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EDITED BY
Marina Chugunova,
Claremont Graduate University, United States

REVIEWED BY
Youssri Hassan Youssri,
Cairo University, Egypt
Dudkin Mykola,
Kyiv Polytechnic Institute, Ukraine

*CORRESPONDENCE
Ivan Kovalyov
i.m.kovalyov@gmail.com

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Darboux transformation of symmetric Jacobi matrices and Toda lattices

Ivan Kovalyov[1]* and Oleksandra Levina[2]

[1]Institut für Mathematik, Universität Osnabrück, Osnabrück, Germany, [2]Faculty of Mathematics, Informatics and Physics, Mykhailo Drahomanov Ukrainian State University, Kyiv, Ukraine

Let J be a symmetric Jacobi matrix associated with some Toda lattice. We find conditions for Jacobi matrix J to admit factorization $J = LU$ (or $J = \mathfrak{U}\mathfrak{L}$) with L (or $\mathfrak{L}$) and U (or $\mathfrak{U}$) being lower and upper triangular two-diagonal matrices, respectively. In this case, the Darboux transformation of J is the symmetric Jacobi matrix $J^{(p)} = UL$ (or $J^{(d)} = \mathfrak{L}\mathfrak{U}$), which is associated with another Toda lattice. In addition, we found explicit transformation formulas for orthogonal polynomials, **m**-functions and Toda lattices associated with the Jacobi matrices and their Darboux transformations.

KEYWORDS

Jacobi matrix, Darboux transformation, orthogonal polynomials, moment problem, Toda lattice

1 Introduction

Let a sequence of real numbers $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ be associated with a measure μ on $(-\infty, +\infty)$, i.e.

$$s_n = \int_{-\infty}^{+\infty} \lambda^n d\mu(\lambda), \quad n \in \mathbb{Z}_+.$$

However, in the general case, $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ is associated with a linear functional $\mathfrak{S}$ by

$$s_n = \mathfrak{S}(\lambda^n), \quad n \in \mathbb{Z}_+. \tag{1.1}$$

We consider the sequence $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ such that

$$D_n \neq 0, \quad \text{for all } n \in \mathbb{N},$$

where $D_n = \det(s_{i+j})_{i,j=0}^{n-1}$. Note, if $D_n > 0$ for all $n \in \mathbb{N}$, then there exists measure μ associated with $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$, otherwise, the sequence $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ is associated with only linear functional $\mathfrak{S}$.

On the other hand (see [1, 2]), the real sequence $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ is associated with the symmetric Jacobi matrix J and the sequence of orthogonal polynomials of the first kind $\{P_n(\lambda)\}_{n=0}^{\infty}$, which can defined by

$$P_0(\lambda) \equiv 1 \quad \text{and} \quad P_n(\lambda) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & \dots & s_{2n-1} \\ 1 & \lambda & \dots & \lambda^n \end{vmatrix}.$$

[3, 4] Moreover, the sequence $\{P_n(\lambda)\}_{n=0}^{\infty}$ satisfies a three-term recurrence relation

$$\lambda P_n(\lambda) = a_{n+1}P_{n+1}(\lambda) + b_nP_n(\lambda) + a_nP_{n-1}(\lambda) \tag{1.2}$$

with the initial conditions

$$P_{-1}(\lambda) \equiv 0 \quad \text{and} \quad P_0(\lambda) \equiv 1. \tag{1.3}$$

In the short form we can rewrite Equation (1.2) as

$$JP(\lambda) = \lambda P(\lambda),$$

where $P(\lambda) = (P_0(\lambda), \ldots, P_n(\lambda), \ldots)^T$ and the symmetric Jacobi matrix J is defined by

$$J = \begin{pmatrix} b_0 & a_1 & & \\ a_1 & b_1 & a_2 & \\ & a_2 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{1.4}$$

On the other hand, the symmetric Jacobi matrix J is associated with the moment sequence $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$, the following relation holds (see [2, 5])

$$s_n = (e_0, J^n e_0) \quad \text{for all } n \in \mathbb{Z}_+, \tag{1.5}$$

where $e_0 = (1, 0, \ldots)^T$ and $\mathbf{m}-$ function of Jacobi matrix is found by

$$m(z) = \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda - z}. \tag{1.6}$$

There exist two type transformations of orthogonal polynomials, which are the Christoffel and Geronimus transformations. One are studied in the paper Zhedanov [6]. The Christoffel transformation is defined by

$$\tilde{P}(\lambda) = \frac{P_{n+1}(\lambda) - A_nP_n(\lambda)}{\lambda - \alpha}, \quad n \in \mathbb{Z}_+, \tag{1.7}$$

where $A_n = \dfrac{P_{n+1}(\alpha)}{P_n(\alpha)}$ and α is arbitrary parameter. Moreover, Equation (1.7) can be rewritten as follows:

Theorem 1.1. ([7, Theorem 1.5]) Let $\{P_n(\lambda)\}_{n=0}^{\infty}$ be the sequence of the orthogonal polynomials associated with Equation (1.2). Then the Christoffel–Darboux formula takes the following form

$$\sum_{i=0}^{n} P_i(x)P_i(t) = a_{n+1}\frac{P_{n+1}(x)P_n(t) - P_n(x)P_{n+1}(t)}{x - t}. \tag{1.8}$$

The second transformation is a Geronimus transformation of the orthogonal polynomials [6], one is defined by

$$\tilde{P}(\lambda) = P_n(\lambda) - B_nP_{n-1}(\lambda), \quad B_n \in \mathbb{R} \quad \text{and} \quad n \in \mathbb{N}.$$

Toda lattice. The Toda lattice is a system of differential equations

$$x_n''(t) = e^{x_{n-1}-x_n} - e^{x_n - x_{n+1}}, \quad n \in \mathbb{N}, \tag{1.9}$$

which was introduced in Toda [8].

