, so if $\left(1 - (1+\beta x)^{-\theta\alpha}\right) \ge 0$ for all $x > 0$, stochastic dominance holds.

The expression $\left(1 - (1+\beta x)^{-\theta\alpha}\right) \ge 0$ is satisfied whenever $(1+\beta x)^{-\theta\alpha} \le 1$, which is factual since $(1+\beta x)^{-\theta\alpha}$ is a decreasing function in x and for $\theta, \alpha > 0$. This proves the result. □

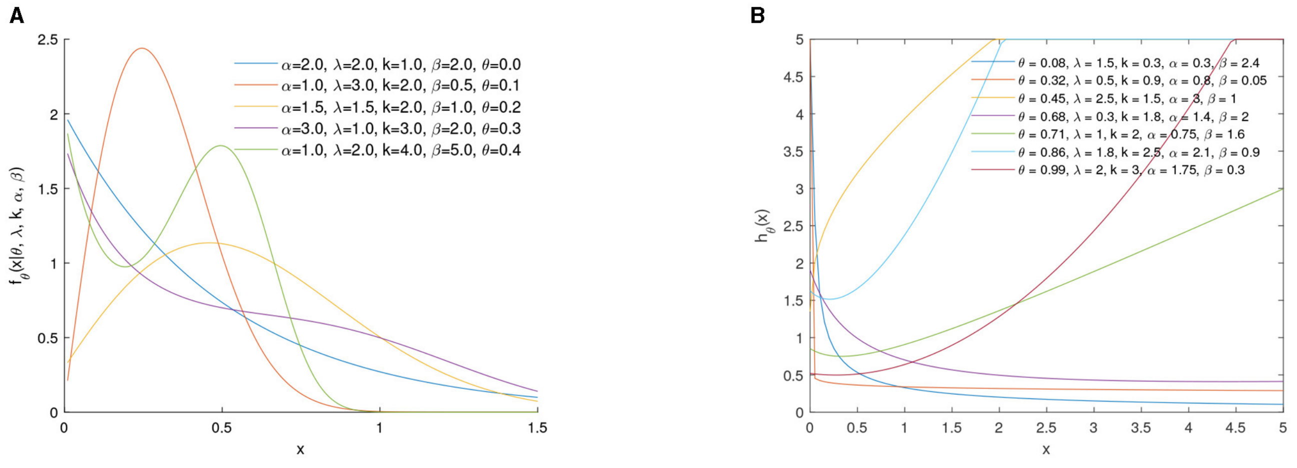

FIGURE 3
(A) PDF of θ-WMWLx distribution. **(B)** HRF of θ-WMWLx distribution.

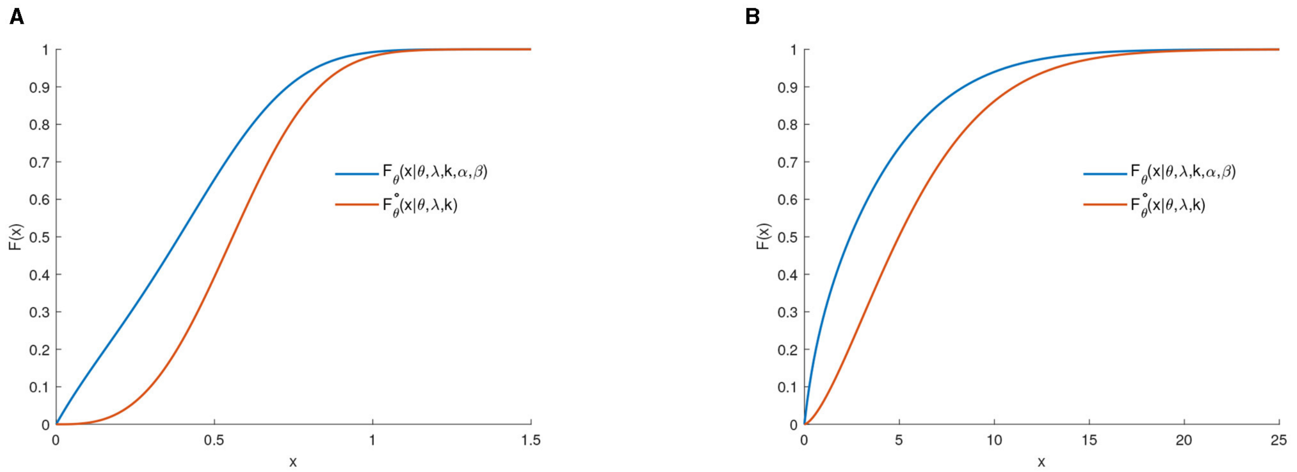

FIGURE 4
(A) CDF $F_{0.5}(x)$ and $F^{\circledast}_{0.5}(x)$. **(B)** CDF $F_{0.3}(x)$ and $F^{\circledast}_{0.3}(x)$.

TABLE 1 **First quartile, median, and third quartile of θ-WMWLx distribution.**

θ	α	β	λ	k	First quartile	Median	Third quartile
0	2	1	3	2	0.1788	0.2775	0.3924
	0.5	3	1	3	0.6601	0.8849	1.1150
0.2	2	1	3	2	0.1759	0.2868	0.4161
	0.5	3	1	3	0.6120	0.8895	1.1562
0.8	2	1	3	2	0.1621	0.3469	0.5956
	0.5	3	1	3	0.3378	0.9291	1.5185
0.9	2	1	3	2	0.1586	0.3716	0.6957
	0.5	3	1	3	0.2947	0.9512	1.7724

TABLE 2 Renyi entropy of θ-WMWLx.

θ	α	β	λ	k	Renyi Entropy for $\delta = 2$
0	0.5	1	0.3	2	1.67133
0.2	0.5	1	0.3	2	1.81029
0.4	0.5	1	0.3	2	1.91527
0.6	0.5	1	0.3	2	1.98275
0.8	0.5	1	0.3	2	2.01838
1	0.5	1	0.3	2	2.07944

Figure 4 clearly shows the first-order stochastic dominance of $F_\theta(x)$ over $F_\theta^{\circledast}(x)$.

```
Given parameters: α = 0.5, β = 1, λ = 0.3, k = 2, δ = 2
θ_values ← [0, 0.2, 0.4, 0.6, 0.8, 1]
for θ in θ_values do
    integrand ← Anonymous function defined as:
        exp(δ · (θ − 1) · (λ · x)^k) · (1 + β · x)^(−δ·θ·α)
        ·((1 − θ) · λ · k · (λ · x)^(k−1) + θ · α · β · (1 + β · x)^(−1))^δ
    Integral value ← integral(integrand, 0, ∞)
    Renyi entropy (I_R) ← (1/(1 − δ)) · log(Integral value)
    Print "For θ = ", θ, ", the Renyi entropy is: ", I_R
end for
```

Algorithm 1. Pseudocode to calculate Renyi entropy for given parameters.

4 Reliability elements and statistical properties of the θ-WMWLx distribution

In this section, we study the fundamental reliability components associated with the θ-WMWLx distribution, along with its basic statistical properties such as survival and hazard functions, mean residual life, mean inactivity time, useful expansions, quantile function, Renyi entropy, moments, order statistics, and estimation methods.

4.1 Survival and hazard functions

In survival analysis, one of the most important functions is the survival function, defined as the probability that an individual will survive beyond time x, as stated by Kartsonaki [25]. It is expressed as

$$\bar{G}(X) = P(X > x) = 1 - F(X), \quad x > 0.$$

Therefore, if X follows a θ-WMWLx $(\theta, \lambda, k, \alpha, \beta)$ distribution, then the SF is represented by the equation:

$$\bar{F}_\theta(x|\theta,\lambda,k,\alpha,\beta) = e^{-(1-\theta)(\lambda x)^k}(1+\beta x)^{-\theta\alpha}, \quad \lambda,k,\alpha,\beta > 0, x > 0. \tag{7}$$

This equation describes the probability of an individual's continued survival beyond time x. The hazard function, also known as the failure rate or hazard function, represents a component's instantaneous probability of failure, assuming failure has not occurred before time x (see Baredar et al. [26]). On the other hand, the reversed hazard rate is defined as the ratio between the probability density function and its distribution function (see Kayid et al. [27]). Therefore, if X follows a θ-WMWLx $(\theta, \lambda, k, \alpha, \beta)$ distribution, the hazard function and the reversed hazard rate are expressed as

$$h_\theta(x|\theta,\lambda,k,\alpha,\beta) = \left[(1-\theta)\lambda k(\lambda x)^{k-1} + \theta\alpha\beta(1+\beta x)^{-1}\right] \tag{8}$$

and

$$r_\theta(x|\theta,\lambda,k,\alpha,\beta) = \frac{e^{(\theta-1)(\lambda x)^k}(1+\beta x)^{-\theta\alpha}\left[(1-\theta)\lambda k(\lambda x)^{k-1} + \theta\alpha\beta(1+\beta x)^{-1}\right]}{1 - e^{-(1-\theta)(\lambda x)^k}(1+\beta x)^{-\theta\alpha}}.$$

4.2 Mean residual life

The mean residual life (MRL) represents the anticipated added duration once a component has endured up to a time t. The MRL is crucial for reliability and survival analysis and characterizes the aging mechanism. It has been established that the MRL exclusively defines the distribution function, encapsulating all model-relevant details (see Alshangiti et al. [28]). If the random variable X represents the life of a component, then the MRL is given by

$$\mu_T(t) = \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x)dx, \quad t \geq 0,$$

where $\bar{F}$ is the survival function.

The MRL function of a lifetime random variable X with θ-WMWLx $(\theta, \lambda, k, \alpha, \beta)$ distribution is given by

$$\mu_T(t|\theta,\lambda,k,\alpha,\beta) = \frac{1}{e^{-(1-\theta)(\lambda t)^k}(1+\beta t)^{-\theta\alpha}} \int_t^\infty e^{-(1-\theta)(\lambda x)^k}(1+\beta x)^{-\theta\alpha}dx, t \geq 0.$$

4.3 Mean inactivity time

The mean inactivity time (MIT) function is a reliability measure with applications in many disciplines, such as reliability theory, survival analysis, and actuarial studies. The MIT allows describing the time elapsed since a failure occurred. Let X be a lifetime random variable with distribution function F. The MIT function of X is defined by

$$m(t) = \frac{1}{F(t)} \int_0^t F(x)dx, \quad t > 0.$$

The MIT function of a lifetime random variable X with θ-WMWLx $(\theta, \lambda, k, \alpha, \beta)$ distribution is given by

$$m(t|\theta,\lambda,k,\alpha,\beta) = \frac{1}{1-e^{-(1-\theta)(\lambda t)^k}(1+\beta t)^{-\theta\alpha}} \int_0^t 1-e^{-(1-\theta)(\lambda x)^k}(1+\beta x)^{-\theta\alpha}dx.$$

4.4 Useful expansions

In this subsection, we provide an infinite mixture representation of the probability density function corresponding to the θ-WMWLx distribution, which will be used in some subsequent calculations.

