frontiers | Frontiers in Applied Mathematics and Statistics

TYPE Original Research
PUBLISHED 16 July 2024
DOI 10.3389/fams.2024.1384559

OPEN ACCESS

EDITED BY
Zailan Siri,
University of Malaya, Malaysia

REVIEWED BY
Jihad H. Asad,
Palestine Technical University Kadoorie,
Palestine
Abdullah Ozbekler,
OSTIM Technical University, Türkiye

*CORRESPONDENCE
Osama Moaaz
✉ o_moaaz@mans.edu.eg

RECEIVED 09 February 2024
ACCEPTED 25 June 2024
PUBLISHED 16 July 2024

CITATION
Hassan AM, Moaaz O, Askar SS and Alshamrani AM (2024) Oscillatory behavior of solutions of second-order non-linear differential equations with mixed non-linear neutral terms.
Front. Appl. Math. Stat. 10:1384559.
doi: 10.3389/fams.2024.1384559

Oscillatory behavior of solutions of second-order non-linear differential equations with mixed non-linear neutral terms

Ahmed M. Hassan[1], Osama Moaaz[2,3]*, Sameh S. Askar[4] and Ahmad M. Alshamrani[4]

[1]Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt, [2]Section of Mathematics, International Telematic University Uninettuno, Rome, Italy, [3]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt, [4]Department of Statistics and Operations Research, College of Science, King Saud University, Riyadh, Saudi Arabia

This study primarily seeks to expand upon these developments by encompassing neutral differential equations of mixed type, incorporating both delay and advanced terms, particularly in the case of the canonical operator. The presented results are derived from the application of the comparison method, Riccati transformation, and integral averaging technique. These methodologies lead to substantial improvements and extensions of existing results found in the literature. Additionally, illustrative examples are provided to demonstrate the practical implications of the developed criteria.

KEYWORDS
neutral differential equations, second-order, oscillatory, mixed terms, Riccati transformation

1 Introduction

In recent years, scholars have shown a growing interest in studying the oscillatory behavior of solutions to second-order differential and dynamic equations. This attention is motivated by the significance of such behavior in real-life applications, such as steam turbine regulation, neural networks, and governing equations which describe the temporal variation of hormones; see [1–9]. Moreover, there exist numerous sophisticated equations that serve as applications or direct representations of problems reliant on both present and future rates of change for further applications in science and technology; see [10]. An advanced argument is characterized by its ability to characterize the impact of a hypothetical future acts. It is commonly observed in phenomena such as population dynamics and economic difficulties.

In this study, we discuss the oscillatory behavior of second-order non-linear neutral differential equations with mixed non-linear neutral terms of the form

$$\left(a(\zeta)\Phi_{\alpha}\left(z'(\zeta)\right)\right)' + q(\zeta)\Phi_{\gamma}(x(\sigma(\zeta))) + r(\zeta)\Phi_{\beta}(x(\varphi(\zeta))) = 0, \quad \zeta \geq \zeta_0 > 0, \quad (1)$$

and

$$z(\zeta) = \Phi_{\eta}(x(\zeta)) + p_1(\zeta)\Phi_{\lambda}(x(\tau_1(\zeta))) + \varepsilon p_2(\zeta)\Phi_{\nu}(x(\tau_2(\zeta))),$$

where $\Phi_*(s) = |s|^{*-1}$ for $* > 0$ and $\varepsilon = \pm 1$. Throughout this article, we assume that the following hypotheses hold

(H$_1$) $\alpha, \eta, \lambda, \nu, \gamma$ and β are ratios of odd positive integers;

(H$_2$) a, p_1, p_2, q and $r \in C\left([\zeta_0,\infty),(0,\infty)\right)$, Equation (1) is in canonical form, i.e.,

$$\int_{\zeta_0}^{\infty} a^{-1/\alpha}(s)\mathrm{d}s = \infty; \tag{2}$$

(H$_3$) $\tau_1, \tau_2, \sigma, \zeta \in C\left([\zeta_0,\infty),\mathbb{R}\right)$ such that $\sigma(\zeta) \leq \zeta, \tau_1', \tau_2' > 0$ and $\varphi(\zeta) \geq \zeta$ with

$$\lim_{\zeta\to\infty} \tau_1(\zeta) = \lim_{\zeta\to\infty} \tau_2(\zeta) = \lim_{\zeta\to\infty} \sigma(\zeta) = \infty.$$

By a solution of Equation (1), we mean a non-trivial function $x \in C\left([T_x,\infty),\mathbb{R}\right)$, $T_x \geq \zeta_0$, which has the properties $z \in C^1\left([T_x,\infty),\mathbb{R}\right)$, $a(z')^{\alpha} \in C\left([T_x,\infty),\mathbb{R}\right)$ and satisfies Equation (1) on $[T_x,\infty)$. Our attention is restricted to those solutions $x(\zeta)$ of Equation (1) satisfying $\sup\left\{|x(\zeta)| : \zeta \geq T\right\} > 0$ for all $T \geq T_x$. We assume that Equation (1) possesses such a solution. A solution of Equation (1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is termed non-oscillatory. Equation (1) is said to be oscillatory if all solutions are oscillatory.

The literature extensively addresses the oscillation and asymptotic behavior of solutions across various classes of delay and advanced differential equations. Recently, Tunç and Özdemir [11] introduced novel sufficient conditions for solutions to second-order half-linear differential equations.

$$\left(a(\zeta)\left(\left(x(\zeta) + p_1(\zeta)x(g_1(\zeta)) + p_2(\zeta)x(g_2(\zeta))\right)'\right)^{\alpha}\right) + q(\zeta)x^{\alpha}\left(h(\zeta)\right) = 0,$$

where $g_1(\zeta) < \zeta, g_2(\zeta) > \zeta, p_1(\zeta) \geq 0$ and $p_2(\zeta) \geq 1, p_2(\zeta) \not\equiv 1$ for large ζ. In the study by Agwa et al. [12], developed new oscillation criteria for the second-order non-linear neutral dynamic equation with mixed arguments

$$\left(a(\zeta)\left(\left(x(\zeta) + p_1(\zeta)x\left(\tau_1(\zeta)\right) + p_2(\zeta)x\left(\tau_2(\zeta)\right)\right)^{\Delta}\right)^{\alpha}\right)^{\Delta} + f\left(\zeta, x\left(\sigma(\zeta)\right)\right) + g\left(\zeta, x\left(\varphi(\zeta)\right)\right) = 0,$$

in the canonical and non-canonical cases. In the study by Moaaz et al. [9], investigated the oscillatory and asymptotic properties of a specific class of delay differential equations of mixed neutral type

$$\left(a(\zeta)\left(\left(x(\zeta) + p_1(\zeta)x(\tau_1(\zeta)) + p_2(\zeta)x(\tau_2(\zeta))\right)'\right)^{\alpha}\right) + q(\zeta)x^{\alpha}\left(\sigma(\zeta)\right) + r(\zeta)x^{\alpha}\left(\varphi(\zeta)\right) = 0.$$

with the non-canonical operator.

Very recently, in the study by Grace et al. [13], introduced novel criteria for the oscillation of second-order non-linear differential equations featuring mixed non-linear neutral terms and mixed deviating arguments

$$\left(a(\zeta)\left(\left(x(\zeta) + p_1(\zeta)x^{\lambda}(\tau_1(\zeta)) - p_2(\zeta)x^{\nu}(\tau_2(\zeta))\right)'\right)^{\alpha}\right) + q(\zeta)x^{\alpha}\left(\sigma(\zeta)\right) + r(\zeta)x^{\alpha}\left(\varphi(\zeta)\right) = 0.$$

Upon reviewing the literature, it becomes evident that numerous results pertain to the oscillation of second-order differential equations with linear neutral terms. In contrast, there is a paucity of articles dedicated to exploring differential equations featuring sublinear or superlinear neutral terms, as evidenced in the studies by Agarwal et al. [14], Bohner et al. [15], Džurina et al. [16], Grace et al. [17], Lin and Tang [18], and Muhib et al. [19]. Furthermore, there is a notable scarcity of results concerning equations incorporating both sublinear and superlinear neutral terms. Motivated by this gap, this study aims to establish oscillation criteria for a specific class of second-order mixed functional differential equations (Equation 1) characterized by sublinear and superlinear neutral terms under the conditions where either $\varepsilon = -1$ or $\varepsilon = +1$, η differs from 1. To the best of our knowledge, there does not appear to be any oscillation results for Equation (1) when $\eta \neq 1$. Additionally, it is worth noting that the findings presented in this study are novel even in the linear case.

2 Preliminaries

In discussing oscillation results for Equation (1), we assume that any functional inequality holds for all large ζ. To lay the groundwork for our key results, we first define a few lemmas. To keep things simple, we will use these symbols

$$A(\zeta,\zeta_1) = \int_{\zeta_1}^{\zeta} a^{-1/\alpha}(s)\mathrm{d}s, \qquad R_1 = \frac{(1-\lambda)p_1^{1/(1-\lambda)}(\zeta)}{\lambda^{\lambda/(\lambda-1)}},$$

$$R_2 = \frac{(\nu-1)}{\nu^{\nu/(\nu-1)}}p_2^{1/(1-\nu)}(\zeta), \qquad \rho_+' = \max\{\rho',0\}$$

$$\delta_i(\zeta) = \begin{cases} \tau^{-1}_{(i+1)/2}(\tau_{(5-i)/2}(\tau^{-1}_{(i+1)/2}(\zeta))), & i=1,3 \\ \tau^{-1}_{i/2}(\tau^{-1}_{i/2}(\zeta)), & i=2,4 \end{cases}, \quad \delta_5 = \tau_2^{-1}(\tau_1^{-1}(\zeta)),$$

$$S_1 = \frac{\frac{\eta}{\lambda}A\left(\delta_2(\zeta)),\zeta_3\right)}{A\left(\tau_1^{-1}(\zeta),\zeta_3\right)p_1^{\eta/\lambda}(\delta_2(\zeta))} + \frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(\zeta))A\left(\delta_1(\zeta),\zeta_3\right)}{A\left(\tau_1^{-1}(\zeta),\zeta_3\right)p_1^{\nu/\lambda}(\delta_1(\zeta))} + \left(\frac{1-\frac{\eta}{\lambda}}{c_*p_1^{\eta/\lambda}(\delta_2(\zeta))} + \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(\zeta))}{c_*p_1^{\nu/\lambda}(\delta_1(\zeta))}\right)$$

$$S_2 = \frac{\frac{\eta}{\nu}}{p_2^{\eta/\nu}(\delta_4(\zeta))} + \frac{\frac{\lambda}{\nu}p_1(\tau_2^{-1}(\zeta))}{p_2^{\lambda/\nu}(\delta_3(\zeta))} + \frac{(1-\frac{\eta}{\nu})}{c_*p_2^{\eta/\nu}(\delta_4(\zeta))} + \frac{(1-\frac{\lambda}{\nu})p_1(\tau_2^{-1}(\zeta))}{c_*p_2^{\lambda/\nu}(\delta_3(\zeta))}$$

$$S_3 = \frac{\frac{\eta}{\nu}}{p_2^{\eta/\nu}(\delta_5(\zeta))} - \frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(\zeta))A\left(\delta_1(\zeta),\zeta_3\right)}{p_1^{\nu/\lambda}(\delta_1(\zeta))A\left(\tau_1^{-1}(\zeta),\zeta_3\right)} + \frac{(1-\frac{\eta}{\nu})}{c_*p_2^{\eta/\nu}(\delta_5(\zeta))} - \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(\zeta))}{c_*p_1^{\nu/\lambda}(\delta_1(\zeta))}, \quad c_* > 0, \ \zeta_3 \in [\zeta_0,\infty).$$

Lemma 1. Baculíková [20] and Philos [21] Let $q_1 : [\zeta_0,\infty) \to (0,\infty)$, $g_1 : [\zeta_0,\infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions, f is non-decreasing with $xf(x) > 0$ for $x \neq 0$ and $g_1(\zeta) \to \infty$ as $\zeta \to \infty$. If

(i) The first-order delay differential inequality (i.e., $g_1(\zeta) \le \zeta$)

$$y_1'(\zeta) + q_1(\zeta) f(y_1(g_1(\zeta))) \le 0$$

has an eventually positive solution, so does the corresponding delay differential equation.