We study the semi-infinite system with $x_{-1} = -\infty$. [9, 10] Flaschka variables are defined by

$$a_k = \frac{1}{2}e^{\frac{x_{k-1}-x_k}{2}} \quad \text{and} \quad b_k = -\frac{1}{2}x_k'. \tag{1.10}$$

Therefore, we obtain the following system in terms of Flaschka variables

$$a_k' = a_k(b_k - b_{k-1}) \quad \text{and} \quad b_k' = 2(a_{k+1}^2 - a_k^2), \quad a_0 = 0. \tag{1.11}$$

Hence, the semi-infinite Toda lattice is associated with the symmetric Jacobi matrix J and Lax pair (J, A), such that

$$[J, A] = JA - AJ,$$

where the matrix $A = J_+ - J_-$, where J_+ and J_- are upper and lower triangular part of J, respectively and

$$A = \begin{pmatrix} 0 & a_1 & & \\ -a_1 & 0 & a_2 & \\ & -a_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

As is known (see [8, 11]), the system (1.11) is equivalent to the following

$$J' = -[J, A].$$

Darboux transformation of the monic classical and generalized Jacobi matrices were studied in Bueno and Marcellán [12], Derevyagin and Derkach [13], and Kovalyov [14, 15]. Darboux transformation involves finding a factorization of a matrix from a certain class such that the new matrix is from the same class. There are two types of Darboux transformation: transformation with and without parameter. Jacobi matrix is associated with many objects. There are moment sequence, measure, linear functional orthogonal polynomials and Toda lattice. Hence, in the current paper, we study not only Darboux transformation of the symmetric Jacobi matrices, but we also study the transformation of the associated objects. Hence, we investigate the Darboux transformation of the symmetric Jacobi matrices J and find relations between associated Toda lattice, orthogonal polynomials, moment sequences and $\mathbf{m}$–functions. We obtain that the Darboux transformation without parameter of the symmetric Jacobi matrices has more additional existence conditions in contrast to case of the monic Jacobi matrices. On the other hand, the Darboux transformation with parameter of the symmetric Jacobi matrices is generated more easily. The results obtained can be applied for further research related to symmetric Jacobi matrices, Toda lattices and inverse problems. Of course, it can also be applied to the Toda lattice hierarchy.

Now, briefly describe the content of the paper. Section 2 contains Darboux transformation without parameter of the symmetric Jacobi matrix J. We find LU–factorization of J and the transformed matrix $J^{(p)}$. Relation between Toda lattices, moment sequences and $\mathbf{m}$–functions associated with the Jacobi matrices was obtained. In this case, the orthogonal polynomials are transformed by the Christoffel formula (1.7). In Section 3, we study the Darboux transformation with parameter of the symmetric Jacobi matrix J. We find $\mathfrak{UL}$–factorization of J and transformed matrix $J^{(d)}$. Moreover, the relations between orthogonal polynomials, **m**–functions, moment sequence and Toda lattices are found according to explicit formulas.

2 Darboux transformation without parameter of symmetric Jacobi matrix

Now we study a Darboux transformation without parameter of symmetric Jacobi matrix J. The goal is to find the transformations of polynomials of the first kind, **m**-functions, measure, moment sequence and Toda lattice, which are associated with the transformed Jacobi matrix.

2.1 *LU*–factorization

Lemma 2.1. Let J be a symmetric Jacobi matrix. Then J admits LU–factorization

$$J = LU, \tag{2.1}$$

where L and U are lower and upper triangular matrices, respectively, which are defined by

$$L = \begin{pmatrix} 1 & & & \\ l_1 & 1 & & \\ & l_2 & 1 & \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} u_1 & v_1 & & \\ & u_2 & v_2 & \\ & & u_3 & \ddots \\ & & & \ddots \end{pmatrix}, \tag{2.2}$$

if and only if the following system is solvable

$$\begin{aligned} &b_0 = u_1, \quad v_1 = a_1, \quad v_j = a_j, \quad l_j u_j = a_j, \\ &l_j v_j + u_{j+1} = b_j, \quad u_j \neq 0 \quad \text{and} \quad l_j \neq 0, \quad j \in \mathbb{N}. \end{aligned} \tag{2.3}$$

Proof. Let us calculate the product LU

$$LU = \begin{pmatrix} u_1 & v_1 & & \\ l_1 u_1 & l_1 v_1 + u_2 & v_2 & \\ & l_2 u_2 & l_2 v_2 + u_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

Comparing the product LU with the Jacobi matrix J

$$\begin{pmatrix} b_0 & a_1 & & \\ a_1 & b_1 & a_2 & \\ & a_2 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} u_1 & v_1 & & \\ l_1 u_1 & l_1 v_1 + u_2 & v_2 & \\ & l_2 u_2 & l_2 v_2 + u_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix},$$

we obtain the system (2.3).

If the system (2.3) is solvable, then J admits the factorization $J = LU$ of the form (2.1–2.3), where L and U are found uniquely. Conversely, if J admit LU—factorization then the system (2.3) is solvable. This completes the proof.

Lemma 2.2. Let J be the symmetric Jacobi matrix and let $J = LU$ be its LU– factorization of the form (2.1–2.3). Let P_j be the polynomials of the first kind associated with the matrix J. Then

$$\frac{P_n(0)}{P_{n-1}(0)} = -\frac{1}{l_n}, \qquad n \in \mathbb{N}. \tag{2.4}$$

Proof. Let J admit the LU–factorization of the form (2.1–2.3). Setting $\lambda = 0$ in Equation (1.2), we obtain

$$a_{n+1}P_{n+1}(0) + b_n P_n(0) + a_n P_{n-1}(0) = 0.$$

By induction, we prove Equation (2.4).