It is well known that the exponential series expansion is given by

$$e^{-x} = \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!}. \quad (9)$$

By applying Equation (9) to the PDF of θ-WMWLx, we obtain

$$f_\theta(x|\theta,\lambda,k,\alpha,\beta) = \sum_{j=0}^{\infty} v_j w_{\theta\alpha,(k-1)}(x), \quad (10)$$

where

$$v_j = \frac{(1-\theta)^j(-1)^j}{j}, \quad \text{and}$$
$$w_{\theta\alpha,(k-1)}(x) = (\lambda x)^{kj}(1+\beta x)^{-\theta\alpha} \left[(1-\theta)\lambda k(\lambda x)^{k-1} + \theta\alpha\beta(1+\beta x)^{-1}\right].$$

4.5 Quantile function

The quantile function is obtained by solving:

$$Q[\varphi(u)] = u, \quad u \in (0,1),$$

where $Q(\cdot)$ is the CDF of θ-WMWLx $(\theta, \lambda, k, \alpha, \beta)$ distribution. In this case, the quantile function is the solution of the non-linear equation:

$$(1-\theta)(\lambda x_u)^k + \theta\alpha\ln(1+\beta x_u) + \ln(1-u) = 0. \quad (11)$$

By setting $u = 0.5$ in Equation (11), we can obtain the median (M_e) of the θ-WMWLx distribution. Furthermore, the lower and higher quartiles can be obtained by setting $u = 0.25$ and $u = 0.75$, respectively.

Table 1 shows the quantiles Q_1, M_e, and Q_3 for different values of the weighting parameter θ. These values were obtained using the root-finding method, specifically implemented computationally through the `uniroot()` function in R software version 4.3.3. The `uniroot()` function is used to find the roots of non-linear equations, and in this context, it was applied to solve an equation that models the relationship between the quantiles and the parameter θ.

4.6 Rényi entropy

The entropy of a random variable X is a measure of the uncertain variation. The Rényi entropy is defined by

$$I_{\mathbb{R}}(\delta) = \frac{1}{1-\delta}\log[I(\delta)],$$

where $I(\delta) = \int_{\mathbb{R}} f^\delta(x)dx, \delta \in \mathbb{R}^+ - \{1\} > 0$.

Let $X \sim \theta$-WMWLx $(\theta, \lambda, k, \alpha, \beta)$. The corresponding Renyi entropy is obtained as

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[\int_0^\infty e^{\delta(\theta-1)(\lambda x)^k}(1+\beta x)^{-\delta\theta\alpha} \left((1-\theta)\lambda k(\lambda x)^{k-1} + \theta\alpha\beta(1+\beta x)^{-1}\right)^\delta dx\right].$$

Table 2 shows the Renyi entropy for θ-WMWLx with $\delta = 2$, $\alpha = 0.5$, $\beta = 1$, $\lambda = 0.3$, $k = 2$, and different choices of parameter θ.

For the calculation of Renyi entropy, the numerical integration method based on the `integral()` function in MATLAB version R2024a was used (see Algorithm 1).

4.7 Moments

Moments in statistical analysis are essential measures that describe diverse probability distribution characteristics. They provide insights into the shape, center, spread, and other important features of the distribution. Specifically, moments help to quantify properties such as mean, variance, skewness, and kurtosis.

Theorem 2. If $X \sim \theta$-WMWLx $(\theta, \lambda, k, \alpha, \beta)$, then the r^{th} moment of X (for $r < \alpha$, when $\theta = 1$) is obtained by

$$\mu_r' = \sum_{j=0}^{\infty} v_j \left(\frac{(1-\theta)k\lambda^{k(j+1)}}{\beta^{r+k(j+1)}} \mathbf{B}\left[r+k(j+1), -r-k(j+1)+\theta\alpha\right] + \frac{\theta\alpha\lambda^{kj}}{\beta^{r+kj}} \mathbf{B}\left[r+kj+1, -r-kj+\theta\alpha\right] \right),$$

where $v_j = \frac{(-1)^j(1-\theta)^j}{j!}$ and $\mathbf{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ is the Beta function.

Proof. We know that

$$\mu_r' = E(x^r) = \int_0^\infty x^r f_\theta(\theta,\lambda,k,\alpha,\beta)dx. \quad (12)$$

Substituting Equation (10) into Equation (12) yields

$$\mu_r' = \sum_{j=0}^{\infty} v_j \left[(1-\theta)k\lambda^{k(j+1)} \int_0^\infty x^{r+k(j+1)-1}(1+\beta x)^{-\theta\alpha}dx + \theta\alpha\beta\lambda^{kj} \int_0^\infty x^{r+kj}(1+\beta x)^{-(\theta\alpha+1)}dx \right].$$

TABLE 3 Mean, variance, kurtosis, and skewness of θ-WMWLx distribution.

θ	α	β	λ	k	Mean	Variance	Kurtosis	Skewness
0	2.89	1.55	3.62	2.14	0.2446	0.0144	3.0933	0.5432
	0.5	3.75	1.89	3.45	0.4756	0.0232	2.7114	0.0378
	2.8	4.5	1.75	6.24	0.5312	0.0098	3.0726	−0.3970
	1.9	0.25	7.25	0.99	0.1385	0.0195	9.2043	2.0303
	3.5	3.8	0.75	1.85	1.1842	0.4410	3.4658	0.7389
0.2	2.89	1.55	3.62	2.14	0.2397	0.0184	3.0744	0.5682
	0.5	3.75	1.89	3.45	0.4753	0.0341	2.8055	−0.2030
	2.8	4.5	1.75	6.24	0.3669	0.0488	1.685	−0.2022
	1.9	0.25	7.25	0.99	0.1708	0.0297	9.2286	2.0333
	3.5	3.8	0.75	1.85	0.5943	0.3865	5.7856	1.6016
0.4	2.89	1.55	3.62	2.14	0.2371	0.0233	3.1849	0.6886
	0.5	3.75	1.89	3.45	0.4809	0.0477	2.5321	−0.1891
	2.8	4.5	1.75	6.24	0.2614	0.0488	2.0194	0.5922
	1.9	0.25	7.25	0.99	0.2226	0.0507	9.2838	2.0403
	3.5	3.8	0.75	1.85	0.3319	0.2013	13.5392	2.7981
0.6	2.89	1.55	3.62	2.14	0.2383	0.0306	3.5391	0.8891
	0.5	3.75	1.89	3.45	0.4978	0.0678	2.2718	−0.0804
	2.8	4.5	1.75	6.24	0.1926	0.0394	3.5502	1.2446
	1.9	0.25	7.25	0.99	0.3196	0.1052	9.4328	2.0593
	3.5	3.8	0.75	1.85	0.2066	0.0970	29.371	4.1098
0.8	2.89	1.55	3.62	2.14	0.2482	0.0452	4.5116	1.2335
	0.5	3.75	1.89	3.45	0.5455	0.1099	2.0946	0.0988
	2.8	4.5	1.75	6.24	0.1470	0.0302	6.4252	1.9140
	1.9	0.25	7.25	0.99	0.5683	0.3423	10.0267	2.1352
	3.5	3.8	0.75	1.85	0.1417	0.0478	60.5299	5.5037

If $u = \beta x$, then $du = \beta dx$, hence

$$\mu_r' = \sum_{j=0}^{\infty} v_j \Bigg[\frac{(1-\theta)k\lambda^{k(j+1)}}{\beta^{r+k(j+1)}} \int_0^\infty u^{r+k(j+1)-1}(1+u)^{-\theta\alpha} du + \frac{\theta\alpha\lambda^{kj}}{\beta^{r+kj}} \int_0^\infty u^{r+kj}(1+u)^{-(\theta\alpha+1)} du \Bigg].$$

Again, if $u = \frac{z}{1-z}$, then $du = \frac{dz}{(1-z)^2}$, hence

$$\mu_r' = \sum_{j=0}^{\infty} v_j \Bigg[\frac{(1-\theta)k\lambda^{k(j+1)}}{\beta^{r+k(j+1)}} \int_0^1 z^{r+k(j+1)-1}(1-z)^{-r-k(j+1)+\theta\alpha-1} dz + \frac{\theta\alpha\lambda^{kj}}{\beta^{r+kj}} \int_0^1 z^{r+kj}(1-z)^{-r-kj+\theta\alpha-1} dz \Bigg],$$

$$\mu_r' = \sum_{j=0}^{\infty} v_j \Bigg(\frac{(1-\theta)k\lambda^{k(j+1)}}{\beta^{r+k(j+1)}} \mathbf{B}\left[r+k(j+1), -r-k(j+1)+\theta\alpha\right] + \frac{\theta\alpha\lambda^{kj}}{\beta^{r+kj}} \mathbf{B}\left[r+kj+1, -r-kj+\theta\alpha\right] \Bigg).$$

□

From the above, the mean of X is given by $E(X) = \mu = \mu_1'$. To determine the variance of X, we use the Koenig-Huygens formula, that is to say, $Var(X) = E(X^2) - \left[E(X)\right]^2$.

The r-th central moment is given by

$$\mu_r = E\left[(X-\mu)^r\right] = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \mu^j \mu_{r-j}'. \quad (13)$$

By using Equation (13), we can obtain the skewness and kurtosis of X as follows:

$$S_k = \frac{\mu_3}{(\mu_2)^{3/2}} \quad \text{and} \quad K_t = \frac{\mu_4}{(\mu_2)^2}.$$

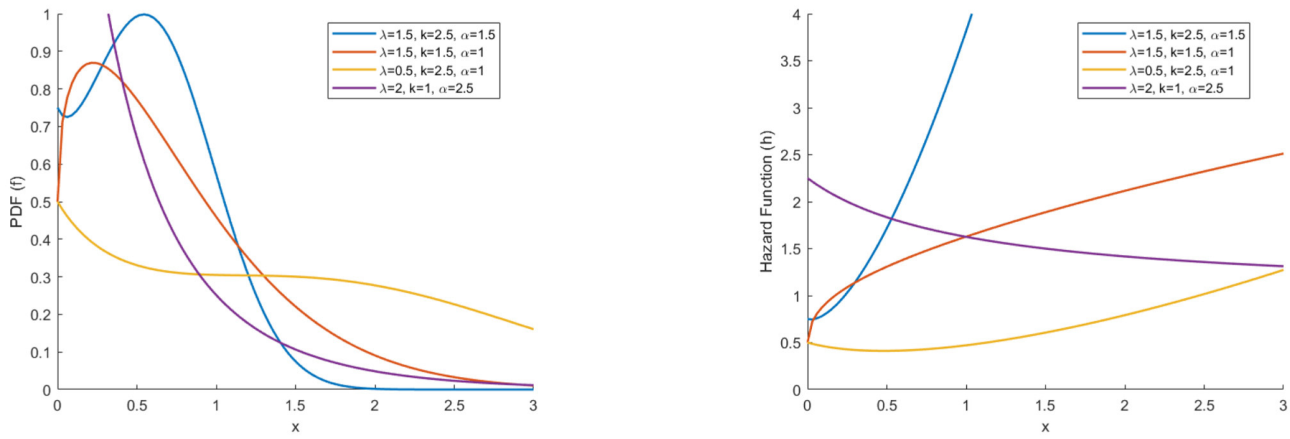

FIGURE 5
PDF and HRF of 1/2-WMWLx1 distribution.

