frontiers | Frontiers in Applied Mathematics and Statistics

TYPE Perspective
PUBLISHED 15 June 2023
DOI 10.3389/fams.2023.1201043

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OPEN ACCESS

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RECEIVED 13 April 2023
ACCEPTED 29 May 2023
PUBLISHED 15 June 2023

CITATION
Abarzhi SI (2023) Invariant forms and control dimensional parameters in complexity quantification.
Front. Appl. Math. Stat. 9:1201043.
doi: 10.3389/fams.2023.1201043

Invariant forms and control dimensional parameters in complexity quantification

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Non-equilibrium dynamics is omnipresent in nature and technology and can exhibit symmetries and order. In idealistic systems this universality is well-captured by traditional models of dynamical systems. Realistic processes are often more complex. This work considers two paradigmatic complexities—canonical Kolmogorov turbulence and interfacial Rayleigh-Taylor mixing. We employ symmetries and invariant forms to assess very different properties and characteristics of these processes. We inter-link, for the first time, to our knowledge, the scaling laws and spectral shapes of Kolmogorov turbulence and Rayleigh-Taylor mixing. We reveal the decisive role of the control dimensional parameters in their respective dynamics. We find that the invariant forms and the control parameters provide the key insights into the attributes of the non-equilibrium dynamics, thus expanding the range of applicability of dynamical systems well-beyond traditional frameworks.

KEYWORDS

fluid instabilities, interfacial mixing, self-similarity, symmetry, invariant forms, dynamical systems

1. Introduction

Non-equilibrium dynamics governs a broad range of processes in nature and technology and is a challenge to study in theory, experiments and simulations. An important aspect advanced our understanding of this complexity is symmetries of the dynamics. For instance, systems with pattern formations—a subject of active research in the field of dynamical systems—are well-described by universal theoretical models, such as the complex Ginzburg-Landau equation and the non-linear Schrödinger equation [1–8].

Realistic processes are often more complicated than idealistic systems studied within the traditional framework. Yet, they are observed to exhibit symmetries, universality and order. Their non-equilibrium dynamics is eligible to the first principle consideration, and can be investigated on the basis of group theory and representation theory. A critical aspect is the link of the theoretical attributes of the non-equilibrium dynamics—the symmetry groups and the invariant forms—to the dimensional parameters that can control the physical process, and to the observable quantities that can be diagnosed in experiments [1, 8–17].

In this work we consider two paradigmatic complexities—the classical fluid dynamics problems of canonical Kolmogorov turbulence and Rayleigh-Taylor interfacial mixing. These non-equilibrium processes have very different physical properties, symmetries and characteristics. We employ the invariant forms of these processes to inter-link their scaling laws and spectral shapes and to reveal the role of the control dimensional parameters in their respective dynamics [8–12, 15–24].

Turbulence and Rayleigh-Taylor mixing are inherent to a broad range of phenomena having considerable scientific and technological importance. Examples include supernovae, solar flares, climate change, plasma fusion, nanofabrication, and purification of water [25–39]. Turbulence is a state of a dissipative system and it decays unless it is driven by an external energy source. Canonical turbulence is self-similar, isotropic and homogeneous, with a non-dissipative energy transport between the scales. It is a stochastic process with strong fluctuations that may fully blackout deterministic conditions. For as much as turbulence is considered to be the last unsolved problem of the classical physics, Rayleigh-Taylor mixing is its more complex counterpart [8–12, 17–19, 40–47].

Rayleigh-Taylor instability develops at the interface between two fluids of different densities accelerated against their density gradient, and it is driven by the acceleration. The amplitude of the interface perturbation grows quickly, and the interface is transformed to a composition of small-scale shear driven vortical structures and a large-scale coherent structure. The scale interaction enhances with time, and the flow transitions to the final stage of intensive interfacial mixing of the fluids. Rayleigh-Taylor mixing is self-similar, like Kolmogorov turbulence, and it is anisotropic, heterogeneous, and sensitive to deterministic conditions, contrary to canonical turbulence [8–12, 20–25, 48–58].