(ii) The first-order advanced differential inequality (i.e., $g_1(\zeta) \ge \zeta$)

$$y_1'(\zeta) - q_1(\zeta) f(y_1(g_1(\zeta))) \ge 0$$

has an eventually positive solution, so does the corresponding advanced differential equation.

Lemma 2. Hardy et al. [22] If $X, Y \ge 0$, then

$$X^{\xi} + (\xi - 1)Y^{\xi} - \xi X Y^{\xi-1} \ge 0, \quad \text{for } \xi > 1 \tag{3}$$

and

$$X^{\xi} - (1 - \xi)Y^{\xi} - \xi X Y^{\xi-1} \le 0, \quad \text{for } 0 < \xi < 1, \tag{4}$$

where equalities hold if and only if $X = Y$.

Lemma 3. Hardy et al. [22] If B, L be non-negative numbers and if $m, n > 1$ are real numbers such that $\frac{1}{n} + \frac{1}{m} = 1$, then

$$BL \le \frac{1}{n} B^n + \frac{1}{m} L^m.$$

Equality holds if and only if $B^n = L^m$.

Lemma 4. Bohner et al. [23] Let $G(U) = AU - B(U - R)^{\frac{\mu+1}{\mu}}$, where $B > 0$, A and B are constants, μ is a ratio of odd positive integers. Then, G attains its maximum value at $U_* = R + (\mu A/((\mu+1)B)^{\mu})$ and

$$\max_{U \in \mathbb{R}} G(U) = G(U_*) = AR + \frac{\mu^{\mu}}{(\mu+1)^{\mu+1}} \frac{A^{\mu+1}}{B^{\mu}}. \tag{5}$$

3 The case $\varepsilon = -1$

In this section, we investigate the oscillatory characteristics of solutions to Equation (1) under the conditions where $\varepsilon = -1$, $\tau_1(\zeta) = \tau_2(\zeta) = \tau(\zeta)$, and either of the following conditions is satisfied either $\lambda < 1$ and $\nu > 1$, or $\lambda < \nu \le 1$.

Theorem 1. Suppose $\tau_1(\zeta) = \tau_2(\zeta) = \tau(\zeta)$, $\lambda < 1$, $\nu > 1$, and conditions (H_1)-(H_3) are satisfied. Additionally, assume that

$$\lim_{\zeta \to \infty} \left[R_1(\zeta) + R_2(\zeta) \right] = 0, \tag{6}$$

and there exists a non-decreasing function $\varrho(\zeta) \in C([\zeta_0, \infty), \mathbb{R})$ such that

$$\tau^{-1}(\sigma(\zeta)) \le \varrho(\zeta) \le \zeta, \tau^{-1}(\varphi(\zeta)) \ge \zeta \text{ and } \tau^{-1}(\varphi(\varrho(\varrho(\zeta)))) \ge \zeta, \tag{7}$$

for $\zeta \ge \zeta_0$. If there exist constant $c_2 \in (0, 1)$ such that the first-order differential equations

$$F'(\zeta) + c_2^{\gamma} q(\zeta) A^{\gamma/\eta}(\sigma(\zeta), \zeta_2) F^{\gamma/\alpha\eta}(\sigma(\zeta)) = 0, \tag{8}$$

$$\Omega'(\zeta) + q(\zeta) \left(\frac{A(\varrho(\zeta), \tau^{-1}(\sigma(\zeta)))}{p_2(\tau^{-1}(\sigma(\zeta)))} \right)^{\gamma/\nu} \Omega^{\gamma/\alpha\nu}(\varrho(\zeta)) = 0, \tag{9}$$

$$\bar{z}'(\zeta) - \left(\frac{1}{a(\zeta)} \int_{\varrho(\zeta)}^{\zeta} \frac{r(s)}{p_2^{\beta/\nu}(\tau^{-1}(\varphi(s)))} ds \right)^{1/\alpha} \bar{z}^{\beta/\nu\alpha}(\tau^{-1}(\varphi(\varrho(\zeta)))) = 0 \tag{10}$$

are oscillatory for sufficiently large $\zeta_2 > \zeta_1 \ge \zeta_0$, then every solution of Equation (1) is oscillatory.

Proof. Consider a non-oscillatory solution $x(\zeta)$ of Equation (1). Without loss of generality, let us assume that $x(\zeta)$ is eventually positive with $\lim_{\zeta \to \infty} x(\zeta) \ne 0$ for $\zeta \ge \zeta_0$. Hence, $x(\zeta) > 0$, $x(\tau(\zeta)) > 0$, $x(\sigma(\zeta)) > 0$, and $x(\varphi(\zeta)) > 0$ for $\zeta \ge \zeta_1 \ge \zeta_0$. It is worth noting that the proof for the case where $x(\zeta)$ is eventually negative follows a similar path and is therefore omitted. Now, from Equation (1), it can be inferred that

$$\left(a(\zeta) \left(z'(\zeta) \right)^{\alpha} \right)' = -q(\zeta) x^{\gamma}(\sigma(\zeta)) - r(\zeta) x^{\beta}(\varphi(\zeta)) < 0.$$

Thus, $\left(a(\zeta) \left(z'(\zeta) \right)^{\alpha} \right)$ is decreasing and eventually of one sign. In other words, there exists $\zeta_2 \ge \zeta_1$ such that $z'(\zeta) > 0$ or $z'(\zeta) < 0$ for $\zeta \ge \zeta_2$. Therefore, we will distinguish the following four cases:

Case(I): $z(\zeta) > 0$ and $z'(\zeta) < 0$,
Case(II): $z(\zeta) > 0$ and $z'(\zeta) > 0$,
Case(III): $z(\zeta) < 0$ and $z'(\zeta) > 0$,
Case(IV): $z(\zeta) < 0$ and $z'(\zeta) < 0$.

Initially, let us examine Case (I). Depending on the fact that $z'(\zeta) < 0$ and $\left(a(\zeta) \left(z'(\zeta) \right)^{\alpha} \right)' < 0$, in accordance with Equation (2), we conclude that $\lim_{\zeta \to \infty} z(\zeta) = -\infty$, contradicting the established condition $z(\zeta) > 0$. Therefore, Case (I) is deemed impossible. For the second case, based on the definition of $z(\zeta)$, we have

$$z(\zeta) = x^{\eta}(\zeta) - p_2(\zeta) x^{\nu}(\tau(\zeta)) + p_1(\zeta) x^{\lambda}(\tau(\zeta)). \tag{11}$$

Utilizing inequality (5) with $U = x^{\eta}(\zeta)$, $A = 1$, $B = p_2(\zeta)$, $R = x^{\eta}(\zeta) - x(\tau(\zeta))$, and $\mu = \frac{1}{\nu-1}$, we deduce

$$\begin{aligned} x^{\eta}(\zeta) - p_2(\zeta) x^{\nu}(\tau(\zeta)) &\le x^{\eta}(\zeta) - x(\tau(\zeta)) \\ &\quad + \frac{\left(\frac{1}{\nu-1} \right)^{1/(\nu-1)}}{p_2^{1/(\nu-1)}(\zeta) \left(\frac{\nu}{\nu-1} \right)^{\nu/(\nu-1)}} \\ &\le x^{\eta}(\zeta) - x(\tau(\zeta)) + \frac{(\nu-1)}{\nu^{\nu/(\nu-1)}} p_2^{1/(1-\nu)}(\zeta). \end{aligned}$$

This with Equation (11) leads to

$$x^{\eta}(\zeta) \ge z(\zeta) + x(\tau(\zeta)) - p_1(\zeta) x^{\lambda}(\tau(\zeta)) - \frac{(\nu-1)}{\nu^{\nu/(\nu-1)}} p_2^{1/(1-\nu)}(\zeta). \tag{12}$$

Applying the inequality (4) to $\left[p_1(\zeta) x^{\lambda}(\tau(\zeta)) - x(\tau(\zeta)) \right]$ with $\xi = \lambda$, $X = x(\tau(\zeta))$ and $Y = \frac{1}{(\lambda p_1(\zeta))^{1/(\lambda-1)}}$, we obtain

$$\begin{aligned} p_1(\zeta) x^{\lambda}(\tau(\zeta)) - x(\tau(\zeta)) &= p_1(\zeta) \left[x^{\lambda}(\tau(\zeta)) - \frac{\lambda}{\lambda p_1(\zeta)} x(\tau(\zeta)) \right] \\ &\le \frac{(1-\lambda) p_1^{1/(1-\lambda)}(\zeta)}{\lambda^{\lambda/(\lambda-1)}}. \end{aligned}$$

It is deduced from Equation (12) that

$$x^{\eta}(\zeta) \geq z(\zeta) - \frac{(\nu-1)}{\nu^{\nu/(\nu-1)}} p_2^{1/(1-\nu)}(\zeta) - \frac{(1-\lambda)}{\lambda^{\lambda/(\lambda-1)}} p_1^{1/(1-\lambda)}(\zeta)$$
$$= \left(1 - \frac{(R_1(\zeta)+R_2(\zeta))}{z(\zeta)}\right) z(\zeta).$$

As $z(\zeta)$ is positive and increasing, there exists a constant $c > 0$, ensuring $z(\zeta) \geq c$ for $\zeta \geq \zeta_2$. This implies

$$x^{\eta}(\zeta) \geq \left(1 - \frac{(R_1(\zeta)+R_2(\zeta))}{c}\right) z(\zeta). \tag{13}$$

Therefore, based on Equations (6, 13), a constant $c_1 \in (0,1)$ exists, such that

$$x(\zeta) \geq c_1^{1/\eta} z^{1/\eta}(\zeta) = c_2 z^{1/\eta}(\zeta); \quad c_2 = c_1^{1/\eta} \quad \text{for } \zeta \geq \zeta_3 \geq \zeta_2. \tag{14}$$

Combining Equation (14) with Equation (1), we get

$$\left(a(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)' \leq -c_2^{\gamma} q(\zeta) z^{\gamma/\eta}(\sigma(\zeta)) - c_2^{\beta} r(\zeta) z^{\beta/\eta}(\varphi(\zeta)). \tag{15}$$

Consequently, since $z(\zeta)$ is increasing,

$$z(\zeta) \geq \int_{\zeta_3}^{\zeta} \frac{a^{1/\alpha}(s) z'(s)}{a^{1/\alpha}(s)} \mathrm{d}s \geq A(\zeta,\zeta_3)(a^{1/\alpha}(\zeta) z'(\zeta)). \tag{16}$$

This with Equation (15) leads to

$$\left(a(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)' + c_2^{\gamma} q(\zeta) A^{\gamma/\eta}(\sigma(\zeta),\zeta_3)$$
$$(a(\sigma(\zeta))(z'(\sigma(\zeta)))^{\alpha})^{\gamma/\alpha\eta} \leq 0.$$

Letting $F(\zeta) := a(\zeta)\left(z'(\zeta)\right)^{\alpha}$, we have

$$F'(\zeta) + c_2^{\gamma} q(\zeta) A^{\gamma/\eta}(\sigma(\zeta),\zeta_3) F^{\gamma/\alpha\eta}(\sigma(\zeta)) \leq 0. \tag{17}$$

By applying Lemma 1 (i), the differential Equation (8) associated with the inequality (17) reveals the existence of a positive solution, leading to a contradiction.

