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Qualitative analysis of solutions for a degenerate partial differential equations model of epidemic spread dynamics

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Compartmental models are widely used in mathematical epidemiology to describe the dynamics of infectious diseases or in mathematical models of population genetics. In this study, we study a time-dependent Susceptible-Infectious-Susceptible (SIS) Partial Differential Equation (PDE) model that is based on a diffusion-drift approximation of a probability density from a well-known discrete-time Markov chain model. This SIS-PDE model is conservative due to the degeneracy of the diffusion term at the origin. The main results of this article are the qualitative behavior of weak solutions, the dependence of the local asymptotic property of these solutions on initial data, and the existence of Dirac delta function type solutions. Moreover, we study the long-term behavior of solutions and confirm our analysis with numerical computations.

KEYWORDS

epidemic modeling, degenerate differential equations, SIS-PDE model, weak solutions, Kimura model, steady states, asymptotic behavior, well-posedness

1 Introduction

Despite undeniable, vast modern improvements in the development of highly efficient antibiotics and vaccines, infectious diseases still contribute significantly to deaths worldwide. The earlier recognized diseases such as cholera or plague still sometimes pose problems in underdeveloped countries, even erupting occasionally in epidemics. In developed countries, new diseases are emerging, such as the case of AIDS (1981) or hepatitis C and E (1989–1990). New variants are constantly surfacing, such as recent bird flu (SARS) epidemic in Asia, the very dangerous Ebola virus in Africa, and the recent worldwide spread of COVID-19. Overall, infectious diseases continue to be one of the most significant and challenging health problems.

Modeling of epidemiological phenomena has a very long history. The first model for smallpox was formulated by Daniel Bernoulli in 1760. A large number of models have been constructed and analyzed from the early 20th century in response to epidemics of various infectious diseases [see for example [1–6] (and references therein)]. Compartmental models are well established as mathematical modeling techniques. It is often applied to the mathematical modeling of infectious diseases. In this type of modeling, the population is subdivided into compartments or categories such as susceptible, infectious, and recovered in the widely used SIR model or susceptible, infectious, and susceptible like in SIS epidemiological scheme. Here, we are interested in analyzing the SIS model that provides the simplest description of the dynamics of a disease that is contact-transmitted and that does not lead to immunity like it is the case for COVID-19. Discrete-time Markov chain-type SIS models are considered to be a classical approach in modern mathematical modeling in epidemiology. The most recent development in mathematical epidemiology is based on the introduction of continuous modeling based on partial differential equations like in [7, 8].

In our study, for $T > 0$ and $\Omega = (0,1)$, $\Omega_T = \Omega \times (0,T)$, we study a time-dependent Susceptible-Infectious-Susceptible (SIS) model derived in the study mentioned in the reference [9], which is a generalized PDE version of a Kimura model [see [10]] in the unknown function $p := p(x,t)$: $\bar{\Omega}_T \to \mathbb{R}$:

$$\frac{\partial p}{\partial t} = \frac{1}{2N}\frac{\partial^2}{\partial x^2}(f(x)p) - \frac{\partial}{\partial x}(g(x)p) \quad \text{in} \quad \Omega_T, \tag{1.1}$$

coupled with the boundary condition

$$\frac{1}{2N}\left[(1-R_0)p(1,t) + \frac{\partial}{\partial x}p(1,t)\right] + p(1,t) = 0, \quad t \in [0,T], \tag{1.2}$$

and initial data

$$p(x,0) = p_0(x) \quad \text{in} \quad \bar{\Omega}. \tag{1.3}$$

Here, $x \in \bar{\Omega}$ represents the fraction of infected, N is the size of the population of interest, p is the probability to find a fraction x at time t in a population of size N, and $R_0 \geqslant 0$ is the basic reproductive factor.

$$f(x) := x(R_0(1-x)+1) \text{ and } g(x) := x(R_0(1-x)-1)$$

are connected with variance and the mean of the change of x in the frame of Kimura model. Note that (1.1) is parabolic equation with non-negative characteristic form, and it is degenerated on the boundary of the domain at $x = 0$. The corresponding Fichera function for (1.1) [see e. g. [11, (1.1.3), p.17]] is $b(x,t) = \frac{1}{2N}(f'(x) - 2Ng(x)) = \frac{R_0+1}{2N} > 0$ on $\{x=0\} \times \{t > 0\}$. Hence, according to [11, 12], the problem (1.1–1.3) is well-posed without any boundary conditions at $x = 0$ for all $t > 0$. Reduced number of boundary conditions required for well-posedness of degenerated problems is a well-known phenomenon, and some interesting examples are shown in the study mentioned in the reference [13, 14]. Imposing zero boundary condition at $x = 0$ makes the problem to be over-determined, and because some weak solutions have this property, the set of solutions for the over-determined problem will not be empty.

It is worth noting that processes defined by similar models were studied by Feller in the early 1950s and used to great effect by Kimura, et al. in the 1960s and 70s to give quantitative answers to a wide range of questions in population genetics. However, rigorous analysis of the analytic properties of these equations is only the focus of applied mathematicians. The study of initial or/and initial-boundary value problems for degenerated equations, including Kimura-type operators, has a long history. Here, we do not provide a complete survey of the published results pertaining to these degenerated equations. Instead, we survey some of them for the benefit of the interested reader. Indeed, the investigation of elliptic and parabolic problems, leading to degenerated equations containing operators such as

$$\mathcal{L} := a(x)\sum_{ij=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}$$

with $a(x) \approx |x|^\alpha$, $\alpha > 0$, and a_{ij} and satisfying ellipticity conditions, are extensively studied by many authors with various analytical approaches [see e.g. [11, 12, 15–26]] including stochastic calculus [27–35].

Under suitable assumptions on the asymptotic behavior of the operator's coefficients at the boundary of the domain, the uniqueness of bounded and unbounded solutions, as well as solutions belonging to the weighted Sobolev spaces, was shown in the study mentioned in the reference [12, 20, 22–24, 36] without prescribing any boundary conditions at the origin. The qualitative properties of the corresponding solutions, including the maximum principle and the Harnack inequality, are discussed in the study mentioned in the reference [31–33, 37–39] (see also references therein). Local asymptotic behavior of solutions for different types of degenerate equations was rigorously studied in the study mentioned in the reference [40–42]. We also refer the reader to the study mentioned in the reference [30–32, 34, 43], where the theories of existence and uniqueness of solutions to stochastic differential equations with degenerate diffusion coefficients are developed. Additionally, the well-posedness of the related problems in the case of $\alpha = 1$ is discussed in the study mentioned in the reference [27–29]. It is worth noting that degenerate diffusion is examined in the context of measure-valued process [see [44–46]] via the semigroup techniques [47–49].