Turbulence and Rayleigh-Taylor mixing are a subject of active research in contemporary science, mathematics and engineering. In-depth understanding of their fundamental properties is achieved over the recent decades. The following aspects are certain now: Turbulence is a super-diffusive stochastic process challenging to implement in practice. Realistic turbulent processes often exhibit anomalous scaling. Properties of self-similar interfacial Rayleigh-Taylor mixing depart from those of canonical turbulence, including scaling laws, spectral shapes, and sensitivity to deterministic conditions [8–12, 42–58].

According to the classical approaches, in canonical turbulence, the velocity scales with length as a power-law with an exponent (1/3) and the wave-vector spectrum has the scaling exponent (−5/3). The group theory approach finds that in Rayleigh-Taylor mixing with constant acceleration the velocity scales with length as a power-law with an exponent (1/2) and the wave-vector spectrum has the scaling exponent (−2). When compared to canonical turbulence, Rayleigh-Taylor mixing has stronger correlations and steeper spectra, and can keep order and sense deterministic conditions even at high Reynolds numbers. The group theory results are consistent with, and explain, experiments on Rayleigh-Taylor mixing in fluids and plasmas. The order in Rayleigh-Taylor mixing is similar in spirit to laminarization of strongly accelerated turbulent flows, including flows in boundary layers and curved pipes [8–12, 17–19, 23, 48–60].

The canonical approaches for Kolmogorov turbulence and the group theory approach for Rayleigh-Taylor mixing are both based on the analysis of symmetries of these processes, including scaling transformations. Questions thus appear: (1) What is the influence of scaling symmetries and invariant forms on theoretical attributes of the non-equilibrium dynamics? (2) Can the properties of very different processes—canonical turbulence and Rayleigh-Taylor mixing—be linked to one another? (3) What is the role of the control dimensional parameters in their respective dynamics? [8–12, 15–19, 23, 48].

The three questions motivate and frame our investigation. We handle mathematical challenges of Kolmogorov turbulence and Rayleigh-Taylor mixing by employing elegant physical concepts. We reveal that these paradigmatic complexities have lucid theoretical representations. We capture the decisive role of the control dimensional parameters in their non-equilibrium dynamics. Our results chart perspectives for future research and expand the range of non-equilibrium processes accessible for analysis, including group theory, representation theory and dynamical systems methodologies.

2. Conservation laws, symmetries and invariant forms

2.1. Governing equations

As in any physical process [17], a dynamics of an ideal fluid is governed by the conservation of mass, momentum and energy represented in continuous approximation in an inertial frame of reference as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0, \frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} + \frac{\partial P}{\partial x_i} = 0,$$
$$\frac{\partial E}{\partial t} + \frac{\partial (E+P) v_i}{\partial x_i} = 0 \quad (1)$$

Here the spatial coordinates and time are $(x_i, t) = (x, y, z, t)$; the fields of density, velocity, pressure and energy density are $(\rho, \mathbf{v}, P, E)$, with $E = \rho\left(e + \mathbf{v}^2/2\right)$ and the specific internal energy e. The closure equation of state related the internal energy and pressure, with constant $(P/\rho e)$. In the presence of kinematic viscosity ν the momentum equation is augmented with the term $\left(-\rho\nu\partial^2 v_i/\partial x_j^2\right)$ and the energy equation is also modified [10–12, 17].

For canonical Kolmogorov turbulence, the density field is uniform, $\rho = \rho_0$, the dynamics is the density independent, and the process is driven by an external source supplying energy at a constant rate per unit mass E_0. This specific power E_0, with the dimension $m^2 s^{-3}$, is the control parameter of the self-similar, isotropic and homogeneous turbulence [17–19, 40–46].