Now, suppose the case (III) holds, we have $z(\zeta) < 0$ and $z'(\zeta) > 0$. Let

$$\bar{z}(\zeta) = -z(\zeta) = -x^{\eta}(\zeta) - p_1(\zeta) x^{\lambda}(\tau(\zeta))$$
$$+ p_2(\zeta) x^{\nu}(\tau(\zeta)) \leq p_2(\zeta) x^{\nu}(\tau(\zeta)),$$

it follows that

$$x(\zeta) \geq \left(\frac{\bar{z}(\tau^{-1}(\zeta))}{p_2(\tau^{-1}(\zeta))}\right)^{1/\nu}. \tag{18}$$

It can be inferred from Equation (1) that

$$\left(a(\zeta)\left(\bar{z}'(\zeta)\right)^{\alpha}\right)' = q(\zeta) x^{\gamma}(\sigma(\zeta)) + r(\zeta) x^{\beta}(\varphi(\zeta)) \geq q(\zeta) x^{\gamma}(\sigma(\zeta))$$
$$\geq q(\zeta)\left(\frac{\bar{z}(\tau^{-1}(\sigma(\zeta)))}{p_2(\tau^{-1}(\sigma(\zeta)))}\right)^{\gamma/\nu}. \tag{19}$$

Since $\bar{z}'(\zeta) < 0$, we have

$$\bar{z}(m) - \bar{z}(n) = -\int_m^n \frac{(a(s)(\bar{z}'(s))^{\alpha})^{1/\alpha}}{a^{1/\alpha}(s)} \mathrm{d}s \geq A(n,m)(-a^{1/\alpha}(n)\bar{z}'(n)),$$
$$\zeta_2 \leq m \leq n.$$

Letting $m = \tau^{-1}(\sigma(\zeta))$ and $n = \varrho(\zeta)$, we conclude

$$\bar{z}(\tau^{-1}(\sigma(\zeta))) \geq A(\varrho(\zeta),\tau^{-1}(\sigma(\zeta)))(-a^{1/\alpha}(\varrho(\zeta))\bar{z}'(\varrho(\zeta))). \tag{20}$$

Combining Equations (19, 20) yields

$$\left(a(\zeta)\left(\bar{z}'(\zeta)\right)^{\alpha}\right)' \geq -q(\zeta)\left(\frac{A(\varrho(\zeta),\tau^{-1}(\sigma(\zeta)))}{p_2(\tau^{-1}(\sigma(\zeta)))}\right)^{\gamma/\nu}$$
$$(a(\varrho(\zeta))(\bar{z}'(\varrho(\zeta)))^{\alpha})^{\gamma/\alpha\nu}.$$

That is as follows:

$$\Omega'(\zeta) + q(\zeta)\left(\frac{A(\varrho(\zeta),\tau^{-1}(\sigma(\zeta)))}{p_2(\tau^{-1}(\sigma(\zeta)))}\right)^{\gamma/\nu} \Omega^{\gamma/\alpha\nu}(\varrho(\zeta)) \leq 0, \tag{21}$$

where $\Omega'(\zeta) := \left(a(\zeta)\left(\bar{z}'(\zeta)\right)^{\alpha}\right)$. By applying Lemma 1 (i), we conclude that the Equation (9) corresponding to the inequality (21) also possesses a positive solution, leading to a contradiction.

Ultimately, examine the case (IV) wherein $z(\zeta) < 0$ and $z'(\zeta) < 0$. As in the preceding case, let us take $\bar{z}(\zeta) = -z(\zeta)$. Utilizing Equation (18) in conjunction with Equation (1), we obtain

$$\left(a(\zeta)\left(\bar{z}'(\zeta)\right)^{\alpha}\right)' = q(\zeta) x^{\gamma}(\sigma(\zeta)) + r(\zeta) x^{\beta}(\varphi(\zeta))$$
$$\geq q(\zeta)\left(\frac{\bar{z}(\tau^{-1}(\sigma(\zeta)))}{p_2(\tau^{-1}(\sigma(\zeta)))}\right)^{\gamma/\nu}$$
$$+ r(\zeta)\left(\frac{\bar{z}(\tau^{-1}(\varphi(\zeta)))}{p_2(\tau^{-1}(\varphi(\zeta)))}\right)^{\beta/\nu}$$
$$\geq r(\zeta)\left(\frac{\bar{z}(\tau^{-1}(\varphi(\zeta)))}{p_2(\tau^{-1}(\varphi(\zeta)))}\right)^{\beta/\nu}.$$

Integrating from $\varrho(\zeta)$ to ζ, we get

$$\bar{z}'(\zeta) \geq \left(\frac{1}{a(\zeta)}\int_{\varrho(\zeta)}^{\zeta} \frac{r(s)}{p_2^{\beta/\nu}(\tau^{-1}(\varphi(s)))} \mathrm{d}s\right)^{1/\alpha} \bar{z}^{\beta/\nu\alpha}(\tau^{-1}(\varphi(\varrho(\zeta)))), \tag{22}$$

which has a positive solution $\bar{z}(\zeta)$. It follows from Lemma 1 (ii) that the Equation 10 corresponds the inequality (22) also has a positive solution. This completes the proof.

Corollary 1. Assume that $\tau_1(\zeta) = \tau_2(\zeta) = \tau(\zeta)$, $\lambda < 1$, $\nu > 1$, and (H_1)-(H_3) hold. Furthermore, assume that Equation (6) holds and there exists a non-decreasing function $\varrho(\zeta) \in C([\zeta_0,\infty),\mathbb{R})$ satisfies Equation (7). If

$$\lim_{\zeta\to\infty} \int_{\zeta_0}^{\zeta} q(s) A^{\gamma/\eta}(\sigma(s),s_3) \mathrm{d}s = \infty \quad \text{for } \gamma < \alpha\eta,$$

$$\lim_{\zeta\to\infty} \int_{\zeta_0}^{\zeta} q(s)\left(\frac{A(\varrho(s),\tau^{-1}(\sigma(s)))}{p_2(\tau^{-1}(\sigma(s)))}\right)^{\gamma/\nu} \mathrm{d}s = \infty \quad \text{for } \gamma < \alpha\nu,$$

and

$$\lim_{\zeta\to\infty} \int_{\zeta_0}^{\zeta} \left(\frac{1}{a(s)}\int_{\varrho(s)}^{s} \frac{r(u)}{p_2^{\beta/\nu}(\tau^{-1}(\varphi(u)))} \mathrm{d}u\right)^{1/\alpha} \mathrm{d}s = \infty \quad \text{for } \beta > \alpha\nu$$

hold for sufficiently large $\zeta_2 > \zeta_1 \geq \zeta_0$, then every solution of Equation (1) is oscillatory.

Theorem 2. Assume that $\tau_1(\zeta) = \tau_2(\zeta) = \tau(\zeta)$, $\lambda < 1$, $\nu > 1$, $\beta > \alpha\eta$, and (H$_1$)-(H$_3$) hold. Furthermore, assume that Equation (6) holds and there exists a non-decreasing function $\varrho(\zeta) \in C\left([\zeta_0,\infty),\mathbb{R}\right)$ satisfies Equation (7). If there exist function $\rho(\zeta) \in C^1\left([\zeta_0,\infty),(0,\infty)\right)$ and $c_* > 0$ such that

$$\limsup_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\left[c_2^{\beta}r(s)\rho(s)-\left(\frac{\alpha\eta a^{1/\alpha}(s)}{\beta c_*^{\beta/(\eta\alpha-1)}\rho(s)}\right)^{\alpha}\left(\frac{\rho'_+(s)}{(\alpha+1)}\right)^{\alpha+1}\right]ds=\infty, \tag{23}$$

and the differential equations (9) and (10) are oscillatory for sufficiently large $\zeta_1 \geq \zeta_0$, then every solution of Equation (1) is oscillatory.

Proof. Let $x(\zeta)$ be a non-oscillatory solution of Equation (1). Following the same proof of Theorem 1, we arrive at Equation (16). Define the Riccati transformation by

$$\omega(\zeta)=\rho(\zeta)\frac{a(\zeta)\left(z'(\zeta)\right)^{\alpha}}{z^{\beta/\eta}(\zeta)},\quad \zeta\geq\zeta_0. \tag{24}$$

Then, $\omega(\zeta) > 0$ and

$$\begin{aligned}\omega'(\zeta)&=\frac{\rho'(\zeta)}{\rho(\zeta)}\omega(\zeta)+\rho(\zeta)\frac{\left(a(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)'}{z^{\beta/\eta}(\zeta)}\\&\quad-\rho(\zeta)\frac{a(\zeta)\left(z'(\zeta)\right)^{\alpha}\frac{\beta}{\eta}z^{\beta/\eta-1}(\zeta)z'(\zeta)}{z^{2\beta/\eta}(\zeta)}\\&=\frac{\rho'(\zeta)}{\rho(\zeta)}\omega(\zeta)+\rho(\zeta)\frac{\left(a(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)'}{z^{\beta/\eta}(\zeta)}-\frac{\beta}{\eta}\frac{z'(\zeta)}{z(\zeta)}\omega(\zeta).\end{aligned}$$

Using the monotonicity of $z(\zeta)$ with Equations (15, 16), we get

$$\begin{aligned}\omega'(\zeta)&\leq\frac{\rho'(\zeta)}{\rho(\zeta)}\omega(\zeta)-c_2^{\beta}r(\zeta)\rho(\zeta)-\frac{\beta}{\eta}\frac{z'(\zeta)}{z(\zeta)}\omega(\zeta).\\&\leq\frac{\rho'(\zeta)}{\rho(\zeta)}\omega(\zeta)-c_2^{\beta}r(\zeta)\rho(\zeta)-\frac{\beta}{\eta}\frac{z^{\beta/(\eta\alpha)-1}(\zeta)}{\rho^{1/\alpha}(\zeta)a^{1/\alpha}(\zeta)}\omega^{(\alpha+1)/\alpha}(\zeta).\end{aligned}$$

Given that $z(\zeta) > 0$ and $z'(\zeta) > 0$, it follows that there exists a positive constant $c_* > 0$ such that $z(\zeta) \geq c_*$ for $\zeta \geq \zeta_2$. Consequently, we have

$$\omega'(\zeta)\leq\frac{\rho'_+(\zeta)}{\rho(\zeta)}\omega(\zeta)-c_2^{\beta}r(\zeta)\rho(\zeta)-\frac{\beta}{\eta}\frac{c_*^{\beta/(\eta\alpha-1)}}{\rho^{1/\alpha}(\zeta)a^{1/\alpha}(\zeta)}\omega^{(\alpha+1)/\alpha}(\zeta).$$

Applying the inequality (5), we get

$$\omega'(\zeta)\leq\left(\frac{\alpha\eta\rho^{1/\alpha}(\zeta)a^{1/\alpha}(\zeta)}{\beta c_*^{\beta/(\eta\alpha-1)}}\right)^{\alpha}\left(\frac{\rho'_+(\zeta)}{(\alpha+1)\rho(\zeta)}\right)^{\alpha+1}-c_2^{\beta}r(\zeta)\rho(\zeta).$$

Integrating from ($\zeta_3 > \zeta_2$) to ζ, we obtain

$$\int_{\zeta_3}^{\zeta}\left[c_2^{\beta}r(s)\rho(s)-\left(\frac{\alpha\eta a^{1/\alpha}(s)}{\beta c_*^{\beta/(\eta\alpha-1)}\rho(s)}\right)^{\alpha}\left(\frac{\rho'_+(s)}{(\alpha+1)}\right)^{\alpha+1}\right]ds\leq\omega(\zeta_3).$$

This contradiction contradicts Equation (23). By completing the proof for the two cases (III) and (IV) in a manner similar to the proof of Theorem 1, we arrive at the conclusion of the theorem.