Finally, for the well-posedness of parabolic degenerate problems, we refer to the study mentioned in the reference [15, 16, 18, 21, 25, 26, 35, 50–52], where the existence of weak and classical solutions is established for different values of $\alpha > 0$. Previous researchers such as Chen and Weth-Wadman [53] and Epstein and Mazzeo [31] restricted their attention to the solutions with the best possible regularity properties, which leads to considerable simplifications and limitations. For real applications, it is important to consider solutions with more complicated behavior, which is the goal of our study.

The outline of the study is as follows: in Section 2, we show the existence of stationary solutions, analyze the dependence of their asymptotic behavior, near the origin, on initial data, confirm numerically their meta-stability, and analyze convergence; in Section 3, we analyze particular classical and weak solutions. We used COMSOL Multiphysics® software to perform the numerical simulations [54].

2 Weak solutions: convergence to steady state and asymptotic behavior as $t \to +\infty$

Throughout the whole article, we encounter the usual spaces $W^{1,p}(\Omega)$, $L^p(\Omega)$, and $L^2_\omega(\Omega)$. It is worth noting that the last class is a weighted space L^2 with a weight ω and the induced norm

$$\|v\|_{L^2_\omega(\Omega)} = \left(\int_\Omega \omega(x) v^2(x)\, dx \right)^{1/2}.$$

Moreover, we use the notations $H^1(\Omega)$ and $H^1_0(\Omega)$ for $W^{1,2}(\Omega)$ and $W^{1,2}_0(\Omega)$, respectively.

In this section, as it is mentioned in the introduction, we discuss the long-term behavior of a weak solution to problem (1.1–1.3). To that end, we first construct the explicit stationary solution $P_s := P_s(x)$: $\bar{\Omega} \to \mathbb{R}$ related to (1.1-1.3), and then, we examine a set of initial data which provide the convergence of the weak solution as $T \to +\infty$. In particular, we consider a case of convergent $p(x,t)$ to $P_s(x)$.

2.1 Existence of a steady state

First, we start with getting an analytical formula for a stationary solution for (1.1):

$$\frac{1}{2N} \frac{d}{dx}\Big(\frac{d}{dx}(f(x)P_s) - 2Ng(x)P_s \Big) = 0 \quad \text{in} \quad \Omega, \tag{2.1}$$

coupled with the boundary condition:

$$\frac{d}{dx} P_s(1) = -(2N - R_0 + 1)P_s(1). \tag{2.2}$$

Integrating (2.1) in x and taking into account (2.2), we get

$$\frac{d}{dx}(f(x)P_s) = 2Ng(x)P_s.$$

It is apparent that this equation has a general solution

$$f(x)P_s(x) = \mathcal{C}\, F(x), \tag{2.3}$$

where

$$F(x) := e^{2N \int_0^x \frac{g(s)}{f(s)} ds} = e^{2Nx} \Big(\frac{R_0(1-x)+1}{R_0+1} \Big)^{\frac{4N}{R_0}} \text{ if } R_0 > 0,$$
$$\text{and } F(x) = e^{-2Nx} \text{ if } R_0 = 0,$$

$$\mathcal{C} := \lim_{x \to 0} f(x) P_s(x).$$

As a result, we obtain the explicit form of the classical stationary solution to (1.1–1.3)

$$P_s(x) = \frac{\mathcal{C}}{\omega(x)} \text{ where } \omega(x) := \frac{f(x)}{F(x)}. \tag{2.4}$$

Observe that the changing-sign convection term for $R_0 = 2$ equals zero at $x = 0.5$, leading to a wave-like solution that moves toward this point, forming a meta-stable steady-state shape. This illustrates that the solution's short-term behavior is driven by the convection, as shown in Figures 1, 2. It takes a long-time for meta-stable steady state (a wave-like solution that slowly changes its shape) to move mass toward the origin. These long-term dynamics are due to a slow diffusion effect, and eventually, the solution blows up at the origin, which is indeed the case for two different sets of parameter values, as shown in Figures 3, 4. All numerical simulations show high accuracy of the mass conservation property even for long-term computations, which suggests the existence of a solution of Delta function type that acts as a global attractor in this dynamical system.

2.2 Long-term behavior of a weak solution

Assuming that $\omega(x)$ is defined by Equation (2.4) and that

$$N \geqslant 1, \quad R_0 \geqslant 0 \quad \text{and} \quad 0 \leqslant p_0(x) \in L^2_\omega(\Omega).$$

We define a weak solution of (1.1–1.3) in the following sense:

Definition 2.1. A non-negative function $p(x,t) \in C([0,T]; L^2_\omega(\Omega))$ is a weak solution of problem (1.1)–(1.3) for any $T > 0$ if

$$p_t \in L^2(0,T; (H^1(\Omega))'), \;\; (\omega(x)p)_x \in L^2(\Omega_T),$$

and p satisfies (1.1) in the sense

$$\int_0^T \Big\langle \frac{\partial p}{\partial t}, \psi \Big\rangle_{(H^1)',H^1} dt + \iint_{\Omega_T} \Big(\frac{1}{2N} \frac{\partial (f(x)p)}{\partial x} - g(x)p \Big) \frac{\partial \psi}{\partial x}\, dxdt = 0$$

for all $\psi \in L^2(0,T; H^1(\Omega))$, and $\psi(0,t) = 0$ for all $t \in [0,T]$. Here, $\langle u, v\rangle_{(H^1)',H^1}$ is a dual pair of elements $u \in (H^1)'$ and $v \in H^1$.

Now, we are ready to state our first main result related to the asymptotic behavior of a weak solution to (1.1–1.3).