1. Let $n = 0$, then

$$a_1 P_1(0) + b_0 P_0(0) + a_0 P_{-1}(0) = 0$$

and due to the initial condition (1.3) and (2.3), we get

$$a_1 P_1(0) + b_0 P_0(0) = 0 \Rightarrow \frac{P_1(0)}{P_0(0)} = -\frac{b_0}{a_1} = -\frac{u_1}{l_1 u_1} = -\frac{1}{l_1}.$$

2. Let $n = 1$, then

$$a_2 P_2(0) + b_1 P_1(0) + a_1 P_0(0) = 0$$

and by Equation (2.3), we have

$$\begin{aligned} &\frac{P_2(0)}{P_1(0)} + \frac{b_1}{a_2} + \frac{a_1}{a_2}\frac{P_0(0)}{P_1(0)} = 0 \Rightarrow \frac{P_2(0)}{P_1(0)} \\ &= -\frac{b_1}{a_2} + \frac{a_1 l_1}{a_2} = \frac{-l_1^2 u_1 - u_2 + l_1^2 u_1}{l_2 u_2} = -\frac{1}{l_2}. \end{aligned}$$

3. Let Equation (2.4) hold for $n = k - 1$.
4. Let us prove Equation (2.4) for $n = k$, we obtain

$$a_{k+1}P_{k+1}(0) + b_n P_k(0) + a_k P_{k-1}(0) = 0.$$

$$\frac{P_{k+1}(0)}{P_k(0)} + \frac{b_k}{a_{k+1}} + \frac{a_k}{a_{k+1}} \cdot \frac{P_{k-1}(0)}{P_k(0)} = 0.$$

Consequently

$$\begin{aligned} \frac{P_{k+1}(0)}{P_k(0)} &= -\frac{b_k}{a_{k+1}} - \frac{a_k}{a_{k+1}} \cdot \frac{P_{k-1}(0)}{P_k(0)} = \{\text{by Section (2.3)}\} \\ &= \frac{-b_k + a_k l_k}{a_{k+1}} = \frac{-l_k^2 u_k - u_{k+1} + l_k^2 u_k}{l_{k+1} u_{k+1}} = -\frac{1}{l_{k+1}}. \end{aligned}$$

So, Equation (2.4) is proven. This completes the proof.

Corollary 2.3. Let J be the symmetric Jacobi matrix and let $J = LU$ be its LU–factorization of the form (2.1–2.3). Let P_j be the polynomials of the first kind associated with the matrix J. Then

$$P_n(0) = (-1)^n \prod_{i=1}^{n} \frac{1}{l_i}. \tag{2.5}$$

Proof. Let J admit the LU–factorization of the form (2.1–2.3) and let P_j be the polynomials of the first kind associated with J. By Lemma 2.2, Equation (2.4) holds and we obtain

$$P_n(0) = \frac{P_n(0)}{P_{n-1}(0)} \cdot \frac{P_{n-1}(0)}{P_{n-2}(0)} \cdot \ldots \cdot \frac{P_1(0)}{P_0(0)} = (-1)^n \prod_{i=1}^{n} \frac{1}{l_i}.$$

So, Equation (2.5) is proven. This completes the proof.

Corollary 2.4. Let J be the symmetric Jacobi matrix and let $J = LU$ be its LU–factorization of the form (2.1–2.3). Let P_j be the polynomials of the first kind associated with the matrix J. Then

$$P_n(0) = (-1)^k \frac{1}{l_n} \cdot \frac{1}{l_{n-1}} \cdot \ldots \cdot \frac{1}{l_{n-(k-1)}} P_{n-k}(0). \tag{2.6}$$

Proof. Let J admit the LU–factorization of the form (2.1–2.3). By Lemma 2.2, we obtain

$$\begin{aligned} P_n(0) &= \frac{P_n(0)}{P_{n-1}(0)} \cdot \frac{P_{n-1}(0)}{P_{n-2}(0)} \cdot \ldots \cdot \frac{P_{n-k-1}(0)}{P_{n-k}(0)} \cdot P_{n-k}(0) \\ &= (-1)^k \frac{1}{l_n} \cdot \frac{1}{l_{n-1}} \cdot \ldots \cdot \frac{1}{l_{n-(k-1)}} P_{n-k}(0). \end{aligned}$$

Hence, Equation (2.6) is proven. This completes the proof.

Theorem 2.5. Let J be the symmetric Jacobi matrix and let P_j be the polynomials of the first kind associated with J. Then J admits LU—factorization of the form (2.1–2.3) if and only if

$$P_j(0) \neq 0 \quad \text{for all } j \in \mathbb{Z}_+. \tag{2.7}$$

Furthermore,

$$b_0 = u_1, \quad v_j = a_j, \quad l_j = -\frac{P_{j-1}(0)}{P_j(0)} \quad \text{and} \quad u_j = -\frac{a_j P_j(0)}{P_{j-1}(0)}. \tag{2.8}$$

Proof. Let $P_j(0) \neq 0$ for all $j \in \mathbb{Z}_+$. By Lemma 2.2 the system (2.8) is equivalent to the system (2.3). Consequently, by Lemma 2.1 the Jacobi matrix J admits LU—factorization of the form (2.1–2.3). Conversely, if the Jacobi matrix J admits LU—factorization of the form (2.1–2.3), then by Lemma 2.1 and Lemma 2.2 the polynomials of the first kind P_j satisfy (2.7). This completes the proof.

2.2 Transformed Jacobi matrix $J^{(p)} = UL$

Definition 2.6. Let the symmetric Jacobi matrix J admit LU—factorization of the form (2.1–2.3). Then a transformation

$$J = LU \to UL = J^{(p)}$$

is called a Darboux transformation without parameter of the symmetric Jacobi matrix J.