Theorem 3. Assume that $\tau_1(\zeta) = \tau_2(\zeta) = \tau(\zeta)$, $\lambda < \nu \leq 1$, and (H$_1$)-(H$_3$) are satisfied. Additionally, assume that all other conditions of Theorem 1 are met, replacing Equation (6) with

$$\lim_{\zeta\to\infty}\left(\frac{\nu-\lambda}{\lambda}p_1^{\nu/(\nu-\lambda)}(\zeta)p_2^{\lambda/(\lambda-\nu)}(\zeta)\right)=0, \tag{25}$$

then the conclusion of Theorem 1remains valid.

Proof. Let $x(\zeta)$ be a non-oscillatory solution of Equation (1). For convenience, assume, without loss of generality, that $x(\zeta)$ becomes eventually positive. Consequently, there exists $\zeta_1 \geq \zeta_0$ such that $x(\zeta) > 0$, $x(\tau(\zeta)) > 0$, $x(\sigma(\zeta)) > 0$, and $x(\varphi(\zeta)) > 0$ for $\zeta \geq \zeta_1$. Employing a similar approach as in the proof of Theorem 1, we analyze the four cases (I)–(IV) for $z(\zeta)$.

Firstly, consider case (I). As demonstrated in the proof of case (I) for Theorem 1, it is evident that this case is not feasible.

Next, let us examine case (II). Expressing $\left[p_1(\zeta)x^{\lambda}(\tau(\zeta)) - p_2(\zeta)x^{\nu}(\tau(\zeta))\right]$ in the form

$$\begin{aligned}&p_1(\zeta)x^{\lambda}(\tau(\zeta))-p_2(\zeta)x^{\nu}(\tau(\zeta))\\&=\frac{\nu}{\lambda}p_2(\zeta)\left[x^{\lambda}(\tau(\zeta))\frac{\lambda}{\nu}\frac{p_1(\zeta)}{p_2(\zeta)}-\frac{\lambda}{\nu}\left(x^{\lambda}(\tau(\zeta))\right)^{\nu/\lambda}\right].\end{aligned}$$

Applying inequality (3) with $n = \frac{\nu}{\lambda}$, $B = x^{\lambda}(\tau(\zeta))$, $L = \frac{\lambda}{\nu}\frac{p_1(\zeta)}{p_2(\zeta)}$ and $m = \frac{\nu}{\nu-\lambda}$, we get

$$\begin{aligned}p_1(\zeta)x^{\lambda}(\tau(\zeta))-p_2(\zeta)x^{\nu}(\tau(\zeta))&\leq\frac{\nu}{\lambda}p_2(\zeta)\left[\frac{\nu-\lambda}{\nu}\right]\left(\frac{\lambda}{\nu}\frac{p_1(\zeta)}{p_2(\zeta)}\right)^{\nu/(\nu-\lambda)}\\&=\frac{\nu-\lambda}{\lambda}p_1^{\nu/(\nu-\lambda)}(\zeta)p_2^{\lambda/(\lambda-\nu)}(\zeta).\end{aligned}$$

It follows from definition of $z(\zeta)$ that

$$x(\zeta)\geq\left(1-\frac{\nu-\lambda}{\lambda}\frac{p_1^{\nu/(\nu-\lambda)}(\zeta)p_2^{\lambda/(\lambda-\nu)}(\zeta)}{z(\zeta)}\right)^{1/\eta}z^{1/\eta}(\zeta).$$

Therefore, considering Equation (25) and the positivity and increasing nature of $z(\zeta)$, it follows that there exists a constant $c_3 \in (0, 1)$ such that

$$x(\zeta)\geq c_3^{1/\eta}z^{1/\eta}(\zeta). \tag{26}$$

Concluding the proof in a manner similar to the proof of Theorem 1, replacing Equation (14) with Equation (26), leading to the conclusion of the theorem.

Remark 1. The outcomes of Corollary 1 and Theorem 2 can be directly applied to Theorem 3, replacing Equation (6) with Equation (25) and considering $\lambda < \nu \leq 1$ instead of $\lambda < 1, \nu > 1$.

4 The case $\varepsilon = +1$

In this section, we investigate the oscillatory behavior of solutions to Equation (1) under the conditions $\varepsilon = +1$, $\tau_1(\zeta) \leq \zeta$, and $\tau_2(\zeta) \geq \zeta$. Specifically, we consider cases where either of the following three conditions holds $\nu < \lambda$ with $\eta < \lambda$, or $\lambda < \nu$ with $\eta < \nu$, or $\eta < \nu < \lambda$.

Theorem 4. Assume that $\tau_1(\zeta) \le \zeta$, $\tau_2(\zeta) \ge \zeta$, $\nu < \lambda$, $\eta < \lambda$, and conditions (H_1)-(H_3) are satisfied. Additionally, suppose that for any $c_* > 0$

$$\lim_{\zeta\to\infty} S_1(\zeta) = 0. \tag{27}$$

If there exists a number $k_3 \in (0,1)$ such that the first-order delay differential equation

$$F'(\zeta) + k_3^{\gamma/\lambda} \frac{q(\zeta)A^{\gamma/\lambda}\left(\sigma(\zeta),\zeta_3\right)}{p_1^{\gamma/\lambda}(\tau_1^{-1}(\sigma(\zeta)))} F^{\gamma/\alpha\lambda}(\sigma(\zeta)) = 0 \tag{28}$$

is oscillatory for sufficiently large $\zeta_2 > \zeta_1 \ge \zeta_0$, then every solution of Equation (1) is oscillatory.

Proof. Let $x(\zeta)$ be a non-oscillatory solution of Equation (1). Without loss of generality, assume that $x(\zeta)$ is eventually positive for $\zeta \ge \zeta_0$. Therefore, there exists $\zeta_1 \ge \zeta_0$ such that $x(\zeta) > 0$, $x(\tau_1(\zeta)) > 0$, $x(\tau_2(\zeta)) > 0$, $x(\sigma(\zeta)) > 0$, and $x(\varphi(\zeta)) > 0$ for $\zeta \ge \zeta_1$. It follows that $z(\zeta) > 0$ for $\zeta \ge \zeta_1$, which means that cases (III) and (IV) mentioned before in the proof of Theorem 1 are impossible here. Consequently, we shall study the two cases, namely, case (I) and case (II) in detail.

From Equation (1), we have Equation (15). First, suppose that case (II) holds. Since $z'(\zeta) < 0$ and $\left(a(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)' < 0$, according to Equation (2), $z(\zeta)$ must be negative, which contradicts the positivity of $z(\zeta)$, making this case impossible.

Now, consider the possibility that case (II) holds. In this case, we have $z'(\zeta) > 0$ for $\zeta \ge \zeta_1$. From the definition of $z(\zeta)$, we have

$$x^{\lambda}(\tau_1(\zeta)) = \frac{1}{p_1(\zeta)}\left[z(\zeta) - x^{\eta}(\zeta) - p_2(\zeta)x^{\nu}(\tau_2(\zeta))\right].$$

It follows that

$$\begin{aligned} x^{\lambda}(\zeta) = \frac{1}{p_1(\tau_1^{-1}(\zeta))}&\left[z(\tau_1^{-1}(\zeta)) - x^{\eta}(\tau_1^{-1}(\zeta))\right.\\ &\left.-p_2(\tau_1^{-1}(\zeta))x^{\nu}(\tau_2(\tau_1^{-1}(\zeta)))\right]. \end{aligned} \tag{29}$$

Hence,

$$\begin{aligned} x^{\nu}(\tau_2(\tau_1^{-1}(\zeta))) = \frac{1}{p_1^{\frac{\nu}{\lambda}}(\delta_1(\zeta))}&\left[z(\delta_1(\zeta)) - x^{\eta}(\delta_1(\zeta))\right.\\ &\left.-p_2(\delta_1(\zeta))x^{\nu}(\tau_2(\delta_1(\zeta)))\right]^{\nu/\lambda} \end{aligned} \tag{30}$$

and

$$\begin{aligned} x^{\eta}(\tau_1^{-1}(\zeta)) = \frac{1}{p_1^{\eta/\lambda}(\delta_2(\zeta))}&\left[z(\delta_2(\zeta)) - x^{\eta}(\delta_2(\zeta))\right.\\ &\left.-p_2(\delta_2(\zeta))x^{\nu}(\tau_2(\delta_2(\zeta)))\right]^{\eta/\lambda}. \end{aligned} \tag{31}$$

Combining Equations (30, 31) with Equation (29), we get

$$\begin{aligned} x^{\lambda}(\zeta) = &\frac{1}{p_1(\tau_1^{-1}(\zeta))}[z(\tau_1^{-1}(\zeta)) - \frac{1}{p_1^{\eta/\lambda}(\delta_2(\zeta))}\left[z(\delta_2(\zeta))\right.\\ &\left.-x^{\eta}(\delta_2(\zeta)) - p_2(\delta_2(\zeta))x^{\nu}(\tau_2(\delta_2(\zeta)))\right]^{\eta/\lambda}\\ &-\frac{p_2(\tau_1^{-1}(\zeta))}{p_1^{\nu/\lambda}(\delta_1(\zeta))}\left[z(\delta_1(\zeta)) - x^{\eta}(\delta_1(\zeta))\right.\\ &\left.-p_2(\delta_1(\zeta))x^{\nu}(\tau_2(\delta_1(\zeta)))\right]^{\nu/\lambda}]. \end{aligned}$$

Applying inequality (4) with $Y = 1$, we get

$$\begin{aligned} &x^{\lambda}(\zeta)\\ &\ge \frac{1}{p_1(\tau_1^{-1}(\zeta))}\Big[z(\tau_1^{-1}(\zeta)) - \frac{1-\frac{\eta}{\lambda}}{p_1^{\eta/\lambda}(\delta_2(\zeta))} - \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(\zeta))}{p_1^{\nu/\lambda}(\delta_1(\zeta))}\\ &-\frac{\frac{\eta}{\lambda}}{p_1^{\eta/\lambda}(\delta_2(\zeta))}\left[z(\delta_2(\zeta)) - x^{\eta}(\delta_2(\zeta)) - p_2(\delta_2(\zeta))x^{\nu}(\tau_2(\delta_2(\zeta)))\right]\\ &-\frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(\zeta))}{p_1^{\nu/\lambda}(\delta_1(\zeta))}\left[z(\delta_1(\zeta)) - x^{\eta}(\delta_1(\zeta)) - p_2(\delta_1(\zeta))x^{\nu}(\tau_2(\delta_1(\zeta)))\right]\Big]\\ &\ge \frac{1}{p_1(\tau_1^{-1}(\zeta))}\Big[z(\tau_1^{-1}(\zeta)) - \frac{\frac{\eta}{\lambda}z(\delta_2(\zeta))}{p_1^{\eta/\lambda}(\delta_2(\zeta))} - \frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(\zeta))z(\delta_1(\zeta))}{p_1^{\nu/\lambda}(\delta_1(\zeta))}\\ &-\Big(\frac{1-\frac{\eta}{\lambda}}{p_1^{\eta/\lambda}(\delta_2(\zeta))} + \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(\zeta))}{p_1^{\nu/\lambda}(\delta_1(\zeta))}\Big)\Big]. \end{aligned} \tag{32}$$

Given Equation (16), which implies that $\frac{z(\zeta)}{A(\zeta,\zeta_3)}$ is decreasing, we can deduce that $\delta_2(\zeta) \ge \tau 1^{-1}(\zeta)$ and $\delta 1(\zeta) \ge \tau_2(\tau_1^{-1}(\zeta)) \ge \tau_1^{-1}(\zeta)$. Therefore, we have

$$z(\delta_2(\zeta)) \le \frac{A\left(\delta_2(\zeta),\zeta_3\right)}{A\left(\tau_1^{-1}(\zeta),\zeta_3\right)} z(\tau_1^{-1}(\zeta)) \quad \text{and}$$
$$z(\delta_1(\zeta)) \le \frac{A\left(\delta_1(\zeta)),\zeta_3\right)}{A\left(\tau_1^{-1}(\zeta),t_3\right)} z(\tau_1^{-1}(\zeta)).$$