For Rayleigh-Taylor dynamics, the equations in the bulk are augmented with the boundary conditions at the interface and at the outside boundaries, so that the normal (tangential) component of velocity and pressure (enthalpy) are continuous (discontinuous) at the interface, and there are no external sources.

$$\lfloor\rho(\mathbf{v}\cdot \mathrm{n} + \theta/|\nabla\theta|)\rfloor = 0, [\mathbf{v}\cdot \mathrm{n}] = 0, [\mathrm{P}] = 0,$$
$$[\mathrm{v}\cdot\tau] = any, [W] = any, \mathrm{v}|_{z\to\pm\infty} = 0 \quad (2)$$

Here the jump of a quantity at the interface is [. . .]; the normal and tangential unit vectors of the interface are $\mathbf{n}$, $\boldsymbol{\tau}$ with $\mathbf{n} = \nabla\theta/|\nabla\theta|$, $(\mathbf{n}\cdot\boldsymbol{\tau}) = 0$, and the function $\theta\left(x, y, z, t\right)$ is $\theta = 0$ at the interface and is $\theta > 0$ (< 0) in the light (heavy) fluid sub-domain. The initial conditions prescribe the perturbations of the interface and the flow fields at some instance of time. The dynamics is specific and is driven by balance per unit mass, as follows from the independence of the boundary condition for the normal velocity from the fluid density.

Rayleigh-Taylor dynamics is driven by the acceleration $\mathbf{g}$, $\mathbf{g} = (0, 0, -g)$, $g = |\mathbf{g}|$. It is due to a body force, is directed from the heavy to the light fluid, and modifies the pressure field as $P \to P - \rho g z$. For constant acceleration, $g = g_0$, in the mixing regime the length scale in the acceleration direction (i.e., the amplitude of the interface perturbation) increases quadratic with time $h \sim g_0 t^2$. The acceleration strength g_0, with the dimension m s^{-2}, is the control parameter of the self-similar, anisotropic and heterogeneous Rayleigh-Taylor mixing [8–12, 23, 25, 61–63].

2.2. Symmetries and invariant forms

Symmetries of isotropic homogeneous turbulence include Galilean transformations, translations in space and time and spatial rotations and reflections. Self-similar canonical turbulence is also invariant under the scaling transformation of the length, $L \to LK$, velocity $v \to vK^n$, and time, $t \to tK^{1-n}$, where n is an exponent and $K > 0$ is a constant. In the governing equations in the limit of vanishing viscosity, $\nu/vL \to 0$, conditional on $\nu \to \nu K^{1+n}$, the exponent of the scaling transformation is $n = 1/3$. Its invariant form is the rate of energy dissipation, $\varepsilon = \nu\left(\partial v_i/\partial x_j\right)^2$, with $\varepsilon \sim v^3/L$ and $\varepsilon \to \varepsilon K^{3n-1}$. The energy dissipation rate and the energy power are similar quantities, $\varepsilon \sim \epsilon_0$ [17–19, 40–47]:

$$n = \frac{1}{3}, \varepsilon \sim \frac{v^3}{L}, \varepsilon \sim \epsilon_0 \tag{3}$$

Symmetries of non-inertial RT mixing include translations, rotations and reflections in the plane normal to the acceleration. Self-similar Rayleigh-Taylor mixing is also invariant with respect to the scaling transformation, $L \to LK$, $v \to vK^n$, , conditional on $g_0 \to g_0 K^{2n-1}$. In the governing equations in the limit of vanishing viscosity, $\nu/vL \to 0$, with $\nu \to \nu K^{1+n}$, the exponent of the scaling transformation is $n = 1/2$. Its invariant form is (the component of) the rate of loss of specific momentum μ in the acceleration direction, with $\mu \sim v^2/L$ and $\mu \to \mu K^{2n-1}$, where the vector of the rate of momentum loss is $\mu_i = \nu\left(\partial^2 v_i/\partial x_j^2\right)$. The rate of momentum loss and the acceleration strength are similar quantities, $\mu \sim g_0$. In RT mixing the rate of energy dissipation is scale-dependent, with $\varepsilon \sim \mu^2 t \sim g_0^2 t$ at time t and $\varepsilon \sim \mu^{3/2} L^{1/2} \sim g_0^{3/2} L^{1/2}$ at length L [8–12, 48, 63]:

$$n = \frac{1}{2}, \quad \mu \sim \frac{v^2}{L}, \quad \mu \sim g_0 \tag{4}$$

Distinct symmetries and invariant forms lead to substantial departures of properties of self-similar Rayleigh-Taylor mixing from those of Kolmogorov turbulence, including their scaling laws, spectral shapes, correlations and fluctuations [8–12, 48, 63].

The properties of Kolmogorov turbulence and Rayleigh-Taylor mixing are identified by the classical approaches and by the group theory approach, respectively, through analyzing symmetries and scaling transformations. We need to clarify whether the group theory approach and the classical approaches are consistent with one another, whether the distinctions in properties of Kolmogorov turbulence and Rayleigh-Taylor mixing are fully captured by their invariant forms, and whether the characteristics of these processes depend on their control dimensional parameters [8–12, 15, 17–19, 48].

3. Interlink of Kolmogorov turbulence and Rayleigh-Taylor mixing

This section directly links the properties of Rayleigh-Taylor mixing and canonical turbulence, demonstrates the full consistency of their theoretical descriptions, and reveals the prominent role of the control dimensional parameters in physics of these processes.

3.1. Velocity scaling

Consider the velocity scaling law, with v (v_l) being the velocity scale at the length scale L (l).

In canonical turbulence, the energy dissipation rate is $\varepsilon \sim v^3/L$ with $v \sim (\varepsilon L)^{1/3}$ at the length scale L, and it is $\varepsilon_l \sim v_l^3/l$ with $v_l \sim (\varepsilon_l l)^{1/3}$ at the length scale l. The invariance of the rate of energy dissipation, $\varepsilon = \varepsilon_l$ with $\varepsilon \sim \epsilon_0$, leads to the velocity scaling law $v_l^3/v^3 \sim l/L$ [17–19, 41, 46, 47].

In Rayleigh-Taylor mixing, the rate of energy dissipation is scale-dependent, with $\varepsilon \sim v^3/L \sim \mu^{3/2} L^{1/2}$ and with $\varepsilon_l \sim v_l^3/l \sim \mu_l^{3/2} l^{1/2}$, where the rate of momentum loss is $\mu \sim v^2/L$ at the length scales L and it is $\mu_l \sim v_l^2/l$ at the length scale l. The rate of momentum loss is an invariant quantity, $\mu = \mu_l$ with $\mu = g_0$, leading to the velocity scaling law $v_l^2/v^2 \sim (\mu_l/\mu)(l/L) \sim l/L$ [10–12, 48].

We directly link the velocity scaling laws in canonical turbulence and Rayleigh-Taylor mixing as:

$$\begin{aligned} &v \sim (\varepsilon L)^{1/3} \sim \left(\mu^{3/2} L^{1/2} L\right)^{1/3} \sim (\mu L)^{1/2} \sim \left(g_0 L\right)^{1/2} \\ &v_l \sim \left(\varepsilon_l\, l\right)^{1/3} \sim \left(\mu_l^{3/2} l^{1/2} l\right)^{1/3} \sim \left(\mu_l\, l\right)^{1/2} \sim \left(g_0 l\right)^{1/2} \\ &\frac{v_l}{v} \sim \frac{(\varepsilon_l\, l)^{1/3}}{(\varepsilon\, L)^{1/3}} \sim \frac{(\mu_l\, l)^{1/2}}{(\mu L)^{1/2}} \sim \frac{(g_0 l)^{1/2}}{(g_0 L)^{1/2}} \quad \Rightarrow \\ &\frac{v_l}{v} \sim \left(\frac{l}{L}\right)^{1/2} \end{aligned} \tag{5}$$

This reveals that the velocity scaling laws in Rayleigh-Taylor mixing and in Kolmogorov turbulence are consistent with each other. Due to their distinct invariant forms—μ and ε – the velocity correlations are stronger in Rayleigh-Taylor mixing than in Kolmogorov turbulence [10–12, 23, 48].