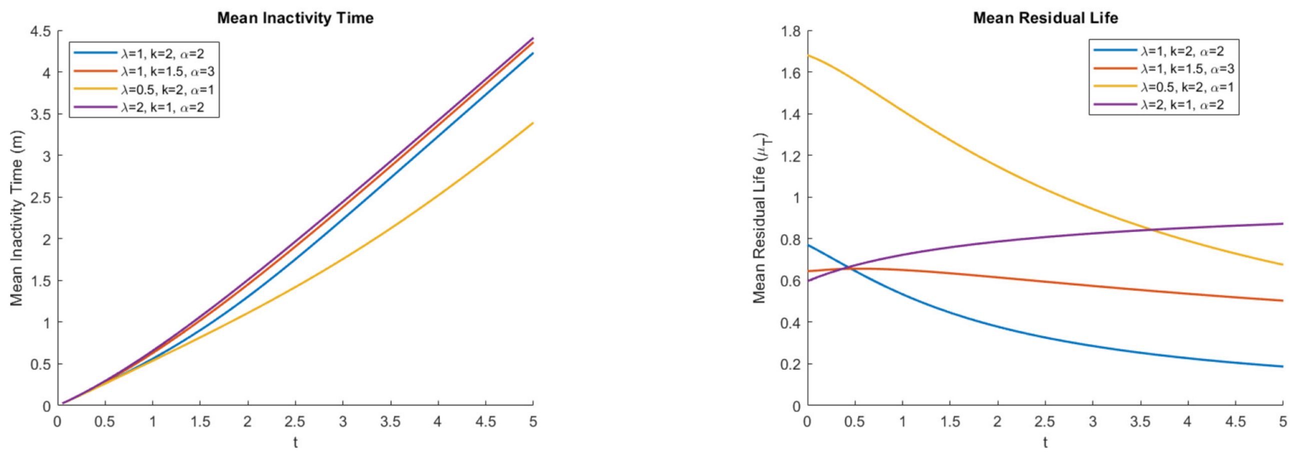

FIGURE 6
Mean inactivity time and residual life of the 1/2-WMWLx1 distribution.

TABLE 4 Basic statistics of the 1/2-WMWLx1 distribution.

Mean	Variance	Kurtosis	Skewness	First quartile	Median	Third quartile
0.7706	0.3715	3.9265	1.0395	0.2815	0.6349	1.1248

Table 3 shows different moments of X for specific parameter combinations. MATLAB version R2024a was employed for their computation.

4.8 Order statistics

Suppose $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ are the order statistics from the θ-WMWLx distribution. The probability density function of the i-th order statistic (with parameters suppressed) is given by

$$f_{x_{(i)}}(x) = \frac{1}{B(i, n-i+1)} f(x) \left[F(x)\right]^{i-1} \left[1 - F(x)\right]^{n-i}.$$

Applying the binomial expansion, we have

$$f_{x_{(i)}}(x) = \frac{1}{B(i, n-i+1)} f(x) \sum_{p=0}^{n-i} (-1)^p \binom{n-i}{p} \left[F(x)\right]^{p+i-1}, \quad (14)$$

where $\left[F(x)\right]^{p+i-1}$ can be written as

$$\left[F(x)\right]^{p+i-1} = \sum_{l=0}^{\infty} (-1)^l \binom{p+i-1}{l} e^{-l(1-\theta)(\lambda x)^k} (1+\beta x)^{-l\theta\alpha}. \quad (15)$$

TABLE 5 Simulation results.

n	Method	$\theta = 0.5, \alpha = 2, \lambda = 1, k = 2, \beta = 1$			$\theta = 0.5, \alpha = 1.5, \lambda = 2, k = 1.5, \beta = 1$		
		Parameter	SE	MSE	Parameter	SE	MSE
20	MLE	α	0.0026766	0.0072713	α	0.0085191	0.0842450
		λ	0.0007385	0.0005584	λ	0.0063004	0.0432720
		k	0.0028428	0.0086853	k	0.0097155	0.1266200
		β	0.0010304	0.0022534	β	0.0069850	0.0490760
	LSE	α	0.130510	23.876700	α	0.0501490	2.9943000
		λ	1.4213000	2019.3611	λ	0.0711850	5.2008000
		k	0.0956240	9.7974000	k	0.1272900	23.316100
		β	0.0401720	1.7466000	β	0.0573600	3.9970000
	WLSE	α	0.2949300	95.935500	α	0.0515200	2.6926000
		λ	0.0589020	3.4702000	λ	0.1336000	27.794600
		k	0.1973700	38.974300	k	0.0852920	21.504100
		β	0.0256380	0.9610400	β	0.0272330	2.0570000
50	MLE	α	0.00066397	0.0004623	α	0.0063901	0.0519880
		λ	0.00056153	0.00036809	λ	0.0059448	0.0423710
		k	0.00145860	0.00234100	k	0.0096493	0.1238600
		β	0.00071411	0.00193460	β	0.0030746	0.0095873
	LSE	α	0.1005900	11.615400	α	0.0162120	0.5901500
		λ	0.0834070	6.9618000	λ	0.0437700	2.0422000
		k	0.0170970	2.6616000	k	0.0618030	5.4477000
		β	0.0206610	0.4285000	β	0.0298590	1.1875000
	WLSE	α	0.1205500	33.601400	α	0.0637040	4.0890000
		λ	0.1556400	25.031800	λ	0.1247200	24.913000
		k	0.087398	10.229000	k	0.1066200	25.128300
		β	0.0249160	0.9178500	β	0.0453830	3.7696000
100	MLE	α	0.0005321	0.0003277	α	0.0066844	0.0511550
		λ	0.0004892	0.0003163	λ	0.0070742	0.0418856
		k	0.0006038	0.0004729	k	0.0103190	0.1212700
		β	0.0003949	0.0017647	β	0.0036105	0.0030830
	LSE	α	0.0690580	4.7766000	α	0.0080895	0.3341300
		λ	0.0359750	1.3115000	λ	0.0298820	1.0003000
		k	0.0062162	2.2903000	k	0.0428510	2.1415000
		β	0.0144910	0.2891600	β	0.0237340	0.6631500
	WLSE	α	0.0629470	28.193600	α	0.0689620	4.9764000
		λ	0.1413400	21.517100	λ	0.1441700	26.481200
		k	0.0054331	3.4600000	k	0.1100500	18.674000
		β	0.0146890	0.5754500	β	0.0372280	2.4712000
	MLE	α	0.0004960	0.0003131	α	0.0077255	0.0462740
		λ	0.0005041	0.0003113	λ	0.0085586	0.0588870
		k	0.0005299	0.0003987	k	0.0110450	0.1164500
		β	0.0002552	0.0015758	β	0.0036089	0.0014591
		α	0.0036444	0.4880200	α	0.0032777	0.1411100

(Continued)

TABLE 5 (Continued)

n	Method	$\theta=0.5,\alpha=3,\lambda=1,k=2.5,\beta=1$			$\theta=0.5,\alpha=1.5,\lambda=2,k=1,\beta=1$		
		Parameter	SE	MSE	Parameter	SE	MSE
500	LSE	λ	0.0506570	2.6432000	λ	0.0142110	0.3668000
		k	0.0026698	1.9585000	k	0.0218630	0.7986100
		β	0.0028420	0.2054200	β	0.0136250	0.2529700
	WLSE	α	0.0276170	30.202400	α	0.0723550	5.8992000
		λ	0.0738480	5.7776000	λ	0.0898350	8.8818000
		k	0.0035250	3.6516000	k	0.0571360	3.4679000
		β	0.0088773	0.6856100	β	0.0252230	0.6458100

Thus, by substituting Equation (15) into Equation (14), we obtain

$$f_{x_{(i)}}(x)=\frac{1}{B(i,n-i+1)}f(x)\sum_{p=0}^{n-i}(-1)^p\binom{n-i}{p}\cdot\sum_{l=0}^{\infty}(-1)^l\binom{p+i-1}{l}e^{-l(1-\theta)(\lambda x)^k}(1+\beta x)^{-l\theta\alpha}. \quad (16)$$

Subsequently, by substituting Equation (10) into Equation (16), we get

$$f_{x_{(i)}}(x)=\frac{1}{B(i,n-i+1)}\sum_{l,j=0}^{\infty}\sum_{p=0}^{n-i}t_{l,j,p}\delta_{\theta\alpha(l+1),k-1}(x), \quad (17)$$

where $t_{l,j,p}$ and $\delta_{\theta\alpha(l+1),k-1}(x)$ are, respectively, given by

$$t_{l,j,p}=\binom{n-i}{p}\binom{p+i-1}{l}\frac{(1-\theta)^j(-1)^{p+l+j}}{j!}$$

and

$$\delta_{\theta\alpha(l+1),k-1}(x)=e^{-l(1-\theta)(\lambda x)^k}(\lambda x)^{kj}(1+\beta x)^{-\theta\alpha(l+1)}\left[(1-\theta)\lambda k(\lambda x)^{k-1}+\theta\alpha\beta(1+\beta x)^{-1}\right].$$

If we set $i=1$ and $i=n$ in Equation (17), we obtain the PDF of the minimum and the PDF of the maximum of the θ-WMWLx distribution, respectively.

4.9 Estimation methods

This section presents three different methods for estimating the parameters of the proposed model: maximum likelihood estimators (MLE), least squares estimators (LSE), and weighted least squares estimators (WLSE).

4.9.1 Maximum likelihood estimators

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from θ-WMWLx distribution, the likelihood function is given by

$$\mathcal{L}(\mathbf{X}|\theta,\lambda,k,\alpha,\beta)=\prod_{i=1}^{n}e^{(\theta-1)(\lambda x_i)^k}(1+\beta x_i)^{-\theta\alpha}\left[(1-\theta)\lambda k(\lambda x_i)^{k-1}+\theta\alpha\beta(1+\beta x_i)^{-1}\right].$$

Then, the logarithm of the likelihood function is

$$l(\mathbf{X}|\theta,\lambda,k,\alpha,\beta)=(\theta-1)\lambda^k\sum_{i=1}^{n}x_i^k-\theta\alpha\sum_{i=1}^{n}\ln(u_i)+\sum_{i=1}^{n}\ln\left((1-\theta)\lambda k(\lambda x_i)^{k-1}+\theta\alpha\beta u_i^{-1}\right) \quad (18)$$

where $u_i=(1+\beta x_i)$.