Theorem 1. (i) Let $0 \leqslant p_0(x) \in L^2_\omega(\Omega)$ and $\lim\limits_{x\to 0} \omega(x)p(x,t) = 0$, a weak solution $p(x,t)$ satisfies the relation

$$\omega^{\frac{1}{2}}(x)p(x,t) \to 0 \text{ strongly in } L^2(\Omega) \text{ as } t \to +\infty.$$

Moreover, if $(\omega(x)p_0(x))_x \in L^2(\Omega)$, $\omega(x)p(x,t) \in C([0,+\infty); H^1(\Omega))$, and there is convergence

$$\omega(x)p(x,t) \to 0 \text{ strongly in } H^1(\Omega) \text{ as } t \to +\infty. \tag{2.5}$$

(ii) Let $\omega^{\frac{1}{2}}(x)p_0(x) \in L^2_\omega(\Omega)$, if $p(x,t)$ is a weak solution to (1.1–1.3) and $\lim\limits_{x\to 0} \omega(x)p(x,t) = \mathcal{C} > 0$, where $\mathcal{C}$ is the same constant as in Equation (2.3), there exists a constant $\mathcal{C}_1 > 0$, depending on R_0 and N, such that

$$\|\omega(x)p(x,t) - \mathcal{C}\|_{L^2(\Omega)} \leqslant C_1 \|\omega(x)p_0(x) - \mathcal{C}\|_{L^2(\Omega)} \text{ for all } t \geqslant 0. \tag{2.6}$$

Moreover, if $\omega(x)(\omega(x)p_0(x))_x \in L^2(\Omega)$, there exist a constant $C_1 > 0$ and a time $T^* > 0$, depending on R_0 and N, such that

$$\|\omega(x)\frac{\partial}{\partial x}(\omega(x)p(x,t))\|_{L^2(\Omega)} \leqslant C_2 \|\omega(x)p_0(x) - \mathcal{C}\|_{L^2(\Omega)} \text{ for all } t \geqslant T^*.$$

Numerical simulations in Figures 5, 6 illustrate the convergence result in Equation (2.6).

Note that Theorem 1 describes a behavior of a weak solution to direct well-posed problem (1.1)–(1.3), depending on the different types of behavior $\omega(x)p(x,t)$ at $x = 0$, taking into account two explicit solutions: steady state (subsection 2.1) and Fourier series solutions (subsection 3.1). In other words, our main result has a conditional characteristic via inserting additional assumptions on the term $\omega(x)p(x,t)$ as $x \to 0$ in the statement of the Theorem 1 but not to the statement of the problem (1.1)–(1.3). In the context of infectious disease spreading dynamic, Theorem 1 says that a different regularity of the initial data at $x = 0$ leads to a different rate of the disease extinction, i. e., more regular initial data give us faster decay of infection.

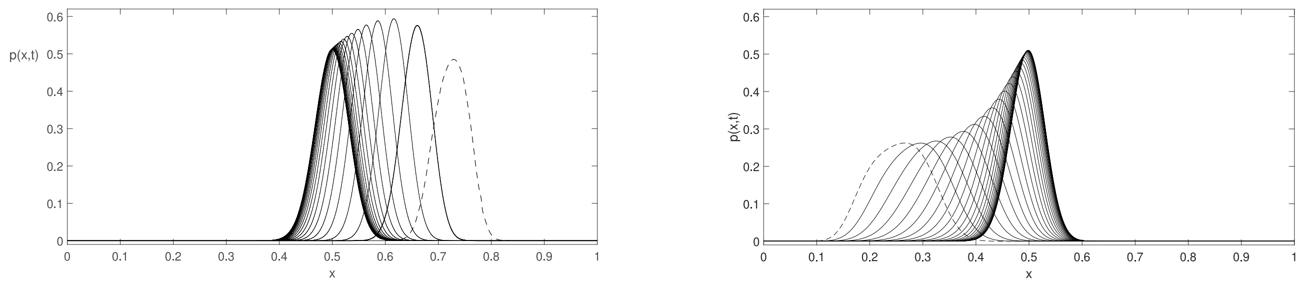

FIGURE 1
These two pictures illustrate the dominant behavior of convection in the short-term $t \in [0, 0.1]$. **(Left)** Convection moves the solutions toward the steady state from the right side to the left one for $R_0 = 2$ and $N = 200$, **(Right)** convection moves the solutions toward the steady state from the left side to the right one for the same parameter values. The initial data are plotted with a dashed line.

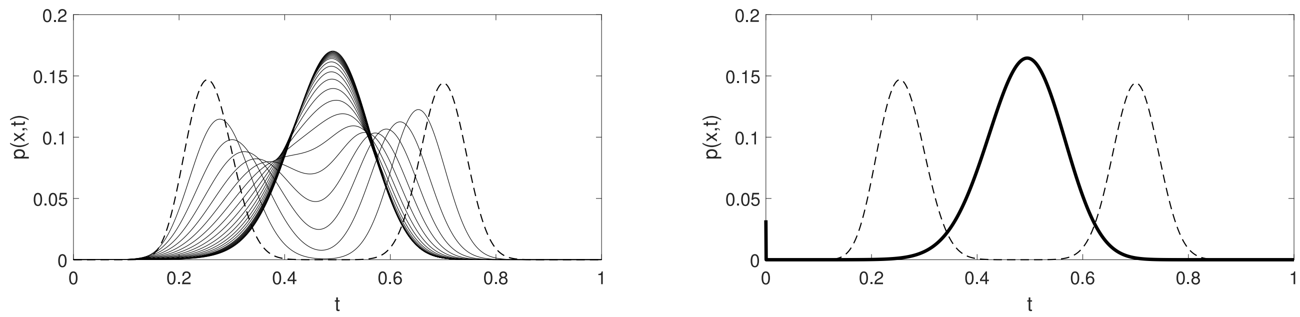

FIGURE 2
These two pictures illustrate $t \in [0, 0.1]$ short-term dynamics for $R_0 = 2$ and $N = 100$ **(Left)** and $t \in [0, 2000]$ long-term dynamics with blow up at the origin **(Right)**. The initial data are plotted with a dashed line.