Theorem 2.7. Let J be the symmetric Jacobi matrix (1.4) and let $J = LU$ be its LU–factorization of the form (2.1–2.3). Then the Darboux transformation without parameter of the matrix J is the symmetric Jacobi matrix

$$J^{(p)} = UL = \begin{pmatrix} b_1 & a_1 & & \\ a_1 & b_2 & a_2 & \\ & a_2 & b_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \tag{2.9}$$

if and only if

$$u_j = b_0 \quad \text{and} \quad \frac{a_j^2 + b_0^2}{b_0} = b_j \quad \text{for all } j \in \mathbb{N}. \tag{2.10}$$

Proof. Calculating UL, we obtain

$$J^{(p)} = UL = \begin{pmatrix} u_1 & v_1 & & \\ & u_2 & v_2 & \\ & & u_3 & \ddots \\ & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & \\ l_1 & 1 & & \\ & l_2 & 1 & \\ & & \ddots & \ddots \end{pmatrix} =$$

$$= \begin{pmatrix} u_1 + v_1 l_1 & v_1 & & \\ l_1 u_2 & u_2 + v_2 l_2 & v_2 & \\ & l_2 u_3 & u_3 + v_3 l_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix} = \{\text{ by Equation (2.3)}\}$$

$$= \begin{pmatrix} u_1 + v_1 l_1 & a_1 & & \\ l_1 u_2 & u_2 + v_2 l_2 & a_2 & \\ & l_2 u_3 & u_3 + v_3 l_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

Consecuently, $J^{(p)}$ is the symmetric Jacobi matrix if and only if

$$l_j u_{j+1} = a_j \quad \text{for all } j \in \mathbb{N}. \tag{2.11}$$

Comparing Equation (2.3) with Equation (2.11), we get

$$l_j u_j = a_j = l_j u_{j+1} \Rightarrow u_j = u_{j+1} \Rightarrow u_j = b_0 \quad \text{for all } j \in \mathbb{N}.$$

By Equation (2.3), $u_j + v_j l_j = b_j$ for all $j \in \mathbb{N}$, we obtain Equations (2.9, 2.10) and $J^{(p)}$ is the symmetric Jacobi matrix. This completes the proof.

Theorem 2.8. Let the symmetric Jacobi matrix J satisfy (2.7) and let $J = LU$ be its LU–factorization of the form (2.1–2.3). Let $J^{(p)} = UL$ be the Darboux transformation without parameter of J. Then the polynomials of the first kind $P_n^{(p)}$ associated with $J^{(p)}$ can be found by Christoffel–Darboux formula

$$P_n^{(p)}(\lambda) = \frac{1}{P_n(0)} \frac{P_{n+1}(\lambda)P_n(0) - P_n(\lambda)P_{n+1}(0)}{\lambda}, \tag{2.12}$$

where P_j are the polynomials of the first kind associated with the symmetric Jacobi matrix J.

Proof. Let the Jacobi matrix J satisfy (2.7) and admit LU–factorization of the form (2.1–2.3). Calculating the inverse matrix of L, we obtain

$$L^{-1} = \begin{pmatrix} 1 & & & & & & \\ -l_1 & 1 & & & & & \\ l_1 l_2 & -l_2 & 1 & & & & \\ -l_1 l_2 l_3 & l_2 l_3 & -l_3 & 1 & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ (-1)^n \prod_{i=1}^{n} l_i & (-1)^{n-1} \prod_{i=2}^{n} l_i & \dots & l_{n-1} l_n & -l_n & 1 & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

On the other hand,

$$J^{(p)} P(\lambda) = ULP^{(p)}(\lambda) = \lambda P^{(p)}(\lambda) \Rightarrow$$
$$\Rightarrow LULP^{(p)}(\lambda) = J\left(LP^{(p)}(\lambda)\right) = \lambda\left(LP^{(p)}(\lambda)\right) = \lambda P(\lambda).$$

Consequently, we obtain the relation between the polynomials of the first kind

$$P^{(p)}(\lambda) = L^{-1} P(\lambda) = \begin{pmatrix} 1 & & & & \\ -l_1 & 1 & & & \\ l_1 l_2 & -l_2 & 1 & & \\ -l_1 l_2 l_3 & l_2 l_3 & -l_3 & 1 & \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} P_0(\lambda) \\ P_1(\lambda) \\ P_2(\lambda) \\ P_3(\lambda) \\ \vdots \end{pmatrix}$$
$$= \begin{pmatrix} P_0(\lambda) \\ P_1(\lambda) - l_1 P_0(\lambda) \\ P_2(\lambda) - l_2 P_1(\lambda) + l_1 l_2 P_0(\lambda) \\ P_3(\lambda) - l_3 P_2(\lambda) + l_2 l_3 P_1(\lambda) - l_1 l_2 l_3 P_0(\lambda) \\ \vdots \end{pmatrix}$$
$$= \begin{pmatrix} P_0^{(p)}(\lambda) \\ P_1^{(p)}(\lambda) \\ P_2^{(p)}(\lambda) \\ P_3^{(p)}(\lambda) \\ \vdots \end{pmatrix}.$$

By Corollary 2.4, we obtain

$$P_n^{(p)}(\lambda) = P_n(\lambda) + \sum_{i=0}^{n-1} (-1)^{n-i} P_i(\lambda) \prod_{j=i+1}^{n} l_j = P_n(\lambda) + \sum_{i=0}^{n-1} \frac{P_i(0)}{P_n(0)} P_i(\lambda). \tag{2.13}$$

However, we can rewrite Equation (2.13) and by Christoffel–Darboux formula (1.8), we obtain

$$P_n^{(p)}(\lambda) = P_n(\lambda) + \sum_{i=0}^{n-1} \frac{P_i(0)}{P_n(0)} P_i(\lambda) = \frac{1}{a_{n+1} P_n(0)} \sum_{i=0}^{n} P_i(0) P_i(\lambda) =$$
$$= \frac{1}{P_n(0)} \frac{P_{n+1}(\lambda) P_n(0) - P_n(\lambda) P_{n+1}(0)}{\lambda}.$$

Hence, Equation (2.12) holds. This completes the proof.