We obtain maximum likelihood estimates by differentiating l with respect to each parameter λ, k, α, and β and setting the result equal to zero. The partial derivatives of l with respect to each parameter or the score function are given by

$$U_n(\Theta)=\left[\frac{\partial l}{\partial\lambda},\frac{\partial l}{\partial k},\frac{\partial l}{\partial\alpha},\frac{\partial l}{\partial\beta}\right]$$

These equations cannot be solved analytically. Therefore, statistical software can be used to solve them numerically.
The components of the score vector U_n are

$$\frac{\partial l}{\partial\lambda}=\sum_{i=1}^{n}\frac{(1-\theta)k^2(\lambda x_i)^{k-1}}{(1-\theta)\lambda k(\lambda x_i)^{k-1}+\theta\alpha\beta u_i^{-1}}+(\theta-1)k\lambda^{k-1}\sum_{i=1}^{n}u_i^{-\theta\alpha}x_i^k=0,$$

$$\frac{\partial l}{\partial k}=\sum_{i=1}^{n}\frac{\left((1-\theta)\lambda(\lambda x_i)^{k-1}\right)(1+k\ln(\lambda x_i))}{(1-\theta)\lambda k(\lambda x_i)^{k-1}+\theta\alpha\beta u_i^{-1}}+(\theta-1)\sum_{i=1}^{n}(\lambda x_i)^k\ln(\lambda x_i)u_i^{-\theta\alpha}=0,$$

$$\frac{\partial l}{\partial\alpha}=\sum_{i=0}^{n}\frac{\beta\theta u_i^{-1}}{(1-\theta)\lambda k(\lambda x_i)^{k-1}+\theta\alpha\beta u_i^{-1}}+(\theta-\theta^2)\sum_{i=1}^{n}(\lambda x_i)^k u_i^{-\theta\alpha}\ln(u_i)=0,$$

$$\frac{\partial l}{\partial\beta}=\sum_{i=1}^{n}\frac{\frac{\alpha\theta}{u_i}-\frac{\alpha\beta\theta x_i}{u_i^2}}{(1-\theta)\lambda k(\lambda x_i)^{k-1}+\theta\alpha\beta u_i^{-1}}+\alpha\theta(1-\theta)\sum_{i=1}^{n}x_i(\lambda x_i)^k u_i^{-(\alpha\theta+1)}=0.$$

TABLE 6 Simulation results.

n	Method	$\theta = 0.5, \alpha = 3, \lambda = 1, k = 2.5, \beta = 1$			$\theta = 0.5, \alpha = 1.5, \lambda = 2, k = 1, \beta = 1$		
		Parameter	SE	MSE	Parameter	SE	MSE
20	MLE	α	0.0011070	0.0012246	α	0.0011128	0.0147610
		λ	0.0002768	0.0000776	λ	0.0013653	0.0161620
		k	0.0009670	0.0009633	k	0.0012026	0.0014458
		β	0.0006565	0.0021980	β	0.0007689	0.0107400
	LSE	α	0.1452300	22.801200	α	0.0558450	3.2350000
		λ	0.1927900	37.219700	λ	0.0988380	9.8013000
		k	0.1327000	18.136900	k	0.1511600	25.221500
		β	0.0533450	3.0154000	β	0.4200700	176.70910
	WLSE	α	0.3597600	130.44330	α	0.0841070	8.1153000
		λ	0.0674760	4.5578000	λ	0.1439700	22.006700
		k	0.1094400	13.416400	k	0.2070100	90.893800
		β	0.0312300	1.0327000	β	0.1574800	30.894700
50	MLE	α	0.0001972	0.0000400	α	0.0008142	0.0139120
		λ	0.0001323	0.0000181	λ	0.0011190	0.0145790
		k	0.0002709	0.0007500	k	0.0007818	0.0006513
		β	0.0004474	0.0020303	β	0.0005266	0.0105994
	LSE	α	0.0952910	9.1703000	α	0.0290570	1.1189000
		λ	5.007500	250.82230	λ	0.0936530	8.7629000
		k	0.0602630	7.1737000	k	0.0569820	3.8532000
		β	0.0224860	0.5053600	β	0.0757400	5.8482000
	WLSE	α	0.0526380	11.463100	α	0.0261940	1.3786000
		λ	0.0472900	2.6152000	λ	0.973180	95.021410
		k	0.0261020	5.1386000	k	0.2988400	57.597400
		β	0.0142220	0.3255200	β	0.5918000	36.56631
100	MLE	α	0.0002964	0.00003781	α	0.0005952	0.0135270
		λ	0.0001111	0.00001277	λ	0.0012344	0.0149200
		k	0.0003472	0.0006123	k	0.0006046	0.0004130
		β	0.0003401	0.0020079	β	0.0003768	0.0119290
	LSE	α	0.0693930	5.5122000	α	0.0149730	0.4775300
		λ	0.6670600	217.69450	λ	0.0440010	1.9351000
		k	0.0163140	3.8753000	k	0.0409350	1.8926000
		β	0.0160810	0.3346700	β	0.0391740	1.5504000
	WLSE	α	0.0380350	11.324400	α	0.0284430	1.3295000
		λ	0.0420900	2.5314000	λ	0.1606700	27.023900
		k	0.0169390	5.0278000	k	0.5428600	69.637900
		β	0.0166280	0.4637700	β	0.5542800	24.266100
	MLE	α	0.0000642	0.0000041	α	0.0002565	0.0122180
		λ	0.0000475	0.0000023	λ	0.0007173	0.0137670
		k	0.0004744	0.0002305	k	0.0002789	0.0000819
		β	0.0002030	0.0019787	β	0.0002509	0.011455
		α	0.0281670	2.8674000	α	0.0007836	0.2356200

(Continued)

TABLE 6 (Continued)

n	Method	$\theta = 0.5, \alpha = 3, \lambda = 1, k = 2.5, \beta = 1$			$\theta = 0.5, \alpha = 1.5, \lambda = 2, k = 1, \beta = 1$		
		Parameter	SE	MSE	Parameter	SE	MSE
500	LSE	λ	0.0305320	1.0213000	λ	0.0171830	0.3044100
		k	0.0231020	2.9810000	k	0.0262760	0.6958100
		β	0.0078387	0.1982000	β	0.0214690	0.4822300
	WLSE	α	0.0566390	11.455400	α	0.0135730	0.5329400
		λ	0.0305110	1.2966000	λ	0.0867940	8.8674000
		k	0.0317100	4.2489000	k	1.0813000	122.70340
		β	0.0131850	0.3860800	β	1.0710000	117.36960

TABLE 7 MLEs for the first application.

Models	$\hat{\theta}$	$\hat{\lambda}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$
θ-WMWLx	0.99999	0.071822	6.7431	0.18317	0.55999
	(6.75e-6)	(0.0074)	(0.6109)	(0.0505)	(0.3795)
1/2-WMWLx1	–	0.014	7.134	0.296	–
	–	(0.001)	(2.592)	(0.070)	–
W	–	–	–	0.9490	44.9125
	–	–	–	(0.1208)	(7.0518)
Lx	–	84.7502	–	3847.6691	–
	–	(415.8422)	–	(5.3629)	–
EKIW	0.35329	5.4053	176.062	74.4398	0.219
	(0.0545)	(0.1266)	(10.7016)	(4.5385)	(0.011)
OKIW	0.10591	19.0126	1.0102	1.2801	0.25141
	(1.1751)	(0.2505)	(0.8501)	(3.4699)	(0.181)
LED	–	0.36228	0.053935	0.11279	–
	–	(0.2132)	(0.0302)	(0.9965)	–
FTLLx	78.0083	2.3655	0.0015422	0.12127	71.8406
	(0.2478)	(0.00870)	(0.00035)	(0.069235)	(0.17954)
APPLx	98.6969	1277.533	–	5.8648	0.78438
	(24.7343)	(340.9696)	–	(3.1649)	(0.03121)
KPL	3.9588	15504.8704	2180.0369	921.436	0.25076
	(0.58035)	(19968.1378)	(2806.8895)	(1184.8074)	(0.04277)

The normal approximation of the maximum likelihood estimator (MLE) of Θ can be used to construct approximate confidence intervals and test hypotheses about the parameters. Under regularity conditions (see Schafer [29]), we have that

$$\sqrt{n}(\hat{\Theta} - \Theta) \approx N_4(0, K_\Theta^{-1}),$$

where $\approx$ means “approximately distributed" and K_Θ^{-1} is the unit information matrix. The asymptotic behavior remains valid if $K_\Theta^{-1} = \lim_{n\to\infty} n^{-1} J_n(\Theta)$, where $J_n(\Theta)$ is the observed information matrix, which is replaced by the average sample information matrix evaluated at $\hat{\Theta}$, that is, $n^{-1}J_n(\Theta)$.

$$J_n(\Theta) = -\begin{bmatrix} L_{\lambda\lambda} & L_{\lambda k} & L_{\lambda\alpha} & L_{\lambda\beta} \\ L_{k\lambda} & L_{kk} & L_{k\alpha} & L_{k\beta} \\ L_{\alpha\lambda} & L_{\alpha k} & L_{\alpha\alpha} & L_{\alpha\beta} \\ L_{\beta\lambda} & L_{\beta k} & L_{\beta\alpha} & L_{\beta\beta} \end{bmatrix}$$

The elements of the main diagonal in the observed information matrix are given as follows:

$$L_{\lambda\lambda} = (\theta-1)k^2\lambda^{k-1}\sum_{i=1}^{n}\frac{(1-\theta)(k-1)k^2x_i(\lambda x_i)^{k-2}}{(1-\theta)k\lambda(\lambda x_i)^{k-1}\alpha\beta\theta(1+\beta x_i)^{-1}} + \frac{(1-\theta)k^2(\lambda x_i)^{k-1}\left[(1-\theta)(k-1)k\lambda x_i(\lambda x_i)^{k-2} + (1-\theta)k(\lambda x_i)^{k-1}\right]}{\left((1-\theta)k\lambda(\lambda x_i)^{k-1} + \alpha\beta\theta(1+\beta x_i)^{-1}\right)^2}$$

TABLE 8 Goodness-of-fit statistics and information criteria for the first application.

Models	AIC	CAIC	HQIC	BIC	D	A^*	W^*
θ-WMWLx	444.9610	445.6424	448.6016	454.5212	0.1013	1.2123	0.1028
1/2-WMWLx1	440.5440	440.8049	442.7283	446.2801	0.098657	1.3027	0.13798
W	486.0036	486.1313	487.4599	489.8277	0.8703	124.9	0.5589
Lx	486.4697	486.5974	487.9259	490.2938	0.97245	123.856	1.2153
EKIW	500.3307	501.0125	503.9712	509.8908	0.5035	19.317	4.1673
OKIW	568.2197	568.9016	571.8603	577.7799	0.8373	53.067	10.7454
LED	480.2970	480.5579	482.4813	486.0331	0.2682	−82.875	1.3988
FTLLx	477.0685	477.7503	480.7091	486.6286	0.15984	2.5019	0.36016
APPLx	487.7325	488.1769	490.6449	495.3806	0.16357	3.5885	0.4542
KPL	493.4107	494.0926	497.0513	502.9709	0.81153	−47.5227	18.104

$$L_{kk} = \sum_{i=1}^{n} \frac{(1-\theta)(\lambda x_i)^{k-1}\ln(\lambda x_i)(k\ln(\lambda x_i)+1)\lambda(\lambda x_i)^{k+1}}{(1-\theta)k\lambda(\lambda x_i)^{k-1}+\alpha\beta\theta(1+\beta x_i)^{-1}} + \frac{(1-\theta)\ln(\lambda x_i)\lambda(\lambda x_i)^{k-1}}{(1-\theta)k\lambda(\lambda x_i)^{k-1}+\alpha\beta\theta(1+\beta x_i)^{-1}} - \frac{(1-\theta)\lambda(\lambda x_i)^{k-1}(k\ln(\lambda x_i)+1)\left[(1-\theta)\lambda(\lambda x_i)^{k-1}+(1-\theta)k\lambda(\lambda x_i)^{k-1}\ln(\lambda x_i)\right]}{\left((1-\theta)k\lambda(\lambda x_i)^{k-1}+\alpha\beta\theta(1+\beta x_i)^{-1}\right)^2} + (\theta-1)\sum_{i=1}^{n}(\lambda x_i)^k\ln^2(\lambda x_i)(1+\beta x_i)^{-\theta\alpha}$$