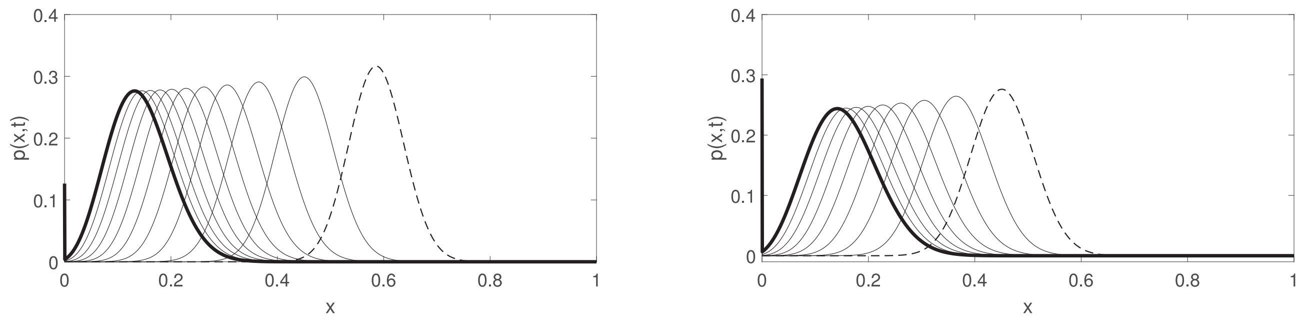

FIGURE 3
These two pictures illustrate the dominant behavior of convection in the short-term $t \in [0, 0.1]$. **(Left)** Convection moves solutions toward the origin, here $R_0 = 0.5$ and $N = 100$ and where solutions blow up. **(Right)** Convection again moves solutions toward the origin, here $R_0 = 1$ and $N = 100$ and where solutions blow up. The initial data are plotted with a dashed line.

Remark 2.1. In this study, we do not discuss the existence and uniqueness of weak solutions vanishing at the origin. As for these issues, we refer the interested readers to Section 7 in the study mentione4d in the reference [51], where the related questions are analyzed.

Remark 2.2. In particular, Theorem 1 provides the following properties:

(i) (2.5) implies

$$\int_{\Omega} p(x,t)\,dx \to 0 \text{ as } t \to +\infty,$$

where we deduce that $\lim\limits_{t\to+\infty} p(x,t) = 0$ a. e. $x \in \bar{\Omega}$;

(ii) (2.6) gives the stability of the steady state P_s.

Proof of Theorem 1. Introducing a new function $z := \omega(x)p(x,t)$ and rewriting problem (1.1)–(1.3) in the more suitable form:

$$\begin{cases} \omega^{-1}(x)\frac{\partial z}{\partial t} = \frac{1}{2N}\frac{\partial}{\partial x}\big(F(x)\frac{\partial z}{\partial x}\big), & (x,t) \in \Omega_T, \\ \frac{\partial z}{\partial x}|_{x=1} = 0 \quad \text{and} \quad z|_{x=0} = 0, & t \in [0,T], \\ z(x,0) = z_0(x) := \omega(x)p_0(x), & x \in \bar{\Omega}. \end{cases} \tag{2.7}$$

Note that if $z|_{x=0} = \mathcal{C} > 0$, we can define a new function $\tilde{z} = z - \mathcal{C}$, and we reduce the case to a problem

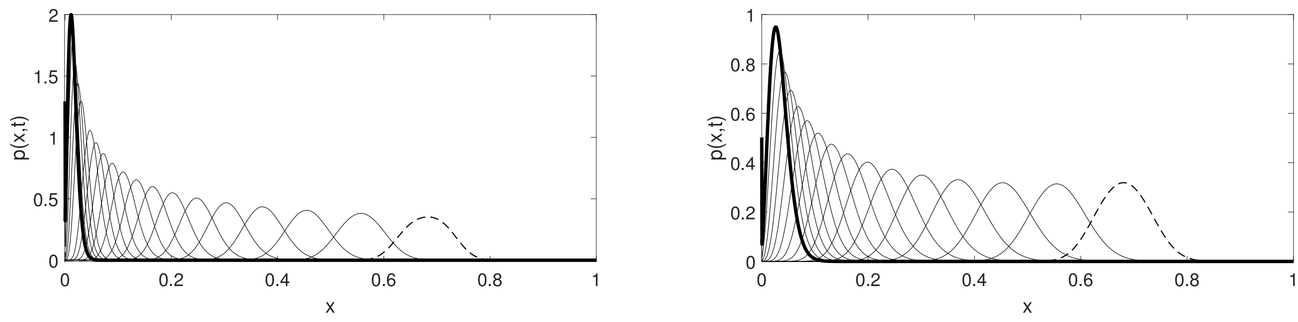

FIGURE 4
These two pictures illustrate the dominant behavior of convection in the short term $t \in [0, 0.1]$. **(Left)** Convection moves solutions toward the origin, here $R_0 = 0$ and $N = 100$, where the solutions blow up. **(Right)** Convection again moves solutions toward the origin, here $R_0 = 0$ and $N = 200$, where the solutions blow up. The initial data are plotted with a dashed line.

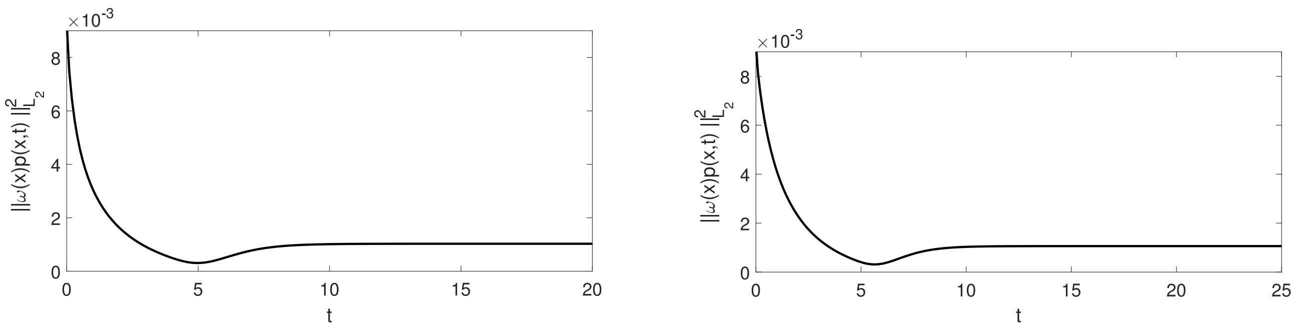

FIGURE 5
These two pictures illustrate convergence of weighted L^2-norm of $p(x, t)$ to a constant for $R_0 = 0$ and $N = 100$, $\mathcal{C} = 4.3 * 10^{-5}$ **(left)** and $R_0 = 0$ and $N = 200$, $\mathcal{C} = 1.7 * 10^{-4}$ **(right)**.