In the following statements we find the connection between orthogonal polynomials, moment sequences, measures, linear functionals, **m**–functions and Toda lattices according to the transformation Darboux transformation without parameter of the symmetric Jacobi matrix.

Proposition 2.9. Let the symmetric Jacobi matrix J admit LU–factorization of the form (2.1–2.3) and let the symmetric Jacobi matrix $J^{(p)} = UL$ be the Darboux transformation without parameter of J. Let $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ and $\mathbf{s}^{(p)} = \{s_n^{(p)}\}_{n=0}^{\infty}$ be the moment sequences associated with the matrices J and $J^{(p)}$, respectively. Then the moment sequence $\mathbf{s}^{(p)} = \{s_n^{(p)}\}_{n=0}^{\infty}$ can be found by the following formula

$$s_{n-1}^{(p)} = \frac{s_n}{b_0} \quad \text{for all } n \in \mathbb{N}. \tag{2.14}$$

Proof. Let the symmetric Jacobi matrix J admit LU–factorization of the form (2.1–2.3) and let the symmetric Jacobi matrix $J^{(p)} = UL$ be its Darboux transformation without parameter. By Equation (1.5), we obtain

$$s_n = (e_0, J^n e_0) = (e_0, (LU)^n e_0) = (e_0, L(UL)^{n-1} U e_0) =$$
$$= (L^T e_0, (J^{(p)})^{n-1} b_0 e_0) = b_0 (e_0, (J^{(p)})^{n-1} e_0) = b_0 s_{n-1}^{(p)}.$$

Consequently, the moments $s_{n-1}^{(p)}$ can be found by Equation (2.14). This completes the proof.

Corollary 2.10. Let the symmetric Jacobi matrix J admit LU–factorization of the form (2.1)–(2.3) and let the symmetric Jacobi matrix $J^{(p)} = UL$ be the Darboux transformation without parameter of J. Let $\mathfrak{S}$ and $\mathfrak{S}^{(p)}$ be the linear functionals associated with the matrices J and $J^{(p)}$, respectively. Then

$$\mathfrak{S}^{(p)} = \frac{\lambda}{b_0} \mathfrak{S}. \tag{2.15}$$

Proof. Let $\mathfrak{S}$ and $\mathfrak{S}^{(p)}$ be the linear functionals associated with the symmetric Jacobi matrices $J = LU$ and $J^{(p)} = UL$, respectively, where L and U are defined by Equations (2.1–2.3). By Equation (1.1), we obtain

$$\mathfrak{S}^{(p)}(\lambda^{n-1}) = s_{n-1}^{(p)} = \frac{s_n}{b_0} = \frac{1}{b_0} \mathfrak{S}(\lambda^n) \quad \text{for all } n \in \mathbb{N}.$$

Consequently, Equation (1.19) holds. This completes the proof.

Corollary 2.11. Let the symmetric Jacobi matrix J admit LU–factorization of the form (2.1–2.3) and let the symmetric Jacobi matrix $J^{(p)} = UL$ be the Darboux transformation without parameter of J. Let $d\mu$ and $d\mu^{(p)}$ be the measures associated with the matrices J and $J^{(p)}$, respectively. Then

$$d\mu^{(p)}(\lambda) = \frac{\lambda}{b_0} d\mu(\lambda). \tag{2.16}$$

Proof. Let μ and $\mu^{(p)}$ be the measures associated with the symmetric Jacobi matrices $J = LU$ and $J^{(p)} = UL$, respectively, where L and U are defined by Equation (2.1–2.3). Then

$$\int_{-\infty}^{+\infty} \lambda^{n-1} d\mu^{(p)}(\lambda) = s^{(p)}_{n-1} = \frac{s_n}{b_0} = \frac{1}{b_0}\int_{-\infty}^{+\infty} \lambda^n d\mu(\lambda) \quad \text{for all } n \in \mathbb{N}.$$

Consequently, we find transformation of the measure and Equation (2.16) holds. This completes the proof.

Proposition 2.12. Let the symmetric Jacobi matrix J admit LU–factorization of the form (2.1–2.3) and let the symmetric Jacobi matrix $J^{(p)} = UL$ be the Darboux transformation without parameter of J. Let m and $m^{(p)}$ be the **m**–functions associated with the matrices J and $J^{(p)}$, respectively. Then

$$m^{(p)}(z) = \frac{s_0 + zm(z)}{b_0}. \tag{2.17}$$

Proof. By Equation (1.6)

$$m^{(p)}(z) = \int_{-\infty}^{+\infty} \frac{d\mu^{(p)}(\lambda)}{\lambda - z} =$$
$$= \frac{1}{b_0}\int_{-\infty}^{+\infty} \frac{\lambda d\mu(\lambda)}{\lambda - z} = \frac{1}{b_0}\int_{\mathbb{R}} \frac{\lambda - z}{\lambda - z} d\mu(\lambda) + \frac{1}{b_0}\int_{-\infty}^{+\infty} \frac{z d\mu(\lambda)}{\lambda - z}$$
$$= \frac{1}{b_0}\int_{-\infty}^{+\infty} 1 d\mu(\lambda) + \frac{z}{b_0}\int_{-\infty}^{+\infty} \frac{d\mu(\lambda)}{\lambda - z} = \frac{s_0 + zm(z)}{b_0}.$$

Hence, **m**–function is transformed by Equation (2.17). This completes the proof.