Hence, Equation (32) takes the form

$$\begin{aligned} x^{\lambda}(\zeta) \ge &\frac{z(\tau_1^{-1}(\zeta))}{p_1(\tau_1^{-1}(\zeta))}\Big[1 - \frac{\frac{\eta}{\lambda}A\left(\delta_2(\zeta)),\zeta_3\right)}{A\left(\tau_1^{-1}(\zeta),\zeta_3\right)p_1^{\eta/\lambda}(\delta_2(\zeta))}\\ &-\frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(\zeta))A\left(\delta_1(\zeta),\zeta_3\right)}{A\left(\tau_1^{-1}(\zeta),\zeta_3\right)p_1^{\nu/\lambda}(\delta_1(\zeta))}\\ &-\Big(\frac{1-\frac{\eta}{\lambda}}{p_1^{\eta/\lambda}(\delta_2(\zeta))} + \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(\zeta))}{p_1^{\nu/\lambda}(\delta_1(\zeta))}\Big)\frac{1}{z(\tau_1^{-1}(\zeta))}\Big]. \end{aligned}$$

By virtue of the positivity and increasing fact of $z(\zeta)$, there a constant $c_* > 0$ such that $z(\zeta) \ge c_*$ for $\zeta \ge \zeta_4 \ge \zeta_3$ and consequently, we have

$$\begin{aligned} x^{\lambda}(\zeta) \ge &\frac{z(\tau_1^{-1}(\zeta))}{p_1(\tau_1^{-1}(\zeta))}\Big[1 - \frac{\frac{\eta}{\lambda}A\left(\delta_2(\zeta)),\zeta_3\right)}{A\left(\tau_1^{-1}(\zeta),\zeta_3\right)p_1^{\eta/\lambda}(\delta_2(\zeta))}\\ &-\frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(\zeta))A\left(\delta_1(\zeta),\zeta_3\right)}{A\left(\tau_1^{-1}(\zeta),\zeta_3\right)p_1^{\nu/\lambda}(\delta_1(\zeta))}\\ &-\Big(\frac{1-\frac{\eta}{\lambda}}{c_*p_1^{\eta/\lambda}(\delta_2(\zeta))} + \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(\zeta))}{c_*p_1^{\nu/\lambda}(\delta_1(\zeta))}\Big)\Big]. \end{aligned}$$

It follows from Equation (27) that there exists a constant $c_4 \in (0,1)$ such that

$$x(\zeta) \ge \frac{c_4^{1/\lambda}z^{1/\lambda}(\tau_1^{-1}(\zeta))}{p_1^{1/\lambda}(\tau_1^{-1}(\zeta))} \quad \text{for } \zeta \ge \zeta_4. \tag{33}$$

This with Equation (15) leads to

$$\begin{aligned}\left(a(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)' &\le -c_4^{\gamma/\lambda} q(\zeta)\frac{z^{\gamma/\lambda}(\tau_1^{-1}(\sigma(\zeta)))}{p_1^{\gamma/\lambda}(\tau_1^{-1}(\sigma(\zeta)))}\\ &\quad - c_4^{\beta/\lambda} r(\zeta)\frac{z^{\beta/\lambda}(\tau_1^{-1}(\varphi(\zeta)))}{p_1^{\beta/\lambda}(\tau_1^{-1}(\varphi(\zeta)))}\\ &\le - c_4^{\gamma/\lambda} q(\zeta)\frac{z^{\gamma/\lambda}(\tau_1^{-1}(\sigma(\zeta)))}{p_1^{\gamma/\lambda}(\tau_1^{-1}(\sigma(\zeta)))}. \end{aligned} \tag{34}$$

Given that $z(\zeta)$ is an increasing function and $\tau_1^{-1}(\sigma(\zeta)) \ge \sigma(\zeta)$, we can conclude that $z(\tau_1^{-1}(\sigma(\zeta))) \ge z(\sigma(\zeta))$. Hence, Equation (34) takes the form

$$\left(a(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)' \le -c_4^{\gamma/\lambda} q(\zeta)\frac{z^{\gamma/\lambda}(\sigma(\zeta))}{p_1^{\gamma/\lambda}(\tau_1^{-1}(\sigma(\zeta)))}. \tag{35}$$

Since $\left(a(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)'$ is positive and decreasing, a conclusion analogous to the proof of Theorem 1 leads us to Equation (16). By substituting Equation (16) into Equation (35), we obtain

$$F'(\zeta) + c_4^{\gamma/\lambda}\frac{q(\zeta)A^{\gamma/\lambda}\left(\sigma(\zeta),\zeta_3\right)}{p_1^{\gamma/\lambda}(\tau_1^{-1}(\sigma(\zeta)))}F^{\gamma/\alpha\lambda}(\sigma(\zeta)) \le 0, \tag{36}$$

where $F(\zeta) := a(\zeta)\left(z'(\zeta)\right)^{\alpha}$. According to Lemma 2 (i), the differential equation (27) associated with the inequality (36) also possesses a positive solution. However, this contradicts our earlier assertion. As a result, the proof is completed.

Theorem 5. Assume that $\tau_1(\zeta) \le \zeta$, $\tau_2(\zeta) \ge \zeta$, $\lambda < \nu$, $\eta < \nu$, and conditions (H_1)-(H_3) are satisfied. Additionally, suppose that

$$\lim_{t\to\infty} S_2(\zeta) = 0. \tag{37}$$

If there exists a number $k_4 \in (0, 1)$ such that the first-order delay differential equation

$$F'(\zeta) + c_5^{\gamma/\nu}\frac{q(\zeta)A^{\gamma/\nu}\left(\tau_2^{-1}(\sigma(\zeta)),\zeta_3\right)}{p_2^{\gamma/\nu}(\tau_2^{-1}(\sigma(\zeta)))}F^{\gamma/\alpha\lambda}(\sigma(\zeta)) = 0, \tag{38}$$

is oscillatory for sufficiently large $\zeta_2 > \zeta_1 \ge \zeta_0$, then every solution of Equation (1) is also oscillatory.

Proof. Let $x(\zeta)$ be a non-oscillatory solution of Equation (1). Following the approach outlined in the proof of Theorem 4, we deduce that the possible case for $z(\zeta)$ is $z(\zeta) > 0$, $z'(\zeta) > 0$ and $(a(\zeta)(z'(\zeta))^{\alpha})' < 0$ holds for $\zeta \ge \zeta_1 \ge \zeta_0$. From the definition of $z(\zeta)$, we have

$$x^{\nu}(\tau_2(\zeta)) = \frac{1}{p_2(\zeta)}\left[z(\zeta) - x^{\eta}(\zeta) - p_1(\zeta)x^{\lambda}(\tau_1(\zeta))\right],$$

It follows that

$$\begin{aligned} x^{\nu}(\zeta) = \frac{1}{p_2(\tau_2^{-1}(\zeta))}&\left[z(\tau_2^{-1}(\zeta)) - x^{\eta}(\tau_2^{-1}(\zeta))\right.\\ &\left.-p_1(\tau_2^{-1}(\zeta))x^{\lambda}(\tau_1(\tau_2^{-1}(\zeta)))\right]. \end{aligned} \tag{39}$$

Hence,

$$\begin{aligned} x^{\lambda}(\tau_1(\tau_2^{-1}(\zeta))) = \frac{1}{p_2^{\lambda/\nu}(\delta_3(\zeta))}&\left[z(\delta_3(\zeta)) - x^{\eta}(\delta_3(\zeta))\right.\\ &\left.-p_1(\delta_3(\zeta))x^{\lambda}(\tau_1(\delta_3(\zeta)))\right]^{\lambda/\nu}, \end{aligned} \tag{40}$$

and

$$\begin{aligned} x(\tau_2^{-1}(\zeta)) = \frac{1}{p_2^{\eta/\nu}(\delta_4(\zeta))}&\left[z(\delta_4(\zeta))\right.\\ &\left.-x^{\eta}(\delta_4(\zeta)) - p_1(\delta_4(\zeta))x^{\lambda}(\tau_1(\delta_4(\zeta)))\right]^{\eta/\nu}. \end{aligned} \tag{41}$$

Combining Equations (40, 41) with Equation (39), we get

$$\begin{aligned} x^{\nu}(\zeta) =& \frac{1}{p_2(\tau_2^{-1}(\zeta))}[z(\tau_2^{-1}(\zeta)) - \frac{1}{p_2^{\eta/\nu}(\delta_4(\zeta))}\left[z(\delta_4(\zeta))\right.\\ &\left.-x^{\eta}(\delta_4(\zeta)) - p_1(\delta_4(\zeta))x^{\lambda}(\tau_1(\delta_4(\zeta)))\right]^{\eta/\nu}\\ &-\frac{p_1(\tau_2^{-1}(\zeta))}{p_2^{\lambda/\nu}(\delta_3(\zeta))}\left[z(\delta_3(\zeta)) - x^{\eta}(\delta_3(\zeta))\right.\\ &\left.-p_1(\delta_3(\zeta))x^{\lambda}(\tau_1(\delta_3(\zeta)))\right]^{\lambda/\nu}]. \end{aligned}$$

Applying inequality (4) with $Y = 1$, we get

$$\begin{aligned} x^{\nu}(\zeta) \ge& \frac{1}{p_2(\tau_2^{-1}(\zeta))}[z(\tau_2^{-1}(\zeta))\\ &-\frac{\frac{\eta}{\nu}}{p_2^{\eta/\nu}(\delta_4(\zeta))}\left[z(\delta_4(\zeta)) - x^{\eta}(\delta_4(\zeta))\right.\\ &\left.-p_1(\delta_4(\zeta))x^{\lambda}(\tau_1(\delta_4(\zeta)))\right]\\ &-\frac{\frac{\lambda}{\nu}p_1(\tau_2^{-1}(\zeta))}{p_2^{\lambda/\nu}(\delta_3(\zeta))}\left[z(\delta_3(\zeta)) - x^{\eta}(\delta_3(\zeta))\right.\\ &\left.-p_1(\delta_3(\zeta))x^{\lambda}(\tau_1(\delta_3(\zeta)))\right]\\ &-\frac{(1-\frac{\eta}{\nu})}{p_2^{\eta/\nu}(\delta_4(\zeta))} - \frac{(1-\frac{\lambda}{\nu})p_1(\tau_2^{-1}(\zeta))}{p_2^{\lambda/\nu}(\delta_3(\zeta))}]\\ \ge& \frac{1}{p_2(\tau_2^{-1}(\zeta))}[z(\tau_2^{-1}(\zeta)) - \frac{\frac{\eta}{\nu}z(\delta_4(\zeta))}{p_2^{\eta/\nu}(\delta_4(\zeta))}\\ &-\frac{\frac{\lambda}{\nu}p_1(\tau_2^{-1}(\zeta))z(\delta_3(\zeta))}{p_2^{\lambda/\nu}(\delta_3(\zeta))}\\ &-\frac{(1-\frac{\eta}{\nu})}{p_2^{\eta/\nu}(\delta_4(\zeta))} - \frac{(1-\frac{\lambda}{\nu})p_1(\tau_2^{-1}(\zeta))}{p_2^{\lambda/\nu}(\delta_3(\zeta))}]. \end{aligned} \tag{42}$$