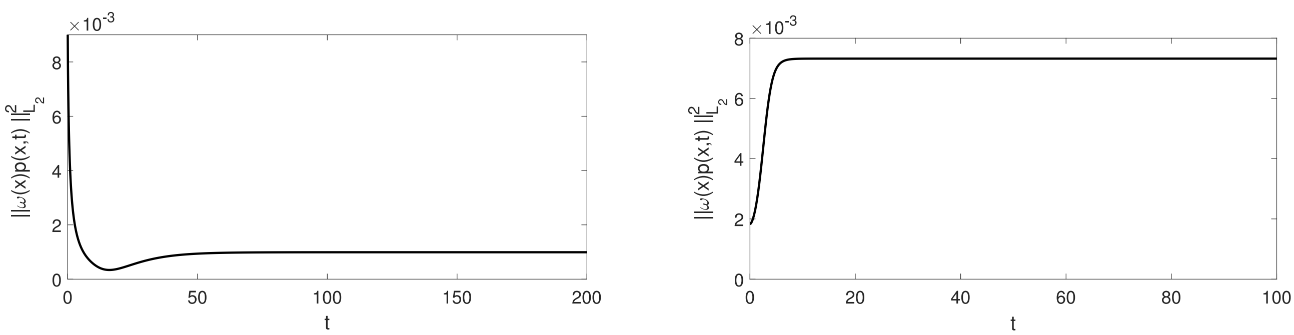

FIGURE 6
These two pictures illustrate convergence of weighted L^2-norm of $p(x, t)$ for $R_0 = 0.5$ and $N = 100$, $\mathcal{C} = 4.2 * 10^{-5}$ **(left)** and $R_0 = 2$ and $N = 200$, $\mathcal{C} = 1.6 * 10^{-4}$ **(right)**.

similar to Equation (2.7). Since the approximation approach is well developed for this type of problem, to avoid technical details, we proceed with formal computations. Our formal computations can be rigorously justified by introducing a sequence of approximate solutions with extra regularity property, taking advantage of the standard approximation arguments, and passing to the limit in the final estimates. The weak solution will be obtained as a limit as $\varepsilon \to 0$ of smooth solutions for the corresponding approximating problems. For any $\varepsilon > 0$, we consider the approximating problems of Equation (2.7), where instead of $\omega(x)$ and $z_0(x)$, we take $\omega_\varepsilon(x) = \frac{f(x)+\varepsilon}{F(x)}$ and $z_{\varepsilon,0}(x) \in C^\infty(\bar{\Omega})$ such that $z_{\varepsilon,0}(x) \to z_0(x)$ strongly in $H^1(\Omega)$ as $\varepsilon \to 0$. As these approximating problems are uniformly parabolic, by general PDE theory for the second order parabolic equations (see, e.g. [55]), we find a solution $z_\varepsilon(x, t) \in C^\infty(\Omega_T)$. By going through all routine calculations for z_ε, and then passing to the limit with respect to $\varepsilon \to 0$, we arrive at the required estimates for the corresponding limit solution z.

We now verify claim **(i)** of Theorem 1. To this end, multiplying the equation in (2.7) by $z(x,t)$ and integrating over Ω, we obtain as follows:

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}\omega^{-1}(x)z^2\,dx+\tfrac{1}{2N}\int\limits_{\Omega}F(x)\Big(\frac{\partial z}{\partial x}\Big)^2dx=\frac{1}{2N}F(x)z\frac{\partial z}{\partial x}\Big|_0^1=0. \tag{2.8}$$

Next, we take advantage of Hardy inequality [56, p. 22, (1.25) with $p=q=2$]

$$\int\limits_{\Omega}\omega^{-1}(x)z^2\,dx\leqslant C_H(R_0)\int\limits_{\Omega}F(x)\Big(\frac{\partial z}{\partial x}\Big)^2dx$$

with $z(0)=0$. Here, the constant $C_H(R_0)$ satisfies the inequalities:

$$A(R_0)\leqslant C_H(R_0)\leqslant 4A(R_0)\quad\text{with}\quad A(R_0)=\sup_{r\in(0,1)}\Big(\int\limits_0^r\tfrac{dx}{F(x)}\Big)\Big(\int\limits_r^1\tfrac{dx}{\omega(x)}\Big).$$

Note that

$$\Big(\int\limits_0^r\tfrac{dx}{F(x)}\Big)\Big(\int\limits_r^1\tfrac{dx}{\omega(x)}\Big)=\Big(\int\limits_0^r e^{-2Nx}(R_0(1-x)+1)^{-\frac{4N}{R_0}}dx\Big)\Big(\int\limits_r^1 x^{-1}e^{2Nx}(R_0(1-x)+1)^{\frac{4N}{R_0}-1}dx\Big)\leqslant$$
$$r^{-1}e^{2N}\Big(\int\limits_0^r(R_0(1-x)+1)^{-\frac{4N}{R_0}}dx\Big)\Big(\int\limits_r^1(R_0(1-x)+1)^{\frac{4N}{R_0}-1}dx\Big)$$
$$\leqslant e^{2N}(R_0(1-r)+1)^{-\frac{4N}{R_0}}\Big(\int\limits_r^1(R_0(1-x)+1)^{\frac{4N}{R_0}-1}dx\Big)\leqslant e^{2N},$$

where it follows that $A(R_0)\leqslant e^{2N}$. Thus, statement (2.8) along with Hardy inequality, see [9], leads to the relation

$$\int\limits_{\Omega}\omega^{-1}(x)z^2(x,t)\,dx\leqslant e^{-\frac{t}{NC_H(R_0)}}\int\limits_{\Omega}\omega^{-1}(x)z_0^2(x)\,dx\to 0\text{ as }t\to+\infty.$$

Multiplying the equation in (2.7) by $-\omega(x)\frac{\partial}{\partial x}\big(F(x)\frac{\partial z}{\partial x}\big)$ and integrating over Ω, we obtain the equation

$$\tfrac{1}{2}\tfrac{d}{dt}\int\limits_{\Omega}F(x)\Big(\tfrac{\partial z}{\partial x}\Big)^2dx+\tfrac{1}{2N}\int\limits_{\Omega}\omega(x)\Big(\tfrac{\partial}{\partial x}\big(F(x)\tfrac{\partial z}{\partial x}\big)\Big)^2dx=F(x)\tfrac{\partial z}{\partial t}\tfrac{\partial z}{\partial x}\Big|_0^1,$$

which implies

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}F(x)\Big(\frac{\partial z}{\partial x}\Big)^2dx+\frac{1}{2N}\int\limits_{\Omega}\omega(x)\Big(\frac{\partial}{\partial x}\Big(F(x)\frac{\partial z}{\partial x}\Big)\Big)^2dx=0.$$