Toda latice. The last statement is the following theorem of this section. One is described the Toda lattice associated with the symmetric Jacobi matrices $J^{(p)}$.

Theorem 2.13. Let the symmetric Jacobi matrix J admit LU–factorization of the form (2.1–2.3) and J be associated with the Toda lattice (1.9–1.11). Let the symmetric Jacobi matrix $J^{(p)} = UL$ be the Darboux transformation without parameter of J. Then $J^{(p)}$ is associated with the following Toda lattice

$$x_k''(t) = e^{x_{k-1}-x_k} - e^{x_k - x_{k+1}}, \tag{2.18}$$

$$a_k = \frac{1}{2}e^{\frac{x_{k-1}-x_k}{2}} \quad \text{and} \quad b_{k+1} = -\frac{1}{2}x_k'. \tag{2.19}$$

$$a_k' = a_k(b_{k+1} - b_k) \quad \text{and} \quad b_{k+1}' = 2(a_{k+1}^2 - a_k^2), \quad a_0 = 0. \tag{2.20}$$

Furthermore, the matrix A does not change.

Proof. Let the symmetric Jacobi matrix be associated be associated with the Toda (1.9–1.11) and let $J = LU$, where L and U are defined by Equations (2.2, 2.3, 2.10). Consequently, the symmetric Jacobi matrix $J^{(p)} = UL$ is the Darboux transformation without parameter of J. By Equation (2.9), we obtain $J_+ = J_+^{(p)}$, $J_- = J_-^{(p)}$ and the matrix A does not change in the Lax pair, i.e.

$$A = J_+ - J_- = J_+^{(p)} - J_-^{(p)}.$$

Moreover, similar to Equation (1.9–1.11), the symmetric Jacobi matrix $J^{(p)} = UL$ is associated with the Toda lattice (2.18–2.20). This completes the proof.

3 Darboux transformation with parameter of the Jacobi matrix

The next step is the Darboux transformation with parameter of the symmetric Jacobi matrix J. We study the transformations of the polynomials of the first kind, **m**–functions, measure, moment sequence and Toda lattice, which are associated with the transformed Jacobi matrix.

3.1 $\mathfrak{U}\mathfrak{L}$–factorization

Theorem 3.1. Let J be the symmetric Jacobi matrix and let S_0 be a some real parameter. Then J admits the following $\mathfrak{U}\mathfrak{L}$–factorization

$$J = \mathfrak{U}\mathfrak{L}, \tag{3.1}$$

where $\mathfrak{L}$ and $\mathfrak{U}$ are lower and upper triangular matrices, respectively, which are defined by

$$\mathfrak{L} = \begin{pmatrix} 1 & & & \\ \frac{S_0 + b_0}{a_1} & 1 & & \\ & \frac{S_1 + b_1}{a_2} & 1 & \\ & & \ddots & \ddots \end{pmatrix}$$
$$\text{and} \quad \mathfrak{U} = \begin{pmatrix} -S_0 & a_1 & & \\ & -S_1 & a_2 & \\ & & -S_2 & \ddots \\ & & & \ddots \end{pmatrix}, \tag{3.2}$$

if and only if the following system is solvable

$$S_i(S_{i-1} + b_{i-1}) = -a_i^2, \quad S_{i-1} + b_{i-1} \neq 0 \quad \text{and} \quad S_{i-1} \neq 0, \quad \text{for all } i \in \mathbb{N}. \tag{3.3}$$

Proof. Let J be the Jacobi matrix. Let $\mathfrak{L}$ and $\mathfrak{U}$ are defined by Equation (3.2), where the parameter $S_0 \in \mathbb{R} \setminus \{0, -b_0\}$.

Calculating the product $\mathfrak{U}\mathfrak{L}$, we obtain

$$\mathfrak{U}\mathfrak{L} = \begin{pmatrix} -S_0 & a_1 & & \\ & -S_1 & a_2 & \\ & & -S_2 & \ddots \\ & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & \\ \frac{S_0+b_0}{a_1} & 1 & & \\ & \frac{S_1+b_1}{a_2} & 1 & \\ & & \ddots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} b_0 & a_1 & & \\ -\frac{S_1(S_0+b_0)}{a_1} & b_1 & a_2 & \\ & -\frac{S_2(S_1+b_1)}{a_2} & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

Comparing the product $\mathfrak{U}\mathfrak{L}$ with the Jacobi matrix J, we obtain the system (3.3). This completes the proof.

3.2 Transformed Jacobi matrix $J^{(d)} = \mathfrak{U}\mathfrak{L}$

Definition 3.2. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3). Then a transformation

$$J = \mathfrak{U}\mathfrak{L} \to \mathfrak{L}\mathfrak{U} = J^{(d)}$$

is called a Darboux transformation with parameter of the Jacobi matrix J.

Theorem 3.3. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3) with parameter $S_0 \in \mathbb{R}\backslash\{0, -b_0\}$. Then the Darboux transformation with parameter of the Jacobi matrix J is the symmetric Jacobi matrix

$$J^{(d)} = \begin{pmatrix} -S_0 & a_1 & & \\ a_1 & b_0 & a_2 & \\ & a_2 & b_1 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \tag{3.4}$$

if and only if

$$S_0 = S_i \quad \text{for all } i \in \mathbb{N}. \tag{3.5}$$

Proof. Let J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3). Calculating the product $\mathfrak{L}\mathfrak{U}$, we obtain

$$J^{(d)} = \mathfrak{L}\mathfrak{U} = \begin{pmatrix} -S_0 & a_1 & & \\ -\frac{S_0(S_0+b_0)}{a_1} & S_0+b_0-S_1 & a_2 & \\ & -\frac{S_1(S_1+b_1)}{a_2} & S_1+b_1-S_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

Hence, $J^{(d)}$ is the symmetric Jacobi matrix if and only if

$$-S_{i-1}(S_{i-1}+b_{i-1}) = a_i^2 \quad \text{for all } i \in \mathbb{N}.$$

On the other hand, by Equation (3.3), we know

$$-S_i(S_{i-1}+b_{i-1}) = a_i^2 \quad \text{for all } i \in \mathbb{N}.$$

Consequently, we obtain Equation (3.5). This completes the proof.