Since $\frac{z(\zeta)}{A(\zeta,\zeta_3)}$ is decreasing, $\delta_4(\zeta) \ge \tau_2^{-1}(\zeta)$ and $\delta_3(\zeta) \ge \tau_1(\tau_2^{-1}(\zeta)) \ge \tau_2^{-1}(\zeta)$, we have $z(\delta_4(\zeta)) \le z(\tau_2^{-1}(\zeta))$ and $z(\delta_3(\zeta)) \le z(\tau_2^{-1}(\zeta))$. Hence, Equation (42) takes the form

$$\begin{aligned} x^{\nu}(\zeta) \ge& \frac{z(\tau_2^{-1}(\zeta))}{p_2(\tau_2^{-1}(\zeta))}\Big[1 - \Big(\frac{\frac{\eta}{\nu}}{p_2^{\eta/\nu}(\delta_4(\zeta))} + \frac{\frac{\lambda}{\nu}p_1(\tau_2^{-1}(\zeta))}{p_2^{\lambda/\nu}(\delta_3(\zeta))}\\ &+\Big(\frac{(1-\frac{\eta}{\nu})}{p_2^{\eta/\nu}(\delta_4(\zeta))} - \frac{(1-\frac{\lambda}{\nu})p_1(\tau_2^{-1}(\zeta))}{p_2^{\lambda/\nu}(\delta_3(\zeta))}\Big)\frac{1}{z(\tau_2^{-1}(\zeta))}\Big)\Big]. \end{aligned}$$

By virtue of the positivity and increasing fact of $z(\zeta)$, there a constant $c_* > 0$ such that $z(\zeta) \ge c_*$ for $\zeta \ge \zeta_4 \ge \zeta_3$ and consequently, we have

$$x^{\nu}(\zeta) \geq \frac{z(\tau_2^{-1}(\zeta))}{p_2(\tau_2^{-1}(\zeta))}\Big[1-\Big(\frac{\frac{\eta}{\nu}}{p_2^{\eta/\nu}(\delta_4(\zeta))}+\frac{\frac{\lambda}{\nu}p_1(\tau_2^{-1}(\zeta))}{p_2^{\lambda/\nu}(\delta_3(\zeta))} + \frac{(1-\frac{\eta}{\nu})}{c_* p_2^{\eta/\nu}(\delta_4(\zeta))}-\frac{(1-\frac{\lambda}{\nu})p_1(\tau_2^{-1}(\zeta))}{c_* p_2^{\lambda/\nu}(\delta_3(\zeta))}\Big)\Big].$$

It follows from Equation (37) that there exists a constant $c_5 \in (0,1)$ such that

$$x(\zeta) \geq \frac{c_5^{1/\nu} z^{1/\nu}(\tau_2^{-1}(\zeta))}{p_2^{1/\nu}(\tau_2^{-1}(\zeta))}.$$

This with Equation (15) leads to

$$\begin{aligned}\left(a(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)' &\leq -c_4^{\gamma/\nu} q(\zeta)\frac{z^{\gamma/\nu}(\tau_2^{-1}(\sigma(\zeta)))}{p_2^{\gamma/\nu}(\tau_2^{-1}(\sigma(\zeta)))} - c_4^{\frac{\beta}{\nu}} r(\zeta)\frac{z^{\beta/\nu}(\tau_2^{-1}(\varphi(\zeta)))}{p_2^{\beta/\nu}(\tau_2^{-1}(\varphi(\zeta)))}\\ &\leq -c_4^{\gamma/\nu} q(\zeta)\frac{z^{\gamma/\nu}(\tau_2^{-1}(\sigma(\zeta)))}{p_2^{\gamma/\nu}(\tau_2^{-1}(\sigma(\zeta)))}. \end{aligned} \tag{43}$$

Since $\left(a(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)'$ is positive and decreasing, a conclusion analogous to the proof of Theorem 1 leads us to Equation (16). By substituting Equation (16) into Equation (43), we obtain

$$F'(\zeta) + c_5^{\gamma/\nu}\frac{q(\zeta)A^{\gamma/\nu}\left(\tau_2^{-1}(\sigma(\zeta)),\zeta_3\right)}{p_2^{\gamma/\nu}(\tau_2^{-1}(\sigma(\zeta)))}F^{\gamma/\alpha\lambda}(\sigma(\zeta)) \leq 0, \tag{44}$$

where $F(\zeta) := a(\zeta)\left(z'(\zeta)\right)^{\alpha}$. According to Lemma 2 (i), the differential equation (37) associated with the inequality (44) also possesses a positive solution. However, this contradicts our earlier assertion. As a result, the proof is completed.

Theorem 6. Assume that $\tau_1(\zeta) \leq \zeta$, $\tau_2(\zeta) \geq \zeta$, $\eta < \nu < \lambda$, and conditions (H_1)-(H_3) are satisfied. Additionally, suppose that

$$\lim_{\zeta\to\infty} S_3(\zeta) = 0. \tag{45}$$

If there exists a number $k_3 \in (0,1)$ such that the first-order delay differential equation (28) is oscillatory for sufficiently large $\zeta_2 > \zeta_1 \geq \zeta_0$, then every solution of Equation (1) is also oscillatory.

Proof. Let $x(\zeta)$ be a non-oscillatory solution of Equation (1). Following the approach outlined in the proof of Theorem 4, we deduce that the possible case for $z(\zeta)$ is $z(\zeta) > 0$, $z'(\zeta) > 0$ and $(a(\zeta)(z'(\zeta))^{\alpha})' < 0$ holds for $\zeta \geq \zeta_1 \geq \zeta_0$. From the definition of $z(\zeta)$, we have Equation (29). Hence,

$$x^{\eta}(\tau_1^{-1}(\zeta)) = \frac{1}{p_2^{\eta/\nu}(\delta_5(\zeta))}\left[z(\delta_5(\zeta)) - x^{\eta}(\delta_5(\zeta)) - p_1(\delta_5(\zeta))x^{\lambda}(\tau_1(\delta_5(\zeta)))\right]^{\eta/\nu}, \tag{46}$$

and

$$x^{\nu}(\tau_2(\tau_1^{-1}(\zeta))) = \frac{1}{p_1^{\nu/\lambda}(\delta_1(\zeta))}\left[z(\delta_1(\zeta)) - x^{\eta}(\delta_1(\zeta)) - p_2(\delta_1(\zeta))x^{\nu}(\tau_2(\delta_1(\zeta)))\right]^{\nu/\lambda}. \tag{47}$$

Combining Equations (46, 47) with Equation (29), we get

$$\begin{aligned}x^{\lambda}(\zeta) =& \frac{z(\tau_1^{-1}(\zeta))}{p_1(\tau_1^{-1}(\zeta))} - \frac{1}{p_1(\tau_1^{-1}(\zeta))}\left[\frac{\left[z(\delta_5(\zeta)) - x^{\eta}(\delta_5(\zeta)) - p_1(\delta_5(\zeta))x^{\lambda}(\tau_1(\delta_5(\zeta)))\right]^{\eta/\nu}}{p_2^{\eta/\nu}(\delta_5(\zeta))}\right]\\ &- \frac{p_2(\tau_1^{-1}(\zeta))}{p_1(\tau_1^{-1}(\zeta))}\left[\frac{\left[z(\delta_1(\zeta)) - x^{\eta}(\delta_1(\zeta)) - p_2(\delta_1(\zeta))x^{\nu}(\tau_2(\delta_1(\zeta)))\right]^{\nu/\lambda}}{p_1^{\nu/\lambda}(\delta_1(\zeta))}\right].\end{aligned}$$

Applying the inequality (4) with $Y = 1$, we obtain

$$\begin{aligned}x^{\lambda}(\zeta) =& \frac{z(\tau_1^{-1}(\zeta))}{p_1(\tau_1^{-1}(\zeta))} - \frac{\frac{\eta}{\nu}z(\delta_5(\zeta))}{p_1(\tau_1^{-1}(\zeta))p_2^{\eta/\nu}(\delta_5(\zeta))} - \frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(\zeta))z(\delta_1(\zeta))}{p_1(\tau_1^{-1}(\zeta))p_1^{\nu/\lambda}(\delta_1(\zeta))}\\ &- \frac{(1-\frac{\eta}{\nu})}{p_1(\tau_1^{-1}(\zeta))p_2^{\eta/\nu}(\delta_5(\zeta))} - \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(\zeta))}{p_1(\tau_1^{-1}(\zeta))p_1^{\nu/\lambda}(\delta_1(\zeta))}.\end{aligned} \tag{48}$$

Since $\frac{z(\zeta)}{A(\zeta,\zeta_3)}$ is decreasing, $\delta_5(\zeta) \geq \tau_1^{-1}(\zeta)$ and $\delta_1(\zeta) \geq \tau_1^{-1}(\zeta)$, we have $z(\delta_5(\zeta)) \leq z(\tau_1^{-1}(\zeta))$ and $z(\delta_1(\zeta)) \leq \frac{A(\delta_1(\zeta),\zeta_3)}{A(\tau_1^{-1}(\zeta),\zeta_3)}z(\tau_1^{-1}(\zeta))$. Hence, Equation (48) takes the form

$$\begin{aligned}x^{\lambda}(\zeta) =& \frac{z(\tau_1^{-1}(\zeta))}{p_1(\tau_1^{-1}(\zeta))}\Big[1 - \frac{\frac{\eta}{\nu}}{p_2^{\eta/\nu}(\delta_5(\zeta))} - \frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(\zeta))A\left(\delta_1(\zeta),\zeta_3\right)}{p_1^{\nu/\lambda}(\delta_1(\zeta))A\left(\tau_1^{-1}(\zeta),\zeta_3\right)}\\ &- \Big(\frac{(1-\frac{\eta}{\nu})}{p_2^{\eta/\nu}(\delta_5(\zeta))} + \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(\zeta))}{p_1^{\nu/\lambda}(\delta_1(\zeta))}\Big)\frac{1}{z(\tau_1^{-1}(\zeta))}\Big]\\ \geq& \frac{z(\tau_1^{-1}(\zeta))}{p_1(\tau_1^{-1}(\zeta))}\Big[1 - \frac{\frac{\eta}{\nu}}{p_2^{\eta/\nu}(\delta_5(\zeta))} - \frac{\frac{\nu}{\lambda}p_2(\tau_1^{-1}(\zeta))A\left(\delta_1(\zeta),\zeta_3\right)}{p_1^{\nu/\lambda}(\delta_1(\zeta))A\left(\tau_1^{-1}(\zeta),\zeta_3\right)}\\ &- \frac{(1-\frac{\eta}{\nu})}{c_* p_2^{\eta/\nu}(\delta_5(\zeta))} - \frac{(1-\frac{\nu}{\lambda})p_2(\tau_1^{-1}(\zeta))}{c_* p_1^{\nu/\lambda}(\delta_1(\zeta))}\Big]\end{aligned}$$

for $\zeta \geq \zeta_4 \geq \zeta_3$. Now from Equation (45), it follows that there exists a constant $c_6 \in (0,1)$ such that

$$x(\zeta) \geq \frac{c_6^{1/\lambda} z^{1/\lambda}(\tau_1^{-1}(\zeta))}{p_1^{1/\lambda}(\tau_1^{-1}(\zeta))} \quad \text{for } \zeta \geq \zeta_4 \tag{49}$$

Completing the proof following the steps outlined in the proof of Theorem 4 and substituting Equation (33) with Equation (49), we arrive at the same conclusion as stated in the theorem. This completes the proof.

Corollary 2. Assume that all the hypotheses of Theorem 4 are satisfied, with the modification that the condition

$$\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\frac{q(s)A^{\gamma/\lambda}\left(\sigma(s),\zeta_1\right)}{p_1^{\gamma/\lambda}(\tau_1^{-1}(\sigma(s)))}\mathrm{d}s = \infty \ \text{ for } \gamma < \alpha\lambda$$

is used instead of the condition in Equation (28), and then, the conclusion of Theorem 4 holds.

Corollary 3. Assume that all the hypotheses of Theorem 5 are satisfied, with the modification that the condition

$$\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\frac{q(s)A^{\gamma/\nu}\left(\tau_2^{-1}(\sigma(s)),\zeta_1\right)}{p_2^{\gamma/\nu}(\tau_2^{-1}(\sigma(s)))}\mathrm{d}s = \infty \ \text{ for } \gamma < \alpha\nu$$

is used instead of the condition in Equation (38), and then, the conclusion of Theorem 5 holds.