To handle the second term in the left-hand side of this equality, we apply to $v=F(x)\frac{\partial z}{\partial x}$ the following inequality:

$$\int\limits_{\Omega}\tfrac{v^2}{F(x)}\,dx\leqslant C_P(R_0)\int\limits_{\Omega}\omega(x)\Big(\tfrac{\partial v}{\partial x}\Big)^2dx\text{ with }v(1)=0,$$
$$\text{where}\quad C_P(R_0)=\int\limits_{\Omega}\tfrac{1}{F(x)}\Big(\int\limits_x^1\tfrac{dy}{\omega(y)}\Big)\,dx.$$

Hence, we end up with the relation

$$\int\limits_{\Omega}F(x)\Big(\frac{\partial z}{\partial x}\Big)^2dx\leqslant e^{-\frac{t}{NC_P(R_0)}}\int\limits_{\Omega}F(x)\Big(\frac{\partial z_0}{\partial x}\Big)^2dx\to 0\text{ as }t\to+\infty. \tag{2.9}$$

As a result, we obtain the following convergence:

$$z(x,t)\to 0\quad\text{strongly in}\quad H^1(\Omega)\text{ as }t\to+\infty$$

provided the following inequality holds:

$$\int\limits_{\Omega}\Big(\omega^{-1}(x)z_0^2(x)+F(x)\Big(\frac{\partial z_0}{\partial x}\Big)^2\Big)dx<+\infty.$$

As a simple consequence of this fact and the convergence of (2.9), we obtain an upper bound on $z(x,t)$:

$$z(x,t)\leqslant x^{\frac{1}{2}}e^{-\frac{t}{2NC_P(R_0)}}\Big(\int\limits_{\Omega}F(x)\Big(\frac{\partial z_0}{\partial x}\Big)^2dx\Big)^{\frac{1}{2}},$$

which, in turn, provides the desired relation

$$p(x,t)\leqslant\tfrac{x^{\frac{1}{2}}}{\omega(x)}e^{-\frac{t}{2NC_P(R_0)}}\Big(\int\limits_{\Omega}F(x)\Big(\frac{\partial z_0}{\partial x}\Big)^2dx\Big)^{\frac{1}{2}}.$$

We now proceed by showing that statement (ii) of Theorem 1 is in fact valid. We multiply (2.7) by $\omega(x)\psi(x)z(x,t)$ and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}\psi(x)z^2\,dx+\frac{1}{2N}\int\limits_{\Omega}f(x)\psi(x)\Big(\frac{\partial z}{\partial x}\Big)^2dx=\frac{1}{2N}\Big(f(x)\psi(x)z\frac{\partial z}{\partial x}-\frac{1}{2}(\omega(x)\psi(x))'F(x)z^2\Big)\Big|_0^1+\frac{1}{4N}\int\limits_{\Omega}z^2\frac{\partial}{\partial x}\Big(F(x)\frac{\partial}{\partial x}\Big(\omega(x)\psi(x)\Big)\Big)\,dx.$$

Then, choosing here

$$\psi(x)=\omega^{-1}(x)\int\limits_0^x\tfrac{dy}{F(y)}=\tfrac{F(x)}{f(x)}\int\limits_0^x\tfrac{dy}{F(y)}\to\tfrac{1}{1+R_0}\text{ as }x\to 0,$$

we arrive at the equality

$$\frac{d}{dt}\int\limits_{\Omega}\psi(x)z^2\,dx+\frac{1}{N}\int\limits_{\Omega}f(x)\psi(x)\Big(\frac{\partial z}{\partial x}\Big)^2dx+\frac{1}{2N}z^2(1,t)=0, \tag{2.10}$$

where

$$\int_\Omega \psi(x)z^2\,dx \leqslant \int_\Omega \psi(x)z_0^2(x)\,dx.$$

Thus, we easily conclude that

$$\int_\Omega z^2(x,t)\,dx \leqslant C_1 \int_\Omega z_0^2(x)\,dx \text{ for all } t \geqslant 0,$$

where $0 < C_1 = \frac{\sup \psi(x)}{\inf \psi(x)} < +\infty$. Now, multiplying the equation in (2.7) by $-\omega(x)\phi(x)\frac{\partial}{\partial x}\left(F(x)\frac{\partial z}{\partial x}\right)$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \phi(x)F(x)\left(\frac{\partial z}{\partial x}\right)^2 dx + \frac{1}{2N}\int_\Omega \omega(x)\phi(x)\left(\frac{\partial}{\partial x}\left(F(x)\frac{\partial z}{\partial x}\right)\right)^2 dx$$
$$= \left(\phi(x)F(x)\frac{\partial z}{\partial t}\frac{\partial z}{\partial x} - \right.$$
$$\left.\frac{1}{4N}\omega(x)\phi'(x)F^2(x)\left(\frac{\partial z}{\partial x}\right)^2\right)\Big|_0^1 + \frac{1}{4N}\int_\Omega (\omega(x)\phi'(x))'F^2(x)\left(\frac{\partial z}{\partial x}\right)^2 dx.$$

Now, consider $\phi(x)$ such that $(\omega(x)\phi'(x))'F^2(x) = 2f(x)\psi(x)$, i. e.,

$$\phi(x) = 2\int_0^x \tfrac{1}{\omega(y)}\Big(\int_0^y \tfrac{1}{F(v)}\Big(\int_0^v \tfrac{ds}{F(s)}\Big)dv\Big)dy \sim \tfrac{x^2}{2(R_0+1)} \text{ as } x \to 0,$$

we have

$$\tfrac{d}{dt}\int_\Omega \phi(x)F(x)\left(\tfrac{\partial z}{\partial x}\right)^2 dx + \tfrac{1}{N}\int_\Omega \omega(x)\phi(x)\left(\tfrac{\partial}{\partial x}\left(F(x)\tfrac{\partial z}{\partial x}\right)\right)^2 dx$$
$$= \tfrac{1}{N}\int_\Omega f(x)\psi(x)\left(\tfrac{\partial z}{\partial x}\right)^2 dx.$$

The above equality, along with (2.10), leads to

$$\tfrac{d}{dt}\int_\Omega \left(\phi(x)F(x)\left(\tfrac{\partial z}{\partial x}\right)^2 + \psi(x)z^2\right)dx + \tfrac{1}{N}\int_\Omega \omega(x)\phi(x)$$
$$\left(\tfrac{\partial}{\partial x}\left(F(x)\tfrac{\partial z}{\partial x}\right)\right)^2 dx + \tfrac{1}{2N}z^2(1,t) = 0. \tag{2.11}$$