Theorem 3.4. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3) and let $J^{(d)} = \mathfrak{L}\mathfrak{U}$ be its Darboux transformation with parameter. Then the polynomials of the first kind transform by the Geronimus formula

$$P_0^{(d)}(\lambda) \equiv P_0(\lambda) \quad \text{and} \quad P_i^{(d)}(\lambda) = P_i(\lambda) + \frac{S_0+b_{i-1}}{a_i} \cdot P_{i-1}(\lambda),\ i \in \mathbb{N}, \tag{3.6}$$

where P_i and $P_i^{(d)}$ are polynomials of the first kind associated with the matrix J and $J^{(d)}$, respectively.

Proof. Let J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3) and let $J^{(d)} = \mathfrak{L}\mathfrak{U}$ be its Darboux transformation with parameter. Then

$$JP(\lambda) = \lambda P(\lambda) \Rightarrow \mathfrak{U}\mathfrak{L}P(\lambda) = \lambda P(\lambda) \Rightarrow \mathfrak{L}\mathfrak{U}\mathfrak{L}P(\lambda) = \lambda \mathfrak{L}P(\lambda) \Rightarrow$$
$$\Rightarrow J^{(d)}P^{(d)}(\lambda) = \lambda P^{(d)}(\lambda),$$

where

$$P^{(d)}(\lambda) = \mathfrak{L}P(\lambda) = \begin{pmatrix} 1 & & & \\ \frac{S_0+b_0}{a_1} & 1 & & \\ & \frac{S_0+b_1}{a_2} & 1 & \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} P_0(\lambda) \\ P_1(\lambda) \\ P_2(\lambda) \\ P_3(\lambda) \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} P_0(\lambda) \\ P_1(\lambda) + \frac{S_0+b_0}{a_1}P_0(\lambda) \\ P_2(\lambda) + \frac{S_1+b_1}{a_2}P_1(\lambda) \\ P_3(\lambda) + \frac{S_2+b_2}{a_3}P_2(\lambda) \\ \vdots \end{pmatrix} = \begin{pmatrix} P_0^{(d)}(\lambda) \\ P_1^{(d)}(\lambda) \\ P_2^{(d)}(\lambda) \\ P_3^{(d)}(\lambda) \\ \vdots \end{pmatrix}.$$

So, the polynomials of the first kind are transformed by the Geronimus formula and Equation (3.6) holds. This completes the proof.

Proposition 3.5. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3) and let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ be the Darboux transformation with parameter of J. Let $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ and $\mathbf{s}^{(d)} = \{s_n^{(d)}\}_{n=0}^{\infty}$ be the moment sequences associated with the matrices J and $J^{(d)}$, respectively. Then the moment sequence $\mathbf{s}^{(d)} = \{s_n^{(d)}\}_{n=0}^{\infty}$ can be found by

$$s_0^{(d)} = 1 \quad \text{and} \quad s_n^{(d)} = -S_0 s_{n-1} \quad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

Proof. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3) and let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$

be its Darboux transformation with parameter. By Equation (1.5), we obtain

$$s_0^{(d)} = (e_0, (J^{(d)})^0 e_0) = (e_0, e_0) = 1$$

and

$$\begin{aligned} s_n^{(d)} &= (e_0, (J^{(d)})^n e_0) = (e_0, \mathfrak{L}\mathfrak{U}^n e_0) = (e_0, \mathfrak{L}(\mathfrak{L}\mathfrak{U})^{n-1}\mathfrak{U}e_0) \\ &= (\mathfrak{L}^T e_0, (J)^{n-1}(-S_0)e_0) = -S_0(e_0, (J)^{n-1}e_0) = -S_0 s_{n-1} \\ &\text{for all } n \in \mathbb{N}. \end{aligned}$$

Hence, Equation (3.7) holds. This completes the proof.

Corollary 3.6. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3) and let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ be the Darboux transformation with parameter of J. Let $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ and $\mathbf{s}^{(d)} = \{s_n^{(d)}\}_{n=0}^{\infty}$ be the moment sequences associated with the matrices J and $J^{(d)}$, respectively. Then

$$s_1^{(d)} = -S_0. \tag{3.8}$$

Proof. By Equation (3.7) and $s_0 = 1$, we obtain

$$s_1^{(d)} = -S_0 s_0 \Rightarrow s_1^{(d)} = -S_0.$$

So, Equation (3.8) holds. This completes the proof.

Corollary 3.7. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3) and let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ be the Darboux transformation with parameter of J. Let $\mathfrak{S}$ and $\mathfrak{S}^{(d)}$ be the linear functionals associated with the matrices J and $J^{(d)}$, respectively. Then

$$\mathfrak{S}^{(d)}(p(\lambda)) = -S_0 \mathfrak{S}\left(\frac{p(\lambda) - p(0)}{\lambda}\right) + p(0), \quad p(\lambda) \in \mathbb{C}[\lambda]. \tag{3.9}$$

Proof. Let $\mathfrak{S}$ and $\mathfrak{S}^{(d)}$ be the linear functionals associated with the symmetric Jacobi matrices $J = \mathfrak{U}\mathfrak{L}$ and $J^{(d)} = \mathfrak{L}\mathfrak{U}$, respectively, where $\mathfrak{L}$ and $\mathfrak{U}$ are defined by Equations (3.2, 3.3). By Equation (1.1), we obtain

$$\mathfrak{S}^{(d)}(\lambda^n) = s_n^{(d)} = -S_0 s_{n-1} = -S_0 \mathfrak{S}\left(\lambda^{n-1}\right), \quad \text{for all } n \in \mathbb{N}.$$

Consequently, Equation (3.9) holds. This completes the proof.