$$L_{\alpha\alpha} = \sum_{i=1}^{n} \frac{-\beta^2\theta^2\lambda(1+\beta x_i)^{-1}}{(1+\beta x_i)\left(k\lambda(1-\theta\lambda)(\lambda x_i)^{k-1}+\alpha\beta\theta\lambda(1+\beta x_i)^{-1}\right)^2} + \theta^2(\theta-1)\sum_{i=1}^{n}(\lambda x_i)^k\ln^2(1+\beta x_i)(1+\beta x_i)^{-\alpha\theta}$$

$$L_{\beta\beta} = \sum_{i=1}^{n} \frac{\alpha\theta x_i^2\left[2(\theta-1)k(1+\beta x_i)(\lambda x_i)^k-\alpha\theta(2\beta x_i+1)\right]}{(1+\beta x_i)^2\left[(\theta-1)k(1+\beta x_i)(\lambda x_i)^k-\alpha\beta\theta x_i\right]^2} + \alpha(\theta-1)\theta\sum_{i=1}^{n}x_i^2\left[(\alpha\theta+1)(\lambda x_i)^k(1+\beta x_i)^{-\alpha\theta-2}\right]$$

Then, the asymptotic variance–covariance matrix $I^{-1}(\hat{\Omega})$ for the maximum likelihood estimators (MLEs) is obtained by inverting the observed information matrix $I(\hat{\Omega})$, or equivalently:

$$I^{-1}(\hat{\Theta}) = \begin{pmatrix} \mathrm{var}(\hat{\lambda}) & \mathrm{Cov}(\hat{\lambda},\hat{k}) & \mathrm{Cov}(\hat{\lambda},\hat{\alpha}) & \mathrm{Cov}(\hat{\lambda},\hat{\beta}) \\ \mathrm{Cov}(\hat{k},\hat{\lambda}) & \mathrm{var}(\hat{k}) & \mathrm{Cov}(\hat{k},\hat{\alpha}) & \mathrm{Cov}(\hat{k},\hat{\beta}) \\ \mathrm{Cov}(\hat{\alpha},\hat{\lambda}) & \mathrm{Cov}(\hat{\alpha},\hat{k}) & \mathrm{var}(\hat{\alpha}) & \mathrm{Cov}(\hat{\alpha},\hat{\beta}) \\ \mathrm{Cov}(\hat{\beta},\hat{\lambda}) & \mathrm{Cov}(\hat{\beta},\hat{k}) & \mathrm{Cov}(\hat{\beta},\hat{\alpha}) & \mathrm{var}(\hat{\beta}) \end{pmatrix}$$

The approximate confidence intervals (ACI) of $(1-\delta)100\%$ for the parameters are

$$\hat{\Theta} \pm z_{\delta/2}\sqrt{\mathrm{Var}(\hat{\Theta})}, \quad \hat{\Theta} = (\hat{\lambda},\hat{k},\hat{\alpha},\hat{\beta})$$

where $\mathrm{Var}(\hat{\Theta})$ are the variances of $\hat{\lambda}$, $\hat{k}$, $\hat{\alpha}$, and $\hat{\beta}$, given by the diagonal elements of $I^{-1}(\hat{\Theta})$, and $z_{\delta/2}$ is the upper $(\delta/2)$ percentile of the standard normal distribution.

4.9.2 Least squares estimators

Suppose $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are ordered observations from the θ-WMWLx distribution with CDF $F_\theta(\theta,\lambda,k,\alpha,\beta)$. Then, the least squares estimators (LSE), as described by Swain et al. [30], are obtained by minimizing

$$L(\theta,\lambda,k,\alpha,\beta) = \sum_{i=1}^{n}\left[F_\theta\left(x_{(i)}|\theta,\lambda,k,\alpha,\beta\right) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\left[\left(1-e^{-(1-\theta)(\lambda x)^k}(1+\beta x)^{-\theta\alpha}\right) - \frac{i}{n+1}\right]^2 \quad (19)$$

with respect to the parameters λ, k, α, and β.

4.9.3 Weighted least squares estimators

Suppose $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are ordered observations from the θ-WMWLx distribution with CDF $F_\theta(\theta,\lambda,k,\alpha,\beta)$. Then, the weighted least squares estimators (WLSE), as described by Swain et al. [30], are obtained by minimizing

$$WL(\theta,\lambda,k,\alpha,\beta) = \sum_{i=1}^{n}\frac{1}{\mathrm{Var}\left[F_\theta\left(x_{(i)}|\theta,\lambda,k,\alpha,\beta\right)\right]}\left[F_\theta\left(x_{(i)}|\theta,\lambda,k,\alpha,\beta\right) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left(1-e^{-(1-\theta)(\lambda x)^k}(1+\beta x)^{-\theta\alpha}\right) - \frac{i}{n+1}\right]^2 \quad (20)$$

with respect to the parameters λ, k, α, and β.

4.10 A parsimonious special case: 1/2-WMWLx1

In this subsection, we consider a special case of the θ-WMWLx model by setting $\theta = 1/2$ and $\beta = 1$. This particular case results in

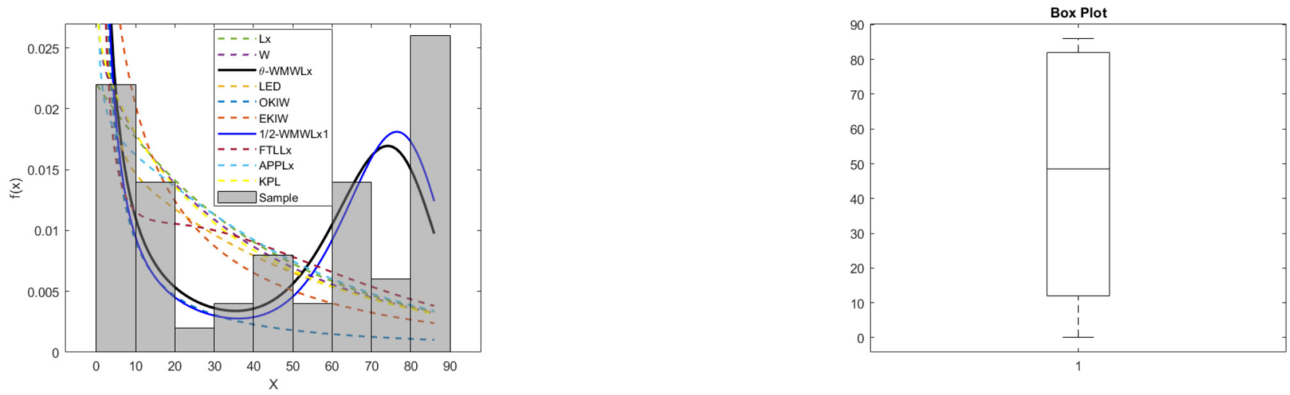

FIGURE 7
Fitted densities and box plot of the θ-WMWLx model for lifetime of a device.

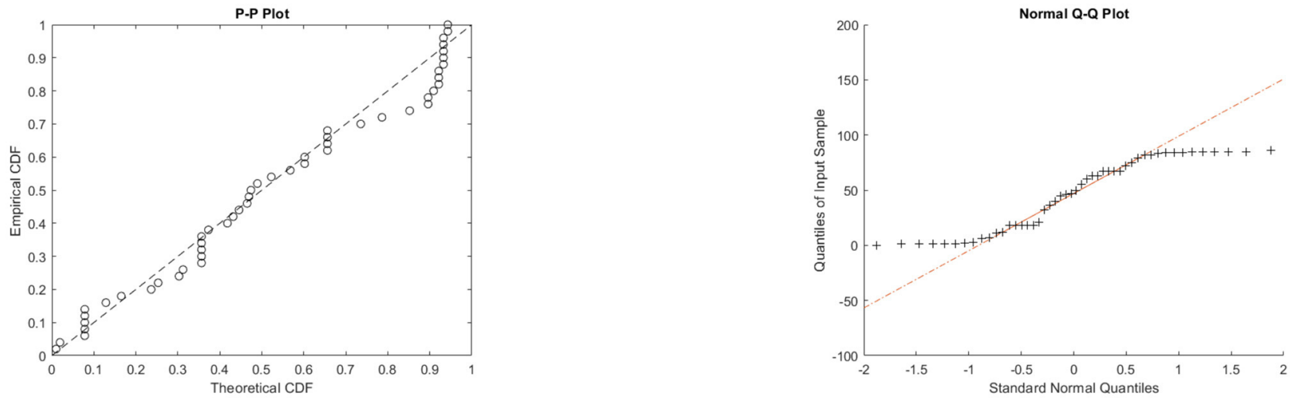

FIGURE 8
The P-P and Q-Q plots of the θ-WMWLx model for lifetime of a device.

a simplified form of the PDF, defined as

$$f(x|\lambda,k,\alpha)=\frac{1}{2}e^{-(1/2)(\lambda x)^k}(1+x)^{-(1/2)\alpha}\left[\lambda k(\lambda x)^{k-1}+\alpha(1+x)^{-1}\right].$$

The choice of $\theta = 1/2$ and $\beta = 1$ is particularly interesting for several reasons:

Equivalent weight in the mixture: Setting $\theta = 1/2$ grants equal weight to the reference distributions in the resulting mixture. This means that the influence of both distributions is balanced, which can simplify the analysis and provide a clearer representation of how each component contributes to the final model.

Model parsimony: Choosing $\beta = 1$ simplifies the model structure by reducing the number of parameters affecting the scale of the distribution. With $\beta = 1$, the parameter λ takes on a more central role as the sole parameter affecting the scale, favoring the parsimony of the resulting model. This parsimony can be beneficial in applications where simpler and more efficient models are preferred without the need for multiple parameters for scale or distribution shape.

Previous studies have shown inconsistencies in the maximum likelihood estimator (ML) and high estimation errors in practical applications. These issues may indicate challenges associated with model identifiability and complexity arising from multiple parameters of scale and shape. Presenting the parsimonious model 1/2-WMLx1 offers a robust and straightforward solution. This model maintains the necessary flexibility for practical applications while reducing complexity and potentially improving the stability and accuracy of estimates. Figures 5, 6 show the plots of the PDF, hazard, mean residual life, and mean inactivity time functions for the 1/2-WMWLx1 distribution with various parameter combinations. Table 4 displays the fundamental statistics of the 1/2-WMWLx1 distribution with $\alpha = 2$, $\lambda = 1$, and $k = 2$.

5 Simulation and application

In this section, we analyze the performance of the proposed model through simulation and its application on two sets of real-world data.