3.2. Reynolds number scaling and viscous scale

Consider the Reynolds number scaling and the viscous scale [17–19].

The Reynolds number is $\mathrm{Re} = vL/\nu$ at the length scale L, and the Reynolds number is $\mathrm{Re}_l = v_l l/\nu$ at the length scale l. Since

$v \sim (\varepsilon L)^{1/3}$ and $v_l \sim (\varepsilon_l l)^{1/3}$, we obtain Re $\sim \varepsilon^{1/3} L^{4/3}/\nu$ and $\mathrm{Re}_l \sim \varepsilon_l^{1/3} l^{4/3}/\nu$. For the viscous length scale $l \sim l_\nu$, the local Reynolds number is $\mathrm{Re}_l \sim 1$ [17–19, 47].

In canonical turbulence, the invariance of the energy dissipation rate, $\varepsilon = \varepsilon_l$ with $\varepsilon \sim \epsilon_0$, leads to the scaling law for the Reynolds number $\mathrm{Re}_l/\mathrm{Re} \sim (l/L)^{4/3}$ and determines the viscous scale $l_\nu \sim (\nu^3/\varepsilon_l)^{1/4} \sim (\nu^3/\varepsilon)^{1/4} \sim (\nu^3/\epsilon_0)^{1/4}$ [17–19, 47].

In Rayleigh-Taylor mixing, with account for the scale-dependence of the energy dissipation rates $\varepsilon \sim \mu^{3/2} L^{1/2}$ and $\varepsilon_l \sim \mu_l^{3/2} l^{1/2}$, we obtain $Re \sim \mu^{1/2} L^{3/2}/\nu$ and $Re_l \sim \mu_l^{1/2} l^{3/2}/\nu$. The invariance of the rate of momentum loss, $\mu = \mu_l$ with $\mu \sim g_0$, leads to the Reynolds number scaling $\mathrm{Re}_l/\mathrm{Re} \sim (l/L)^{3/2}$ and the viscous scale $l_\nu \sim (\nu^2/\mu_l)^{1/3} \sim (\nu^2/\mu)^{1/3} \sim (\nu^2/g_0)^{1/3}$ [10–12, 23, 48].

We directly link these quantities in canonical turbulence and in Rayleigh-Taylor mixing as:

$$\begin{aligned} &\mathrm{Re} \sim \frac{\varepsilon^{1/3} L^{4/3}}{\nu} \sim \frac{\left(\mu^{3/2} L^{1/2}\right)^{1/3} L^{4/3}}{\nu} \sim \frac{\mu^{1/2} L^{3/2}}{\nu} \sim \frac{g_0^{1/2} L^{3/2}}{\nu} \\ &\mathrm{Re}_l \sim \frac{\varepsilon_l^{1/3} l^{4/3}}{\nu} \sim \frac{\left(\mu_l^{3/2} l^{1/2}\right)^{1/3} l^{4/3}}{\nu} \sim \frac{\mu_l^{1/2} l^{3/2}}{\nu} \sim \frac{g_0^{1/2} l^{3/2}}{\nu} \quad \Rightarrow \\ &\frac{\mathrm{Re}_l}{\mathrm{Re}} \sim \left(\frac{l}{L}\right)^{3/2}, \quad l_\nu \sim \left(\frac{\nu^2}{\mu_l}\right)^{1/3} \sim \left(\frac{\nu^2}{\mu}\right)^{1/3} \sim \left(\frac{\nu^2}{g_0}\right)^{1/3} \end{aligned} \tag{6}$$

This reveals that the Reynolds number scaling and the viscous scale in Rayleigh-Taylor mixing are consistent with those in Kolmogorov turbulence. Due to their distinct invariant forms—μ and ε, respectively—the Reynolds number scaling is steeper in Rayleigh-Taylor mixing than in canonical turbulence, whereas the viscous scale is set by the acceleration g_0 in Rayleigh-Taylor mixing and by the energy power ϵ_0 in turbulence [10–12, 23, 48].