Now, applying to $v = F(x)\frac{\partial z}{\partial x}$, the following estimate

$$\int_\Omega \frac{\phi(x)}{F(x)}v^2\,dx \leqslant C_P(R_0)\int_\Omega \omega(x)\phi(x)\left(\frac{\partial v}{\partial x}\right)^2 dx \text{ with } v(1) = 0,$$

where

$$C_P(R_0) = \int_\Omega \tfrac{\phi(x)}{F(x)}\Big(\int_x^1 \tfrac{dy}{\omega(y)\phi(y)}\Big)dx,$$

to (2.11) and conclude that

$$\int_\Omega \phi(x)F(x)\left(\tfrac{\partial z}{\partial x}\right)^2 dx \leqslant e^{-\frac{t}{NC_P(R_0)}}\int_\Omega \phi(x)F(x)\left(\tfrac{\partial z_0}{\partial x}\right)^2 dx$$
$$+\int_\Omega \psi(x)z_0^2(x)\,dx,$$

where

$$\int_\Omega \omega^2(x)\left(\tfrac{\partial z}{\partial x}\right)^2 dx \leqslant \frac{\sup\left(\frac{\phi(x)F(x)}{\omega^2(x)}\right)}{\inf\left(\frac{\phi(x)F(x)}{\omega^2(x)}\right)} e^{-\frac{t}{NC_P(R_0)}}\int_\Omega \omega^2(x)\left(\tfrac{\partial z_0}{\partial x}\right)^2 dx$$
$$+\frac{\sup\psi(x)}{\inf\left(\frac{\phi(x)F(x)}{\omega^2(x)}\right)}\int_\Omega z_0^2(x)\,dx.$$

As a result, there exists a time $T^* > 0$ such that

$$\int_\Omega \omega^2(x)\left(\frac{\partial z}{\partial x}\right)^2 dx \leqslant C_2\int_\Omega z_0^2(x)\,dx \text{ for all } t \geqslant T^*$$

provided the following inequality holds:

$$\int_\Omega \left(\psi(x)z_0^2(x) + \phi(x)F(x)\left(\frac{\partial z_0}{\partial x}\right)^2\right)dx < +\infty.$$

This completes the proof of assertion (ii) and, as a consequence, of Theorem 1.

3 Solutions in weighted L^2-space

In this section, we will illustrate an application of Theorem 1 by constructing solutions, using the spectral decomposition method, in a weighted L^2-space. First, we analyze classical solutions to problem (2.7), and then, we discuss some classes of weak solutions.

3.1 Fourier series solutions in a weighted space

Introducing a new variable

$$s = \sqrt{2N}\int_0^x \frac{dy}{f^{\frac{1}{2}}(y)},$$

and denoting by

$$l(s) := \sqrt{2N}\frac{g(x)}{f^{\frac{1}{2}}(x)} = \sqrt{\frac{2N}{R_0}}\,\frac{\sin\left(\frac{1}{2}\sqrt{\frac{R_0}{2N}}s\right)\left[R_0-1-(R_0+1)\sin^2\left(\frac{1}{2}\sqrt{\frac{R_0}{2N}}s\right)\right]}{\left|\cos\left(\frac{1}{2}\sqrt{\frac{R_0}{2N}}s\right)\right|},$$

$$s_1 := 2\sqrt{\frac{2N}{R_0}}\arcsin\left(\sqrt{\frac{R_0}{R_0+1}}\right),$$

we rewrite problem (2.7) in the form as follows:

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial s^2} + l(s)\frac{\partial z}{\partial s}, & s \in (0, s_1), \quad t \in (0,T), \\ z(0,t) = 0, \quad \frac{\partial z}{\partial s}(s_1, t) = 0, & t \in [0,T]. \end{cases} \tag{3.1}$$

It is worth noting that to establish (3.1), we have made use of the following simple and verifiable relations:

$$s = \begin{cases} 2\sqrt{\frac{2N}{R_0}}\arcsin\left(\sqrt{\frac{R_0}{R_0+1}}x^{\frac{1}{2}}\right) & \text{if} \quad R_0 > 0, \\ 2\sqrt{2N}x^{\frac{1}{2}} & \text{if} \quad R_0 = 0, \end{cases}$$

or as consequence

$$x = \begin{cases} \frac{R_0+1}{R_0}\sin^2\left(\frac{1}{2}\sqrt{\frac{R_0}{2N}}s\right) & \text{if} \quad R_0 > 0, \\ \frac{1}{8N}s^2 & \text{if} \quad R_0 = 0. \end{cases}$$

Separating variables in (3.1):

$$z(s,t) = T(t)S(s),$$

leads to the problems

$$\frac{T'(t)}{T(t)} = \frac{S''(s)+l(s)S'(s)}{S(s)} = -\lambda,$$

where

$$T'(t) = -\lambda T(t),$$

$$S''(s) + l(s)S'(s) = -\lambda S(s) \tag{3.2}$$

with

$$S(0) = 0,\ S'(s_1) = 0.$$

Now, multiplying (3.2) by $p(s) := e^{\int_0^s l(y)\,dy}$, we immediately obtain the equation

$$-(p(s)S'(s))' = \lambda p(s)S(s).$$

Then, setting

$$U(s) = p^{\frac{1}{2}}(s)S(s) \quad q(s) = \frac{(p^{\frac{1}{2}}(s))''}{p^{\frac{1}{2}}(s)} = \frac{1}{2}\left(l'(s) + \frac{1}{2}l^2(s)\right),$$

we arrive at the classical Sturm–Liouville problem with the continuous potential $q(s)$

$$\begin{cases} -U''(s) + q(s)U(s) = \lambda U(s), & s \in (0, s_1), \\ U(0) = 0, \quad U'(s_1) = 0. \end{cases} \tag{3.3}$$

From here, we rely on standard computational methods to obtain the following asymptotic behavior of eigenvalues and eigenfunctions to problem 3.3:

$$\lambda_k \sim \left(\tfrac{\pi}{s_1}\right)^2\left(k + \tfrac{1}{2}\right)^2,\ \ U_k(s) \sim \sin\left(\tfrac{\pi}{s_1}\left(k + \tfrac{1}{2}\right)s\right),$$

or returning to (3.2):

$$\lambda_k \sim \left(\tfrac{\pi}{s_1}\right)^2\left(k + \tfrac{1}{2}\right)^2,\ \ S_k(s) \sim e^{-\frac{1}{2}\int_0^s l(y)\,dy}\sin\left(\tfrac{\pi}{s_1}\left(k + \tfrac{1}{2}\right)s\right).$$