5.1 Simulation

Simulations allow us to visualize the behavior of the proposed model. In this case, we conducted a simulation to analyze the

TABLE 9 MLEs for the second application.

Models	$\hat{\theta}$	$\hat{\lambda}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$
θ-WMWLx	0.87932	0.72072	2.7614	0.015934	10.9032
	(0.1912)	(0.3885)	(0.3041)	(0.0231)	(0.31698)
1/2-WMWLx1	–	0.420	2.850	0.126	–
	–	(0.027)	(0.305)	(0.086)	–
W	–	–	–	2.8629	2.3744
	-	-	–	(0.1377)	(0.2106)
MOPLx	133.9809	66.8539	–	166.3862	0.76174
	(27.7244)	(166.8602)	–	(409.398)	(0.0486)
OKIW	0.8264	1.7365	1693.0013	1907.1241	0.23285
	(0.2763)	(0.3795)	(2038.8)	(2299.7)	(0.0171)
EKIW	0.75686	1.1504	259.8279	227.994	0.36439
	0.0916	(0.1541)	34.199	(30.0648)	(0.0278)
Lx	–	312.0618	–	122.2683	–
	–	(2.3216)	–	(5.236)	–
FTLx	0.3382	1.3688	1.2366e-5	0.61502	1.8457e-7
	(1953.79)	(4246.24)	(7.2584)	(469.0017)	(0.1519)
KPL	0.44752	3603.3834	0.26643	313.0526	3.2451
	(0.10179)	(76.5271)	(0.029143)	(6.6797)	(0.074019)
LIEP	3.1993	0.064092	–	0.69674	4.745e-6
	(0.19965)	(0.011527)	–	(0.1735)	(0.6214)

TABLE 10 Goodness-of-fit statistics and information criteria for the second application.

Models	AIC	CAIC	HQIC	BIC	D	A^*	W^*
θ-WMWLx	262.5444	262.929	267.4302	274.6985	0.0875	0.7417	0.08068
1/2-WMWLx1	259.0392	259.1892	261.9707	266.3317	0.090086	0.77052	0.086364
W	264.1067	264.1807	266.0610	268.9683	0.76149	1.6078	232.2154
MOPLx	265.1646	265.4177	269.0732	274.8878	0.98808	5.2035	28
OKIW	272.3116	272.6962	277.1974	284.4657	0.95463	3.12586	38.2645
EKIW	294.2124	294.597	299.0982	306.3665	1.3253	1520.1358	38.236
Lx	330.3101	330.3842	332.2644	344.0334	0.11905	1.1204	0.28345
FTLx	399.0807	399.4653	403.9665	411.2348	0.30931	11.9305	2.1958
KPL	273.4634	273.8480	278.3492	285.6175	0.95788	−81.81499	27.151
LIEP	284.6981	284.9512	288.6067	294.4213	0.13569	1.9226	0.19922

performance of the maximum likelihood estimator (MLE), least squares estimators (LSE), and weighted least squares estimators (WLSE) in terms of standard errors (SE),and mean squared error (MSE) for different sample sizes and various parameter values. We generated 1,000 replications using different parameter sets with sample sizes of $n = 20$, $n = 50$, $n = 100$, and $n = 500$ from the θ-WMWLx distribution. The pseudo-random numbers are generated using the exponential distribution inverse transform method, where each number x_i is sampled from an exponential distribution with rate parameter λ_{true}. MLE, LSE, and WLS were computed using the `fminsearch` function provided by MATLAB version R2024a to minimize the expressions defined in Equations (18–20) with respect to the parameters λ, k, α, and β.

The simulation results (see Tables 5, 6) indicate that the MLE method showed superior performance in terms of standard error (SE) and mean squared error (MSE) for all parameter combinations. The LSE and WLSE methods exhibited higher MSE, and their estimates were less accurate. These results highlight the robustness of MLE under various conditions.

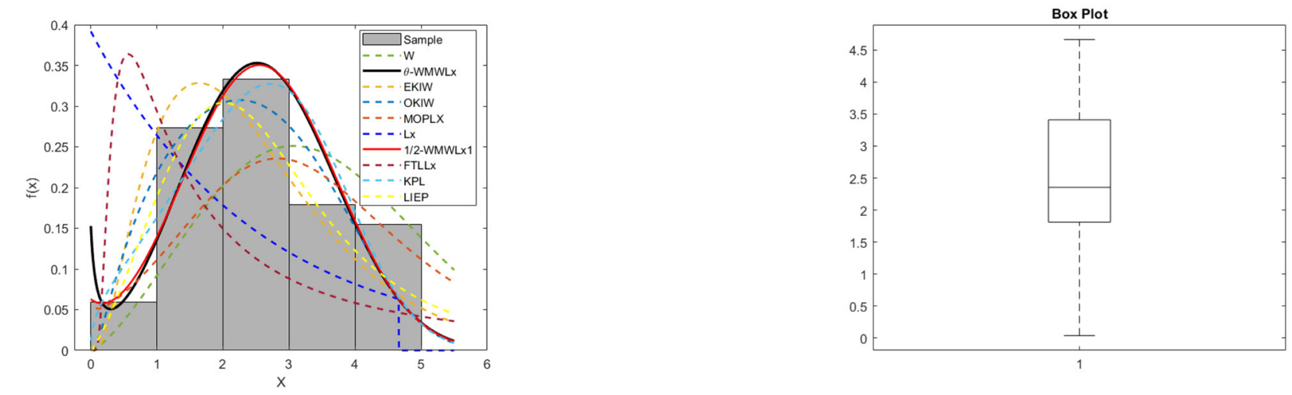

FIGURE 9
Fitted densities and box plot of the θ-WMWLx model for aircraft windshield failure time.

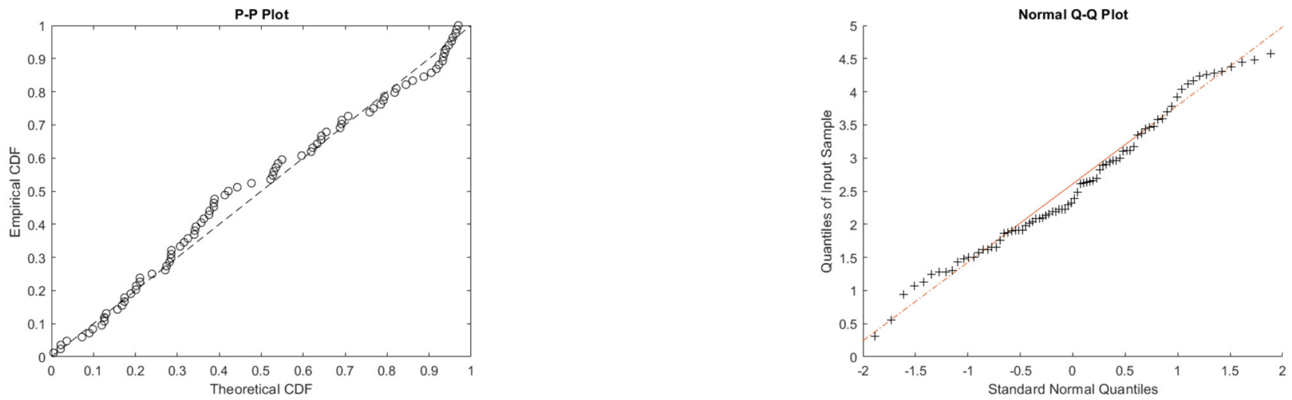

FIGURE 10
P-P and Q-Q plots of the θ-WMWLx model for aircraft windshield failure time.

5.2 Applications on real data

In this section, we study the behavior of the θ-WMWLx distribution applied to three real datasets. We compute the maximum likelihood estimates (MLEs) for the parameters using MATLAB version R2024a's `fmincon` function and compare the fit with the Weibull (W), exponentiated Kumaraswamy inverse Weibull (EKIW) [31], odd Kumaraswamy inverse Weibull (OKIW) [32], Marshall-Olkin power Lomax (MOPLx) [33], Lomax-exponential distribution (LED) [34], Lomax-Rayleigh (LR) [35], Fréchet Topp-Leone Lomax (FTLLx) [36], alpha-power power-Lomax (APPLx) [37], Kumaraswamy generalized power Lomax (KPL) [38], Lomax-inverse exponential power (LIEP) [39], special case (1/2-WMWLx1), and Lomax (Lx) distributions. Below, we present the probability density functions of the mentioned distributions:

1. w: $f(x|\beta,\alpha) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-(x/\alpha)^{\beta}}$.
2. EKIW: $f(x|\beta,\lambda,\theta,k,\alpha) = \beta\lambda\theta k\alpha^{\beta}x^{-(\beta+1)}e^{(-\lambda u)}\left[1-e^{(-\lambda u)}\right]^{k-1}\left[1-\left(1-e^{-\lambda u}\right)^{k}\right]^{\theta-1}$, where $u=(\alpha/x)^{\beta}$.
3. OKIW: $f(x|\Theta) = \beta k\lambda\theta\alpha^{\beta}x^{-(\beta+1)}\left[1-e^{1-\left(1-e^{-\lambda u}\right)^{-k}}\right]^{\theta-1} e^{1-\left(1-e^{-\lambda u}\right)^{-k}-\lambda u}\left(1-e^{-\lambda u}\right)^{-k-1}$, where $\Theta=(\beta,k,\lambda,\theta,\alpha)$, and $u=(\alpha/x)^{\beta}$.
4. MOPLx: $f(x|\theta,\alpha,\beta,\lambda) = \frac{\theta\alpha\beta\lambda^{\alpha}x^{\beta-1}(\lambda+x^{\beta})^{-\alpha-1}}{(1-(1-\theta)\lambda^{\alpha}(\lambda+x^{\beta})^{-\alpha})^{2}}$.
5. LED: $f(x|\lambda,k,\alpha) = \dfrac{\lambda k\alpha^{k}e^{\lambda x}}{\left(e^{\lambda x}+\alpha-1\right)^{k+1}}$.
6. LR: $f(x|\alpha,\lambda) = \dfrac{2\lambda\alpha^{\lambda}x}{(\theta+x^{2})^{\lambda+1}}$.
7. FTLLx:

$$f(x|\alpha,\beta,\theta,k,\lambda) = 2\alpha\beta^{\alpha}\lambda\theta k(1+kx)^{-2\theta-1}\left(1-(1+kx)^{-2\theta}\right)^{-\lambda\alpha-1}\left(1-\left(1-(1+kx)^{-2\theta}\right)^{\lambda}\right)^{\alpha-1}\times \exp\left\{-\beta^{\alpha}\left(\frac{1-(1-(1+kx)^{-2\theta})^{\lambda}}{(1-(1+kx)^{-2\theta})^{\lambda}}\right)^{\alpha}\right\}$$

8. APPLx:

$$f(x|\alpha,\beta,\lambda,\theta) = \begin{cases} \dfrac{\log\alpha}{\alpha-1}\theta\beta\lambda^{\theta}x^{\beta-1}(\lambda+x^{\beta})^{-\theta-1}\alpha^{1-\lambda^{\theta}(\lambda+x^{\beta})^{-\theta}}, & \text{if,}\quad \alpha\neq 1,\alpha>0,\\ \theta\beta\lambda^{\theta}x^{\beta-1}(\lambda+x^{\beta})^{-\theta-1}, & \text{if,}\quad \alpha=1. \end{cases}$$

TABLE 11 MLEs for the third application.