Now, we consider the following special case of Equation (1). We consider $\eta = \lambda = \nu = 1$ and $\alpha = \beta = \gamma$ with $p_2(\zeta) \geq 0$ and $p_1(\zeta) \geq 1$ eventually, namely

$$\begin{aligned}&\left(a(\zeta)\left((x(\zeta)+p_1(\zeta)x(\tau_1(\zeta))+x(\tau_2(\zeta)))'\right)^{\alpha}\right)'\\&+q(\zeta)x^{\alpha}(\sigma(\zeta))+r(\zeta)x^{\alpha}(\varphi(\zeta))=0,\quad \zeta\geq\zeta_0>0, \qquad (50)\end{aligned}$$

For the sake of notation, we define:

$$\begin{aligned}\phi(\zeta)=\frac{1}{p_1(\tau_1^{-1}(\zeta))}\Bigg[1&-\frac{A\left(\delta_2(\zeta)),\zeta_3\right)}{A\left(\tau_1^{-1}(\zeta),\zeta_3\right)p_1(\delta_2(\zeta))}\\&-\frac{p_2(\tau_1^{-1}(\zeta))A\left(\delta_1(\zeta),\zeta_3\right)}{A\left(\tau_1^{-1}(\zeta),\zeta_3\right)p_1(\delta_1(\zeta))}\Bigg].\end{aligned}$$

Theorem 7. Suppose that conditions (H$_1$)- (H$_3$) hold. If there exists a positive function $\rho \in C^1([\zeta_0,\infty),\mathbb{R})$ such that

$$\limsup_{\zeta\to\infty}\int_{\zeta_1}^{\zeta}\left(\rho(s)r(s)\phi^{\alpha}(\tau_1^{-1}(\varphi(s)))-\frac{a(s)(\rho'^{+}(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\right)ds=\infty, \qquad (51)$$

then Equation (50) is oscillatory.

Proof. Let $x(\zeta)$ be a non-oscillatory solution of Equation (50). Employing a procedure akin to the proof of Theorem 4 with $\eta = \lambda = \nu = 1$, Equation (32)can be expressed as follows:

$$\begin{aligned}x(\zeta)\geq\frac{z(\tau_1^{-1}(\zeta))}{p_1(\tau_1^{-1}(\zeta))}\Bigg[1&-\frac{A\left(\delta_2(\zeta)),\zeta_3\right)}{A\left(\tau_1^{-1}(\zeta),\zeta_3\right)p_1(\delta_2(\zeta))}\\-\frac{p_2(\tau_1^{-1}(\zeta))A\left(\delta_1(\zeta),\zeta_3\right)}{A\left(\tau_1^{-1}(\zeta),\zeta_3\right)p_1(\delta_1(\zeta))}\Bigg]&:=\phi(\zeta)z(\tau_1^{-1}(\zeta)).\end{aligned}$$

Let us redefine the Riccati substitution ω according to Equation (24), setting $\alpha = \beta$ and $\eta = 1$, yielding

$$\omega'(\zeta)=\frac{\rho'(\zeta)}{\rho(\zeta)}\omega(\zeta)+\rho(\zeta)\frac{\left(a(\zeta)\left(z'(\zeta)\right)^{\alpha}\right)'}{z^{\alpha}(\zeta)}-\frac{z'(\zeta)}{z(\zeta)}\omega(\zeta).$$

Utilizing the monotonicity properties of $z(\zeta)$ with Equation (16), we get

$$\begin{aligned}\omega'(\zeta)&\leq\frac{\rho'(\zeta)}{\rho(\zeta)}\omega(\zeta)-\rho(\zeta)r(\zeta)\phi^{\alpha}(\tau_1^{-1}(\varphi(\zeta)))\frac{z^{\alpha}(\tau_1^{-1}(\varphi(\zeta)))}{z^{\alpha}(\zeta)}\\&-\frac{\alpha}{\rho^{1/\alpha}(\zeta)a^{1/\alpha}(\zeta)}\omega^{(\alpha+1)/\alpha}(\zeta)\\&\leq\frac{\rho'(\zeta)}{\rho(\zeta)}\omega(\zeta)-\rho(\zeta)r(\zeta)\phi^{\alpha}(\tau_1^{-1}(\varphi(\zeta)))\\&-\frac{\alpha}{\rho^{1/\alpha}(\zeta)a^{1/\alpha}(\zeta)}\omega^{(\alpha+1)/\alpha}(\zeta).\end{aligned}$$

Applying the inequality (5), we get

$$\omega'(\zeta)\leq\frac{a(\zeta)(\rho'^{+}(\zeta))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(\zeta)}-\rho(\zeta)r(\zeta)\phi^{\alpha}(\tau_1^{-1}(\varphi(\zeta))).$$

Integrating from ζ_1 to ζ, we get

$$\int_{\zeta_1}^{\zeta}\left(\rho(s)r(s)\phi^{\alpha}(\tau_1^{-1}(\varphi(s)))-\frac{a(s)(\rho'^{+}(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)}\right)ds\leq\omega(\zeta_1).$$

This contradicts Equation (51) and concludes the proof.

Example 1. Consider the second-order differential equation:

$$\begin{aligned}&\left(\left(\left(x^2(\zeta)+\frac{1}{\zeta}x^{1/2}(\zeta/2)-\zeta^2x(\zeta/2)\right)'\right)^2\right)'\\&+q_0x^3(\zeta/6)+r_0x^5(12\zeta)=0,\;\;\zeta\geq1 \qquad (52)\end{aligned}$$

Here, $a(\zeta) = 1$, $\eta = 2$, $p_1(\zeta) = \frac{1}{\zeta}$, $\lambda = \frac{1}{2}$, $\tau_1(\zeta) = \tau_2(\zeta) = \tau(\zeta) = \frac{\zeta}{2}$, $\varepsilon = -1$, $p_2(\zeta) = \zeta^2$, $\nu = 2$, $\alpha = 2$, $q(\zeta) = q_0$, $\gamma = 2$, $\sigma(\zeta) = \zeta/6$, $r(\zeta) = r_0$, $\beta = 5$ and $\varphi(\zeta) = 12\zeta$. It is clear that $\int_{\zeta_0}^{\infty} a^{-1/\alpha}(s)ds = \infty$ and

$$\lim_{\zeta\to\infty}\left[R_1(\zeta)+R_2(\zeta)\right]=\lim_{\zeta\to\infty}\left[\frac{1}{4\zeta^2}-\frac{1}{4\zeta^2}\right]=0.$$

Choose $\varrho(\zeta) = \frac{\zeta}{2}$, where $\tau^{-1}(\sigma(\zeta)) \leq \varrho(\zeta) \leq \zeta$, $\tau^{-1}(\varphi(\zeta)) \geq \zeta$ and $\tau^{-1}(\varphi(\varrho(\varrho(\zeta)))) \geq \zeta$, and consequently, we have

$$\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}q(s)A^{\gamma/\eta}(s,s_1)ds=\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}q_0(s^{3/2}-s_1^{3/2})ds=\infty,$$

$$\begin{aligned}&\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}q(s)\left(\frac{A(\varrho(s),\tau^{-1}(\sigma(s)))}{p_2(\tau^{-1}(\sigma(s)))}\right)^{\gamma/\nu}ds\\&=\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}q_0\left(\frac{s/6}{s^2/8}\right)^{\frac{3}{2}}ds=\infty\end{aligned}$$

and

$$\begin{aligned}&\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\left(\frac{1}{a(s)}\int_{\varrho(s)}^{s}\frac{r(u)}{p_2^{\beta/\nu}(\tau^{-1}(\varphi(u)))}du\right)^{1/\alpha}ds\\&=\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\left(\int_{s/2}^{s}\frac{r_0}{(24u)^5}du\right)^{1/2}ds=\infty.\end{aligned}$$

Therefore, all the assumptions stated in Corollary 1 are satisfied, indicating that every solution $x(\zeta)$ to Equation (52) exhibits oscillatory behavior.

Example 2. Consider the second-order differential equation

$$\begin{aligned}&\left(\zeta\left(x(\zeta)+\zeta^{\frac{1}{10}}x^{1/5}(\zeta/3)-\zeta^{1/3}x^{1/3}(\zeta/3)\right)'\right)'\\&+\zeta^2x(\zeta/5)+\zeta^5x^5(4\zeta)=0,\;\;\zeta\geq1. \qquad (53)\end{aligned}$$

Here, $a(\zeta) = \zeta^2$, $\eta = 1$, $p_1(\zeta) = \zeta^{1/10}$, $\lambda = \frac{1}{5}$, $\tau_1(\zeta) = \tau_2(\zeta) = \tau(\zeta) = \frac{\zeta}{3}$, $\varepsilon = -1$, $p_2(\zeta) = \zeta^{1/3}$, $\nu = 1/3$, $\alpha = 1$, $q(\zeta) = \zeta^2$, $\gamma = 1$, $\sigma(\zeta) = \zeta/5$, $r(\zeta) = \zeta^5$, $\beta = 5$ and $\varphi(\zeta) = 4\zeta$. It is clear that $\int_{\zeta_0}^{\infty} a^{-1/\alpha}(s)ds = \infty$ and condition (25) becomes

$$\begin{aligned}&\lim_{\zeta\to\infty}\left(\frac{\nu-\lambda}{\lambda}p_1^{\nu/(\nu-\lambda)}(\zeta)p_2^{\frac{\lambda}{\lambda-\nu}}(\zeta)\right)\\&=\lim_{\zeta\to\infty}\left(\frac{1/3-1/5}{1/5}(\zeta^{1/10})^{\frac{1/3}{1/3-1/5}}(\zeta^{1/3})^{\frac{1/5}{1/5-1/3}}\right)=0.\end{aligned}$$

Choose $\varrho(\zeta) = \frac{4\zeta}{5}$, where $\tau^{-1}(\sigma(\zeta)) \leq \varrho(\zeta) \leq \zeta$, $\tau^{-1}(\varphi(\zeta)) \geq \zeta$ and $\tau^{-1}(\varphi(\varrho(\varrho(\zeta)))) \geq \zeta$, consequently we have

$$\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta} q(s)A^{\gamma/\eta}(s,s_1)\mathrm{d}s = \lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta} s^2\ln(s)\mathrm{d}s = \infty,$$

$$\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta} q(s)\left(\frac{A(\varrho(s),\tau^{-1}(\sigma(s)))}{p_2(\tau^{-1}(\sigma(s)))}\right)^{\gamma/\nu}\mathrm{d}s = \lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta} s^2\left(\frac{\ln(4s/5)-\ln(3s/5)}{(3s/5)^{1/3}}\right)^3\mathrm{d}s = \infty,$$

and

$$\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\left(\frac{1}{a(s)}\int_{\varrho(s)}^{s}\frac{r(u)}{p_2^{\beta/\nu}(\tau^{-1}(\varphi(u)))}du\right)^{1/\alpha}\mathrm{d}s = \lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\left(\frac{1}{u}\int_{4s/5}^{s}\frac{u^5}{(12u)^5}du\right)\mathrm{d}s = \infty.$$

It is evident that the requirements outlined in Corollary 1, transferred to Theorem 1 by substituting Equation (25) in place of Equation (6) and $\lambda < \nu \leq 1$ instead of $\lambda < 1, \nu > 1$, are fulfilled. Consequently, each solution to Equation (53) demonstrates oscillatory behavior.

Remark 2. Note that oscillation results presented in the study by Grace et al. [13] fail to apply to the Equation (53), where $0 < \lambda, \nu < 1$ and $\gamma \neq \beta$ unlike in the study by Grace et al. [13].