Thus, problem (3.1) has a particular solution

$$z(s,t) = \sum_{k=0}^{+\infty} c_k e^{-\lambda_k t} S_k(s),$$

which, in turn, means

$$z(x,t) = \sum_{k=0}^{+\infty} c_k e^{-\lambda_k t}\varphi_k(x),$$

where

$$\lambda_k \sim \tfrac{\pi^2}{N}\left(k + \tfrac{1}{2}\right)^2,\ \ \varphi_k(x) \sim e^{-N^{\frac{3}{2}}x}\sin\left(\pi\left(k + \tfrac{1}{2}\right)\frac{\arcsin\left(\sqrt{\frac{R_0}{R_0+1}}x^{\frac{1}{2}}\right)}{\arcsin\left(\sqrt{\frac{R_0}{R_0+1}}\right)}\right).$$

Finally, keeping in mind the relation $z(x,t) = \omega(x)p(x,t)$, we deduce the formal solution

$$p(x,t) = \frac{1}{\omega(x)}\sum_{k=0}^{+\infty} c_k e^{-\lambda_k t}\varphi_k(x)$$

that is a weak solution in a weighted L^2-space in the sense of the Definition 2.1. It is worth noting that the asymptotic behavior of the solution $\frac{C_1}{\sqrt{x e^{C_2 t}}}$ as $x \to 0^+$ is in agreement with Theorem 1 (i).

3.2 The Dirac delta function solutions

In this section, we show that Dirac delta function type solutions belong to our class of weak solutions. The main problem here is that, with zero on the boundary, the integral $\int_0^a f(z)\delta(z)dz$ is *a priori* not well defined (over-determined ill-posed problem was previously considered in the study mentioned in the reference [9]). Now, we denote positive and non-negative cut of functions by $f(x)\chi_{\{x>0\}}$ and $f(x)\chi_{\{x\geqslant 0\}}$, respectively. This corresponds to integrating δ function against the function $f(x)\chi_{\{x>0\}}$ (or possibly $f(x)\chi_{\{x\geqslant 0\}}$), which is not continuous at the origin $x = 0$, where the support of the Dirac delta function lies. With the Dirac delta function at the boundary of the integration, only formal expressions could be found in the literature: $\int_0^a f(z)\delta(z)dz = \int_{-a}^0 f(z)\delta(z)dz = \frac{1}{2}f(0)$. This is the justification for choosing a symmetrization method by considering a problem of extended domain $[-1,1]$ for our Dirac delta function type solutions. Now, we look for a solution to a symmetrically extended problem (1.1)–(1.3) on the interval $(-1,1)$ in the form of $p(x,t) = \eta(t)\delta_0(x)$, where $\delta_0(x)$ is the Dirac delta function concentrated at the origin.

Multiplying symmetrized Equation (1.1) by $\phi(x) \in C^2[-1,1]$ with compact support and $\phi(0) \neq 0$, after integrating by parts in $Q_T := (-1,1)\times(0,T)$, we have

$$\iint_{Q_T} \frac{\partial p}{\partial t}\phi(x)\,dxdt = \frac{1}{2N}\iint_{Q_T}\left(\tilde{f}(x)p\phi''(x) + 2N\tilde{g}(x)p\phi'(x)\right)dxdt,$$

where $\tilde{f}$ and $\tilde{g}$ are even continuation of f and g, respectively. Taking $p(x,t) = \eta(t)\delta_0(x)$ in the last equality, we deduce that

$$(\eta(T) - \eta(0))\phi(0) = \left(\tfrac{1}{2N}f(0)\phi''(0) + g(0)\phi'(0)\right)\int_0^T \eta(t)\,dt = 0.$$

Due to the inequality $\phi(0) \neq 0$, we have

$$\eta(T) = \eta(0) = M > 0.$$

As a result, symmetrized Equation (1.1) has the following solution:

$$p(x,t) = M\delta_0(x) \text{ for all } (x,t) \in (-1,1)\times(0,+\infty).$$

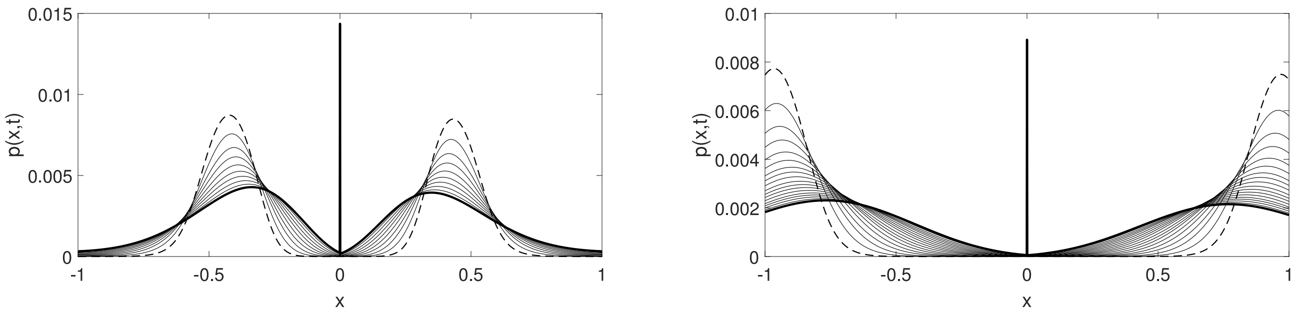

FIGURE 7
These two pictures illustrate the existence of the Dirac delta function type solutions for symmetrized problems with $R_0 = 0$ and $N = 10$. The initial data are plotted with a dashed line.

Convergence of a weak solution to the Dirac delta function is shown in Figure 7. It is interesting to mention that a non-smooth change of variables $y = 2\sqrt{x}$ (for the case $R_0 = 0$) will remove the degeneracy from the equation. However, the whole long-term dynamics will not be recovered in terms of y as a global attractor-type solution. Ce^t that satisfies no-flux boundary conditions in terms of variable y will not be satisfying no-flux boundary conditions in terms of variable x. Although Ce^t solves the original problem with Neumann boundary conditions (which make the original problem ill-posed), it is unstable. Indeed, a slight perturbation will drive the dynamics toward the Dirac delta function.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

RT: Writing – original draft. NV: Writing – original draft. BA-A: Writing – original draft.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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