Models	$\hat{\theta}$	$\hat{\lambda}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$
θ-WMWLx	0.0070987	1.7046	3.9553	78.7351	0.95014
	(0.038911)	(0.0946)	(0.5467)	(49.3484)	(6.2373)
1/2-WMWLx1	–	2.0286	3.9521	1.0661	–
	–	(0.10044)	(0.44333)	(0.36614)	–
W	–	–	–	0.5236	2.6012
	–	–	–	(0.0202)	(0.2107)
MOPLx	12.8395	26.3304	–	214.4599	1.5263
	(8.7693)	(14.1765)	–	(113.5197)	(0.28859)
OKIW	0.65756	0.63636	178.1179	139.8408	0.39516
	(0.0745)	(0.0237)	(20.5559)	(16.1875)	(0.00624)
EKIW	0.62444	46.759	19.6656	0.0069252	0.61243
	(0.98564)	(3.0111)	(1.3263)	(0.00646)	(0.01538)
Lx	–	76.0454	–	162.4385	–
	–	(10.3262)	–	(3.25691)	–
LR	–	182.8232	–	46.7524	–
	–	(458.0913)	–	(117.4975)	–
FTLLx	2700.9811	0.48183	0.0037457	0.24526	3405.4494
	(371.1891)	(0.0868)	(0.0006)	(0.01955)	(468.3646)
APPLx	2.7601	8.888	–	5.9397e-9	2.6226
	(23.3102)	(61.7939)	–	(3.46e-6)	(1.0338)
KPL	0.20242	0.5	1.0125	9.2022	8.7908
	(1.6823)	(3.2578)	(6.3793)	(2.7534)	(0.11828)
LIEP	0.28571	1604.4225	–	0.52375	13007.4722
	(0.013858)	(587.98)	–	(0.062248)	(4779.78)

9. KPL: $f(x|\alpha,\beta,\lambda,\theta,k) = \frac{\theta k\alpha\beta}{\lambda}x^{\beta-1}\left(\frac{\lambda}{\lambda+x^{\beta}}\right)^{\alpha+1}\left(1-\left(\frac{\lambda}{\lambda+x^{\beta}}\right)^{\alpha}\right)^{\theta-1}\left(1-\left(1-\left(\frac{\lambda}{\lambda+x^{\beta}}\right)^{\alpha}\right)^{\theta}\right)^{k-1}$
10. LIEP: $f(x|\alpha,\beta,\lambda,\varpi) = \alpha\beta^{\alpha}\varpi\lambda^{\varpi}x^{-(\varpi+1)}e^{\left(\frac{\lambda}{x}\right)^{\varpi}}\left[\beta-1+e^{\left(\frac{\lambda}{x}\right)^{\varpi}}\right]^{-(\alpha+1)}$.
11. Lx: $f(x|\alpha,\lambda) = \frac{\alpha\lambda^{\alpha}}{(x+\lambda)^{\alpha+1}}$.

The model selection is based on the Akaike information criterion (AIC), the Bayesian information criterion (BIC), the consistent Akaike information criterion (CAIC), and the Hannan-Quinn information criterion (HQIC). In addition, as a measure of the goodness of fit, we consider the Anderson–Darling ($A^{\star}$), Cramer–von Mises ($W^{\star}$), and Kolmogorov–Smirnov (D) statistics.

5.2.1 First application: *lifetime of a device*

The data for this initial application consisted of 50 observations representing the lifetime of a device (see Aarset [40] for more details on the data). Here, we will compare the fits of the θ-WMWLx distribution with those of other competitive models, namely, Lx, W, LED, OKIW, EKIW, FTLLx, APPLx, KPL, and 1/2-WMWLx1. Table 7 shows the MLEs for the parameters, along with their corresponding standard errors (in parentheses), while Table 8 presents the values of the information criteria AIC, BIC, CAIC, and HQIC, as well as the goodness-of-fit statistics for the models. Figures 7, 8 illustrate the fitted densities, the boxplot, and the P–P and Q–Q plots.

5.2.2 Second application: *aircraft windshield failure data*

In this second application, we will use data on aircraft windshield failure times. This dataset contains 84 observations and is available in the study of Murthy et al. [41]. Tables 9, 10 show the MLEs for the parameters, along with their corresponding standard errors (in parentheses), and the goodness-of-fit statistics for the models: θ-WMWLx, W, OKIW, EKIW, MOPLx, Lx, FTLLx, KPL, LIEP, and 1/2-WMWLx1. In addition, Figures 9, 10 illustrate the fitted densities, the boxplot, and the P–P and Q–Q plots.

TABLE 12 Goodness-of-fit statistics and information criteria for the third application.

Models	AIC	CAIC	HQIC	BIC	D	A^*	W^*
θ-WMWLx	–49.5458	–49.2488	–44.1282	–36.1817	0.05318	0.36059	0.031241
1/2-WMWLx1	–53.5458	–53.4293	–50.2952	–45.5273	0.0532	0.36058	0.031255
W	–38.6950	–38.6373	–36.5280	–33.3494	3.2146	2393.5683	0.9776
MOPLx	–45.7155	–45.5194	–41.3814	–35.0242	0.0580	0.51431	0.0344
OKIW	–29.3034	–29.0064	–23.8858	–15.9393	0.10908	2.4512	0.457
EKIW	21.0026	21.2996	26.4203	34.3668	1.92503	228.230	42.3256
Lx	56.4500	56.5077	58.6171	71.1413	–5.0212	358.026	1.9502
LR	–28.9356	–28.8779	–6.7685	–23.5899	0.14108	3.8318	0.76977
FTLLx	–16.7707	–16.4736	–11.353	–3.4065	0.10494	2.9417	0.46645
APPLx	–31.8947	–31.6986	-27.5606	–21.2034	0.090001	2.0316	0.28921
KPL	–45.0621	–44.7651	–39.6445	–31.6980	0.9877	–104.82	34.4056
LIEP	16.7305	16.9266	21.0646	27.4218	0.17291	6.385	1.0671

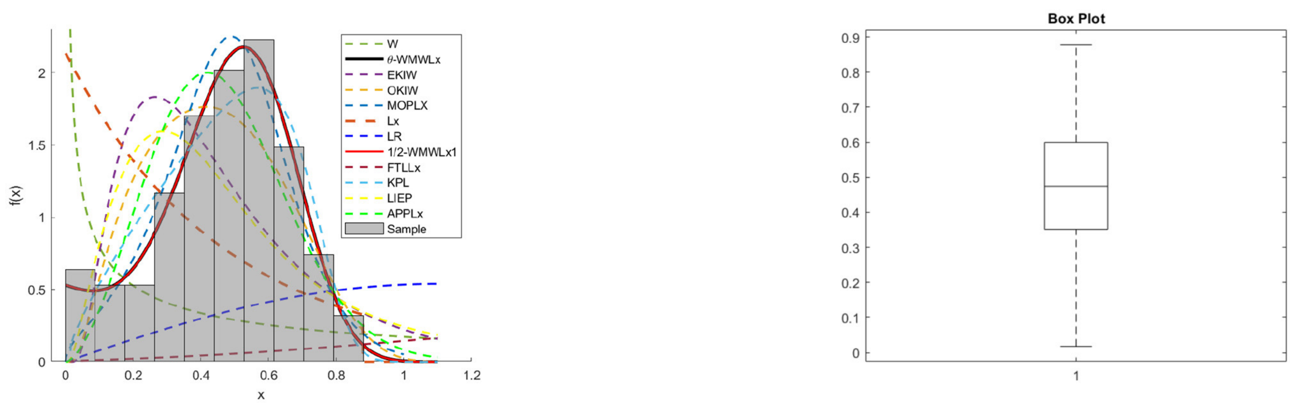

FIGURE 11
Fitted densities and box plot of the θ-WMWLx model for milk production.

5.2.3 Third application: *milk production*

In this application, we utilized a dataset concerning the total milk production in the first birth of 107 cows from SINDI race. Please refer to Yousof et al. [42]. Tables 11, 12 show the MLEs for the parameters, accompanied by their respective standard errors (in parentheses), and the goodness-of-fit statistics for the following models: θ-WMWLx, W, OKIW, EKIW, MOPLx, LR, Lx, FTLLx, APPLx, KPL, LIEP, and 1/2-WMWLx1. In addition, Figure 11 and Table 12 illustrate the fitted densities, the box plot, as well as the P-P and Q-Q plots.

The results presented in Tables 8, 10, 12 indicate that the θ-WMWLx distribution consistently shows the lowest values in information criteria (AIC, BIC, CAIC, and HQIC) compared to its competitor counterparts. This pattern is equally reflected in the goodness-of-fit statistics, where the θ-WMWLx distribution excels. Therefore, we can conclude that the θ-WMWLx distribution offers the most optimal fit for these three datasets.

On the other hand, the graphs in Figures 7–12 reveal that the θ-WMWLx distribution fits more accurately to the three datasets studied compared to other competing models. This observation further reinforces the superiority of the θ-WMWLx distribution in terms of fitting capability and predictive accuracy.

6 Conclusion and comments

The research introduces the family of θ-weighted mixture distributions, offering a flexible approach to modeling the joint distribution of random variables. This family combines survival functions from baseline distributions, resulting in diverse shapes and characteristics determined by the parameters' behavior. A specific case, θ-WMWLx, was investigated, combining the survival functions of the Weibull and Lomax distributions, revealing a spectrum of shapes, from symmetric to asymmetric, with various biases.

Statistical properties and reliability aspects of θ-WMWLx were examined, demonstrating that an increase in sample size enhances the precision of the maximum likelihood estimator (MLE). The

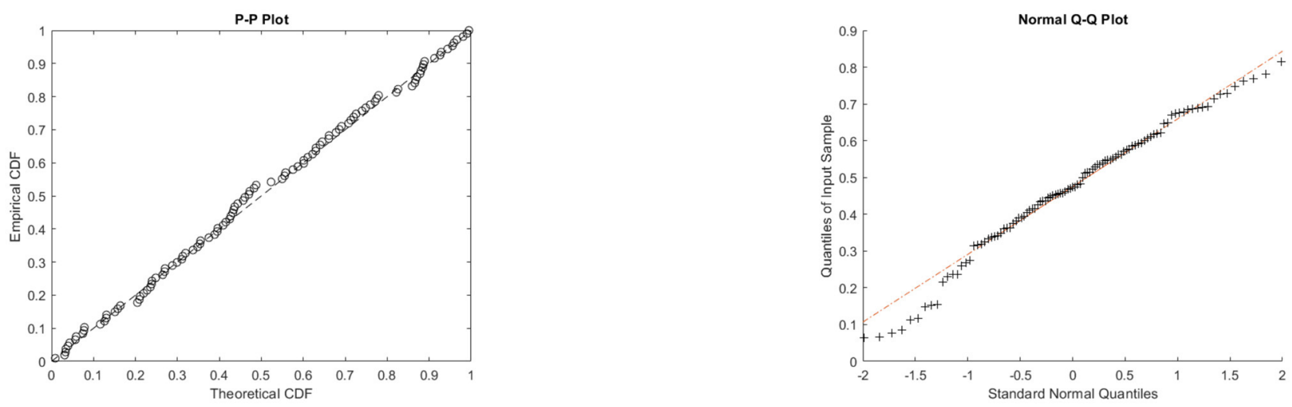

FIGURE 12
P-P and Q-Q plots of the θ-WMWLx model for milk production.

significance of the parameter θ in shaping and fitting the resulting distribution is underscored, posing a future challenge to explore methods for determining its optimal value, which could enhance the utility of this methodology.