Example 3. Consider the second-order differential equation

$$\left(\zeta^3\left(\left(x^{1/5}(\zeta)+\zeta^5x^5(\zeta/2)+\frac{1}{\zeta}x^{1/3}(3\zeta)\right)'\right)^5\right)' + \zeta^2x^3(\zeta/3) + \frac{1}{\zeta^2}x^3(2\zeta) = 0,\quad \zeta \geq 1. \tag{54}$$

Here, $a(\zeta) = \zeta^3$, $\eta = 1/5$, $p_1(\zeta) = \zeta^5$, $\lambda = 5$, $\tau_1(\zeta) = \zeta/2$, $\tau_2(\zeta) = 3\zeta$, $\varepsilon = 1$, $p_2(\zeta) = \frac{1}{\zeta}$, $\nu = 1/3$, $\alpha = 5$, $q(\zeta) = \zeta^2$, $\gamma = 3$, $\sigma(\zeta) = \zeta/3$, $r(\zeta) = \frac{1}{\zeta^2}$, $\beta = 3$ and $\varphi(\zeta) = 2\zeta$. It is clear that $\int_{\zeta_0}^{\infty} a^{-1/\alpha}(s)\mathrm{d}s = \infty$ and $\lim_{\zeta\to\infty} S_1(\zeta) = 0$. Since

$$A(\zeta,\zeta_1) = \int_{\zeta_1}^{\zeta}(s^3)^{-1/5}\mathrm{d}s = \frac{5}{2}(\zeta^{2/5}-\zeta_1^{2/5}),$$

we observe that

$$\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\frac{q(s)A^{\gamma/\lambda}(\sigma(s),\zeta_1)}{p_1^{\gamma/\lambda}(\tau_1^{-1}(\sigma(s)))}\mathrm{d}s = \left(\frac{5}{2}\right)^{3/5}\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\frac{s^2((s/3)^{2/5}-s_1^{2/5})^{\frac{3}{5}}}{(2s/3)^3}\mathrm{d}s = \infty.$$

According to Corollary 2, every solution of Equation (54) is oscillatory.

Example 4. Consider the second-order differential equation

$$\left(\zeta\left(\left(x^3(\zeta)+\frac{1}{\zeta}x(\zeta/4)+5\zeta x^4(6\zeta)\right)'\right)^3\right)' + \zeta x^{3/2}(\zeta/3) + \frac{1}{\zeta}x^5(2\zeta) = 0,\quad \zeta \geq 1.$$

Here, $a(\zeta) = \zeta$, $\eta = 1/5$, $p_1(\zeta) = \zeta^5$, $\lambda = 5$, $\tau_1(\zeta) = \zeta/2$, $\tau_2(\zeta) = 3\zeta$, $\varepsilon = 1$, $p_2(\zeta) = \frac{1}{\zeta}$, $\nu = 4$, $\alpha = 5$, $q(\zeta) = \zeta$, $\gamma = 3/2$, $\sigma(\zeta) = \zeta/3$, $r(\zeta) = \frac{1}{\zeta}$, $\beta = 5$ and $\varphi(\zeta) = 2\zeta$. It is clear that $\int_{\zeta_0}^{\infty} a^{-1/\alpha}(s)\mathrm{d}s = \infty$ and $\lim_{\zeta\to\infty} S_2(\zeta) = 0$. Since

$$A(\zeta,\zeta_1) = \int_{\zeta_1}^{\zeta}(s)^{-1/3}\mathrm{d}s = \frac{3}{2}(\zeta^{2/3}-\zeta_1^{2/3}),$$

we observed that

$$\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\frac{q(s)A^{\gamma/\nu}\left(\tau_2^{-1}(\sigma(s)),\zeta_1\right)}{p_2^{\gamma/\nu}(\tau_2^{-1}(\sigma(s)))}\mathrm{d}s = \lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\frac{s\left(\frac{3}{2}((s/18)^{2/3}-s_1^{2/3})\right)^{3/8}}{(s/18)^{\frac{3}{8}}}\mathrm{d}s = \infty.$$

According to Corollary 3, every solution of Equation (54) is oscillatory.

Example 5. Consider the neutral differential equation

$$(x(\zeta)+\zeta x(\zeta-2\pi)+x(\zeta+\pi))'' + 2x\left(\zeta-\frac{\pi}{2}\right) + \zeta x(\zeta+2\pi) = 0,\quad \zeta \geq 5. \tag{55}$$

Here, we have $\eta = \lambda = \nu = \alpha = \beta = \gamma = 1$, $a(\zeta) = 1$, $p_1(\zeta) = \zeta$, $p_2(\zeta) = 1$ $\tau_1(\zeta) = \zeta - 2\pi$, $\tau_2(\zeta) = \zeta + \pi$, $q(\zeta) = 2$, $r(\zeta) = \zeta$, $\sigma(\zeta) = \zeta - \frac{\pi}{2}$ and $\varphi(\zeta) = \zeta + 2\pi$. It is clear that $A(\zeta,\zeta_0) = \zeta - 5$ and $\phi(\zeta) > 0$. Setting $\rho(\zeta) = 1$, it becomes evident that condition (51) is satisfied. Thus, according to Theorem 7, every solution of Equation (55) is oscillatory. Indeed, $x(\zeta) = \sin(\zeta)$ is such a solution.

Remark 3. It is important to note that the oscillation results outlined in the study by Thandapani et al. [24] and Thandapani and Rama [25] can not directly apply to Equation (55) primarily because $\alpha = \beta = \gamma$ in Equation (55), whereas in the study by Thandapani et al. [24] and Thandapani and Rama [25], this condition does not hold.

5 Conclusion and discussion

The findings of this study are showcased in a fundamentally innovative and broadly applicable manner. These findings not only enrich but also complement the current literature study, as referenced in the studies by Agarwal et al. [14], Bohner et al. [15], Džurina et al. [16], Grace et al. [17], Lin and Tang [18], and Muhib et al. [19]. Furthermore, Equation (1) represents a more general formulation, where ε can take values of either -1 or 1, and (α, β, and γ) are distinct, with η differing from 1.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

AH: Conceptualization, Investigation, Methodology, Writing – original draft. OM: Conceptualization, Formal analysis, Methodology, Writing – original draft. SA: Formal analysis, Investigation, Methodology, Writing – review & editing. AA: Formal analysis, Investigation, Methodology, Writing – review & editing.

Funding

The author(s) declare financial support was received for the research, authorship, and/or publication of this article. This project was funded by King Saud University, Riyadh, Saudi Arabia, Project number (RSPD2024R533).

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

All claims expressed in this article are solely those of the authors and do not necessarily represent those of their affiliated organizations, or those of the publisher, the editors and the reviewers. Any product that may be evaluated in this article, or claim that may be made by its manufacturer, is not guaranteed or endorsed by the publisher.

References

1. Agarwal RP, Grace SR, O'Regan D. *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations*. Heidelberg: Springer Science & Business Media (2013).

2. Agarwal RP, Grace SR, O'Regan D. *Oscillation Theory for Second Order Dynamic Equations*. Boca Raton, FL: CRC Press (2002).

3. Grace SR, Agarwal RP, Bohner M, O'Regan D. Oscillation of second-order strongly superlinear and strongly sublinear dynamic equations. *Commun Nonlinear Sci Numer Simul*. (2009) 14:3463–71. doi: 10.1016/j.cnsns.2009.01.003

4. Grace SR, Agarwal RP, Kaymakçalan B, Sae-Jie W. Oscillation theorems for second order nonlinear dynamic equations. *J Appl Math Comp*. (2010) 32:205. doi: 10.1007/s12190-009-0244-7

5. Grace SR, Agarwal RP, O'Regan D. A selection of oscillation criteria for second-order differential inclusions. *Appl Math Lett*. (2009) 22:153–8. doi: 10.1016/j.aml.2008.01.006

6. Hassan A, Affan S. New oscillation criteria and some refinements for second-order neutral delay dynamic equations on time scales. *J Math Comp Sci*. (2023) 28:192–202. doi: 10.22436/jmcs.028.02.07

7. Hassana A, Affana S. Oscillation criteria for second-order delay dynamic equations with a sub-linear neutral term on time scales. *Filomat*. (2023) 37:7445–54. doi: 10.2298/FIL2322445H

8. Moaaz O, Muhib A, Santra SS. An oscillation test for solutions of second-order neutral differential equations of mixed type. *Mathematics*. (2021) 9:1634. doi: 10.3390/math9141634

9. Moaaz O, Anis M, Baleanu D, Muhib A. More effective criteria for oscillation of second-order differential equations with neutral arguments. *Mathematics*. (2020) 8:986. doi: 10.3390/math8060986

10. Hale JK. Functional differential equations. In: *Analytic Theory of Differential Equations: The Proceedings of the Conference at Western Michigan University, Kalamazoo, from 30 April to 2 May 1970*. Springer (2006). p. 9–22.

11. Tunç E, Özdemir O. On the oscillation of second-order half-linear functional differential equations with mixed neutral term. *J Taibah Univ Sci*. (2019) 13:481–9. doi: 10.1080/16583655.2019.1595948

12. Agwa H, Khodier AM, Arafa HM. Oscillation of second-order nonlinear neutral dynamic equations with mixed arguments on time scales. *J Basic Appl Res Int*. (2016) 17:49–66.

13. Grace SR, Graef JR, Kaleeswari S. *New Criteria on the Oscillatory Behaviour of Second Order Nonlinear Differential Equations with Mixed Neutral Terms*. Kraków: Tatra Mountains Mathematical Publications (2023).

14. Agarwal RP, Bohner M, Li T, Zhang C. Oscillation of second-order differential equations with a sublinear neutral term. *Carpath J Math*. (2014) 30:1–6.

15. Bohner M, Sudha B, Tangavelu K, Thandapani E. Oscillation criteria for second-order differential equations with superlinear neutral term. *Nonlinear Stud*. (2019) 26:425–34.

16. Džurina J, Grace SR, Jadlovská I, Li T. Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term. *Mathematische Nachrichten*. (2020) 293:910–22. doi: 10.1002/mana.201800196

17. Grace SR, Jadlovská I, Zafer A. On oscillation of second order delay differential equations with a sublinear neutral term. *Mediterr J Math*. (2020) 17:1–11. doi: 10.1007/s00009-020-01538-y

18. Lin X, Tang X. Oscillation of solutions of neutral differential equations with a superlinear neutral term. *Appl Math Lett*. (2007) 20:1016–22. doi: 10.1016/j.aml.2006.11.006

19. Muhib A, Elabbasy EM, Moaaz O. New oscillation criteria for differential equations with sublinear and superlinear neutral terms. *Turk J Math*. (2021) 45:919–28. doi: 10.3906/mat-2012-11

20. Baculíková B. Properties of third-order nonlinear functional differential equations with mixed arguments. In: *Abstract and Applied Analysis*. Vol. 2011. New York, NY: Hindawi (2011).

21. Philos CG. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. *Archiv Mathematik*. (1981) 36:168–78. doi: 10.1007/BF01223686

22. Hardy GH, Littlewood JE, Pólya G. *Inequalities*. Cambridge: Cambridge University Press (1952).

23. Bohner M, Grace S, Jadlovská I. Oscillation criteria for second-order neutral delay differential equations. *Electron J Qual Theory Differ Eq*. (2017) 2017:1–12. doi: 10.14232/ejqtde.2017.1.60

24. Thandapani E, Selvarangam S, Vijaya M, Rama R. Oscillation results for second order nonlinear differential equation with delay and advanced arguments. *Kyungpook Math J*. (2016) 56:137–46. doi: 10.5666/KMJ.2016.56.1.137

25. Thandapani E, Rama R. Comparison and oscillation theorems for second order nonlinear neutral differential equations of mixed type. *Serdica Math J*. (2013) 39:1–16. doi: 10.14232/ejqtde.2012.1.75