Corollary 3.8. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3). and let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ be the Darboux transformation with parameter of J. Let $d\mu$ and $d\mu^{(d)}$ be the measures associated with the matrices J and $J^{(d)}$, respectively. Then

$$d\mu(\lambda) = -\frac{\lambda}{S_0} d\mu^{(d)}(\lambda). \tag{3.10}$$

Proof. Let $J = \mathfrak{U}\mathfrak{L}$ and $J^{(d)} = \mathfrak{L}\mathfrak{U}$, where the matrices $\mathfrak{L}$ and $\mathfrak{U}$ are defined by Equations (3.2, 3.3, 3.5). The measures $d\mu$ and $d\mu^{(d)}$ are associated with the matrices J and $J^{(d)}$, respectively. Then

$$-S_0 \int_{-\infty}^{+\infty} \lambda^{n-1} d\mu(\lambda) = -S_0 s_{n-1} = s_n^{(d)} = \int_{-\infty}^{+\infty} \lambda^n d\mu^{(d)}(\lambda).$$

Consequently,

$$\int_{-\infty}^{+\infty} \lambda^{n-1} d\mu(\lambda) = -\int_{-\infty}^{+\infty} \lambda^{n-1} \frac{\lambda}{S_0} d\mu^{(d)}(\lambda).$$

Hence, Equation (3.10) holds. This completes the proof.

Proposition 3.9. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3) and let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ be the Darboux transformation with parameter of J. Let m and $m^{(d)}$ be **m**–functions associated with the matrices J and $J^{(d)}$, respectively. Then

$$m^{(d)}(z) = \frac{1}{z} + \frac{S_0 m(z)}{z}. \tag{3.11}$$

Proof. Let $J = \mathfrak{U}\mathfrak{L}$ and $J^{(d)} = \mathfrak{L}\mathfrak{U}$, where the matrices $\mathfrak{L}$ and $\mathfrak{U}$ are defined by Equations (3.2, 3.3, 3.5). Then **m**–functions of the matrices J and $J^{(d)}$ are related by

$$\begin{aligned} m(z) &= \int_{-\infty}^{+\infty} \frac{d\mu(\lambda)}{\lambda - z} = -\frac{1}{S_0} \int_{-\infty}^{+\infty} \frac{\lambda d\mu^{(d)}(\lambda)}{\lambda - z} \\ &= -\frac{1}{S_0} \int_{-\infty}^{+\infty} \frac{\lambda - z}{\lambda - z} d\mu^{(d)(\lambda)} + \frac{1}{S_0} \int_{-\infty}^{+\infty} \frac{z d\mu^{(d)}(\lambda)}{\lambda - z} \\ &= -\frac{s_0^{(d)}}{S_0} + \frac{z m^{(d)}(z)}{S_0}. \end{aligned}$$

On the other hand

$$\frac{z m^{(d)}(z)}{S_0} = m(z) + \frac{s_0^{(d)}}{S_0} \Rightarrow m^{(d)}(z) = \frac{s_0^{(d)}}{z} + \frac{S_0 m(z)}{z}.$$

By Equation (3.7), $s_0^{(d)} = 1$ and Equation (3.11) holds. This completes the proof.

Toda latice. There is the last target of our investigation.

Theorem 3.10. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$–factorization of the form (3.1–3.3) and J be associated with the Toda lattice (1.9–1.11). Let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ be the Darboux transformation without parameter of J. Then $J^{(d)}$ is associated with the following Toda lattice

$$x_k''(t) = e^{x_{k-1} - x_k} - e^{x_k - x_{k+1}}, \tag{3.12}$$

$$a_k = \frac{1}{2} e^{\frac{x_{k-1} - x_k}{2}}, \quad S_0 = \frac{1}{2} x_0' \quad \text{and} \quad b_{k-1} = -\frac{1}{2} x_k'. \tag{3.13}$$

$$\begin{aligned} &a_0 = 0, \quad a_1' = a_1(b_0 + S_0), \quad a_k' = a_k(b_{k-1} - b_{k-2}), \\ &-S_0' = 2(a_1^2 - a_0^2) \quad \text{and} \quad b_{k-1}' = 2(a_{k+1}^2 - a_k^2), \quad k \in \mathbb{N}. \end{aligned} \tag{3.14}$$

Furthermore, the matrix A does not change.

Proof. Let the symmetric Jacobi matrix J be associated with the Toda lattice (1.9–1.11) and let $J = \mathfrak{U}\mathfrak{L}$, where $\mathfrak{L}$ and $\mathfrak{U}$ are defined by Equations (3.2, 3.3, 3.5). Consequently, the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ is the Darboux transformation with parameter of J. By Equation (3.4), we obtain $J_+ = J_+^{(d)}$, $J_- = J_-^{(d)}$ and the matrix A does not change in the Lax pair, i.e.

$$A = J_+ - J_- = J_+^{(d)} - J_-^{(d)}.$$

Moreover, similar to Equations (1.9–1.11), the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ is associated with the Toda lattice (3.12–3.14). This completes the proof.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

IK: Investigation, Writing – original draft. OL: Investigation, Writing – original draft.

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