A notable special case is the 1/2-WMWLx$_1$, which simplifies the model structure by balancing the weight of the baseline distributions and reducing the number of parameters affecting the scale. This parsimony can be beneficial in practical applications by improving the stability and accuracy of the estimates, facilitating a clearer and more efficient analysis. This was reflected in the application section of the study, where this special case, along with the general case, proved to be the best in terms of performance.

In addition to the aforementioned aspects, several additional research challenges could be addressed in future studies. One such challenge could involve generalizing the proposed model to accommodate a broader range of baseline distributions, allowing for greater flexibility in modeling various data types. Furthermore, a more in-depth analysis of the survival and hazard functions using the θ-WMWLx could be conducted, exploring how these functions vary across different parameter settings and how they can contribute to a better understanding of data behavior in diverse scenarios. These endeavors could advance our comprehension of the joint distribution of random variables and foster the development of more effective tools for their modeling and analysis across various applied contexts.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

CC-M: Writing – original draft, Writing – review & editing. RM: Writing – original draft, Writing – review & editing. EC: Writing – original draft, Writing – review & editing.

Funding

The author(s) declare financial support was received for the research, authorship, and/or publication of this article. This work was supported by Universidad de Playa Ancha, Plan de Fortalecimiento Universidades Estatales - Ministerio de Educación, convenio UPA 1999.

Acknowledgments

The authors would like to thank the Support Network of the General Directorate of Research for their cooperation in reviewing this article.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

All claims expressed in this article are solely those of the authors and do not necessarily represent those of their affiliated organizations, or those of the publisher, the editors and the reviewers. Any product that may be evaluated in this article, or claim that may be made by its manufacturer, is not guaranteed or endorsed by the publisher.

References

1. Pearson K. Contributions to the mathematical theory of evolution. *Philos Trans R Soc Lond A*. (1894) 185:71–110. doi: 10.1098/rsta.1894.0003

2. Amoroso L. Ricerche intorno alla curva dei redditi. *Ann Mat Pura Appl*. (1925) 2:123–59. doi: 10.1007/BF02409935

3. Burr IW. Cumulative frequency functions. *Ann Math Stat*. (1942) 13:215–32. doi: 10.1214/aoms/1177731607

4. Johnson NL. Systems of frequency curves generated by methods of translation. *Biometrika*. (1949) 36:149–76. doi: 10.1093/biomet/36.1-2.149

5. Marshall AW, Olkin I. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*. (1997) 84:641–52. doi: 10.1093/biomet/84.3.641

6. Okasha HM, Kayid M. A new family of Marshall-Olkin extended generalized linear exponential distribution. *J Comput Appl Math*. (2016) 296:576–92. doi: 10.1016/j.cam.2015.10.017

7. Jose KK, Krishna E. Marshall-Olkin extended uniform distribution. In: *ProbStat Forum, Vol. 4*. (2011), p. 78–88.

8. Sengweni W, Oluyede B, Makubate B. The Marshall-Olkin Topp-Leone Half-Logistic-G family of distributions with applications. *Stat Optim Inf Comput*. (2023) 11:1001–26. doi: 10.19139/soic-2310-5070-1082

9. Handiquea L, Chakraborty S, El-Morshedy M, Afifyd AZ, Eliwa MS. Modelling veterinary medical data utilizing a new generalized Marshall-Olkin transmuted generator of distributions with statistical properties. *Thail Stat*. (2024) 22:219–36.

10. Shaw WT, Buckley IR. The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. *arXiv*. (2009). [Preprint]. arXiv:09010434. doi: 10.48550/arXiv:09010434

11. Afify AZ, Al-Mofleh H, Aljohani HM, Cordeiro GM. The Marshall-Olkin-Weibull-H family: estimation, simulations, and applications to COVID-19 data. *J King Saud Univ-Sci*. (2022) 34:102115. doi: 10.1016/j.jksus.2022.102115

12. Barreto-Souza W, Lemonte AJ, Cordeiro GM. General results for the Marshall and Olkin's family of distributions. *An Acad Bras Ciênc*. (2013) 85:3–21. doi: 10.1590/S0001-37652013000100002

13. Souza L, Junior W, De Brito C, Chesneau C, Ferreira T, Soares L. On the Sin-G class of distributions: theory, model and application. *J Math Model*. (2019) 7:357–79. doi: 10.22124/jmm.2019.13502.1278

14. Shama MS, El Ktaibi F, Al Abbasi JN, Chesneau C, Afify AZ. Complete study of an original power-exponential transformation approach for generalizing probability distributions. *Axioms*. (2023) 12:67. doi: 10.3390/axioms12010067

15. Nadarajah S, Kotz S. The exponentiated type distributions. *Acta Applicandae Mathematica*. (2006) 92:97–111. doi: 10.1007/s10440-006-9055-0

16. Eugene N, Lee C, Famoye F. Beta-normal distribution and its applications. *Commun Stat-Theory Methods*. (2002) 31:497–512. doi: 10.1081/STA-120003130

17. Cordeiro GM, de Castro M. A new family of generalized distributions. *J Stat Comput Simul*. (2011) 81:883–98. doi: 10.1080/00949650903530745

18. Mahdavi A, Kundu D. A new method for generating distributions with an application to exponential distribution. *Commun Stat-Theory Methods*. (2017) 46:6543–57. doi: 10.1080/03610926.2015.1130839

19. Iriarte YA, de Castro M, Gómez HW. The Lambert-F distributions class: An alternative family for positive data analysis. *Mathematics*. (2020) 8:1398. doi: 10.3390/math8091398

20. Tahir MH, Cordeiro GM, Mansoor M, Zubair M. The Weibull-Lomax distribution: properties and applications. *Hacet J Math Stat*. (2015) 44:455–74. doi: 10.15672/HJMS.2014147465

21. Afify AZ, Nofal ZM, Yousof HM, El Gebaly YM, Butt NS. The transmuted Weibull Lomax distribution: properties and application. *Pak J Stat Oper Res*. (2015) XI:135–52. doi: 10.18187/pjsor.v11i1.956

22. Hassan AS, Abd-Allah M. Exponentiated Weibull-Lomax distribution: properties and estimation. *J Data Sci*. (2018) 16:277–98. doi: 10.6339/JDS.201804_16(2).0004

23. Ijaz M, Asim SM, Alamgir. Lomax exponential distribution with an application to real-life data. *PLoS ONE*. (2019) 14:e0225827. doi: 10.1371/journal.pone.0225827

24. Alzaghal A, Ghosh I, Alzaatreh A. On shifted Weibull-Pareto distribution. *Int J Stat Probab*. (2016) 5:139–49. doi: 10.5539/ijsp.v5n4p139

25. Kartsonaki C. Survival analysis. *Diagn Histopathol*. (2016) 22:263–70. doi: 10.1016/j.mpdhp.2016.06.005

26. Baredar P, Khare V, Nema S. Design and Optimization of Biogas Energy Systems. Cambridge, MA: Academic Press (2020). doi: 10.1016/B978-0-12-822718-3.00001-0

27. Kayid M, Al-Nahawati H, Ahmad IA. Testing behavior of the reversed hazard rate. *Appl Math Model*. (2011) 35:2508–15. doi: 10.1016/j.apm.2010.11.054

28. Alshangiti AM, Kayid M, Alarfaj B. A new family of Marshall-Olkin extended distributions. *J Comput Appl Math*. (2014) 271:369–79. doi: 10.1016/j.cam.2014.04.020

29. Schafer RE. *Statistical Models and Methods for Lifetime Data*. New York, NY: Taylor & Francis (1983). doi: 10.2307/1267739

30. Swain JJ, Venkatraman S, Wilson JR. Least-squares estimation of distribution functions in Johnson's translation system. *J Stat Comput Simul*. (1988) 29:271–97. doi: 10.1080/00949658808811068

31. Rodrigues J, Silva APCM, Hamedani G. The exponentiated Kumaraswamy inverse Weibull distribution with application in survival analysis. *J Stat Theory Appl*. (2016) 15:8–24. doi: 10.2991/jsta.2016.15.1.2

32. Atem BAM. On the odd Kumaraswamy inverse Weibull distribution with application to survival data. *JKUAT*. (2018). doi: 10.17654/AS0510 50309

33. Ul Haq MA, Hamedani G, Elgarhy M, Ramos PL. Marshall-Olkin Power Lomax distribution: Properties and estimation based on complete and censored samples. *Int J Stat Probab*. (2020). doi: 10.5539/ijsp.v9n1p48

34. Hami Golzar N, Ganji M, Bevrani H. The Lomax-exponential distribution, some properties and applications. *J Stat Res Iran JSRI*. (2017) 13:131–53. doi: 10.18869/acadpub.jsri.13.2.131

35. Venegas O, Iriarte YA, Astorga JM, Gómez HW. Lomax-Rayleigh distribution with an application. *Appl Math Inf Sci*. (2019) 13:741–8. doi: 10.18576/amis/130506

36. Reyad H, Korkmaz MÇ, Afify AZ, Hamedani G, Othman S. The Fréchet Topp Leone-G family of distributions: Properties, characterizations and applications. *Ann Data Sci*. (2021) 8:345–66. doi: 10.1007/s40745-019-00212-9

37. Qura ME, Alqawba M, Al Sobhi MM, Afify AZ. A novel extended power-Lomax distribution for modeling real-life data: properties and inference. *J Math*. (2023) 2023:6661792. doi: 10.1155/2023/66 61792

38. Nagarjuna VB, Vardhan RV, Chesneau C. Kumaraswamy generalized power Lomax distributionand its applications. *Stats*. (2021) 4:28–45. doi: 10.3390/stats4010003

39. Sapkota LP, Kumar V, Gemeay AM, Bakr M, Balogun OS, Muse AH. New Lomax-G family of distributions: statistical properties and applications. *AIP Adv*. (2023) 13:095128. doi: 10.1063/5.0171949

40. Aarset MV. How to identify a bathtub hazard rate. *IEEE Trans Reliab*. (1987) 36:106–8. doi: 10.1109/TR.1987.5222310

41. Murthy DP, Xie M, Jiang R. *Weibull Models*. Hoboken, NJ: John Wiley and Sons, Inc. (2004).

42. Yousof HM, Alizadeh M, Jahanshahi S, Ramires TG, Ghosh I, Hamedani G. The transmuted Topp-Leone G family of distributions: theory, characterizations and applications. *J Data Sci*. (2017) 15:723–40. doi: 10.6339/JDS.201710_15(4). 00008