3.3. Spectral shapes for velocity fluctuations

Consider the spectral shape for fluctuations of the velocity (the specific kinetic energy) [17–19].

In canonical turbulence the spectral density of the velocity fluctuations is *E(k)*. It is defined by the invariance of the energy dissipation rate ε and its independence of the wavevector k, leading to the exponent $-5/3$ of the k spectrum, with $E(k) \sim \varepsilon^{2/3} k^{-5/3} \sim \epsilon_0^{2/3} k^{-5/3}$ [17–19, 47].

In RT mixing, the energy dissipation rate depends on the wavevector, $\varepsilon \sim \mu^{3/2} k^{-1/2}$. We obtain:

$$\begin{aligned} &E(k) \sim \varepsilon^{2/3} k^{-5/3} \sim \left(\mu^{3/2} k^{-1/2}\right)^{2/3} k^{-5/3} \sim \mu k^{-2} \\ &\Rightarrow \quad E(k) \sim \mu k^{-2} \sim g_0 k^{-2} \end{aligned} \tag{7}$$

This demonstrates that the spectral shapes in Rayleigh-Taylor mixing and in Kolmogorov turbulence are consistent with one another. In Rayleigh-Taylor mixing the velocity fluctuations spectra are steeper than in canonical turbulence, due to the distinct invariant forms of these processes, μ and ε respectively [10–12, 23, 48].

In two-dimensional isotropic homogeneous turbulence, with account for invariance properties of the enstrophy Ω, with the dimension s^{-2}, and the rate of enstrophy $\tilde{\Omega}$, with the dimension s^{-3}, the spectral density for the kinetic energy fluctuations v^2 has the form $E(k) \sim \tilde{\Omega}^{2/3} k^{-3}$ [64, 65]. In Rayleigh-Taylor mixing the enstrophy Ω and the rate of enstrophy $\tilde{\Omega}$ depend on the wavevector as $\Omega \sim \mu k$ and $\tilde{\Omega} \sim (\mu k)^{3/2}$. We derive

$$\begin{aligned} &E(k) \sim \tilde{\Omega}^{2/3} k^{-3} \sim \left((\mu k)^{3/2}\right)^{2/3} k^{-3} \sim \mu k^{-2} \\ &\Longrightarrow E(k) \sim \mu k^{-2} \sim g_0 k^{-2} \end{aligned} \tag{8}$$

and find that the spectral shapes in Rayleigh-Taylor mixing and in two dimensional turbulence are consistent with one another. In Rayleigh-Taylor mixing the velocity fluctuations spectra are more gradual than in two-dimensional turbulence, due to distinct invariant forms of these processes, μ and $\tilde{\Omega}$, respectively [10–12, 23, 48, 64–66].

3.4. Spectral shapes for density fluctuations

Consider the spectral shape for the density fluctuations [10, 17, 53].

In canonical turbulence, the spectral shape of the density fluctuations is $E(k) \sim \rho_0 \varepsilon^0 k^{-1}$, since the energy dissipation rate ε is independent of the fluid density ρ_0. In Rayleigh-Taylor mixing, the independence of the rate of momentum loss on the fluid density ρ_0 leads to the spectral shape $E(k) \sim \rho_0 \mu^0 k^{-1}$ [10, 53].

We obtain

$$\begin{aligned} &E(k) \sim \rho_0 \varepsilon^0 k^{-1} \sim \rho_0 \left(\mu^{3/2} k^{-1/2}\right)^0 k^{-1} \sim \rho_0 \mu^0 k^{-1} \\ &\Rightarrow \quad E(k) \sim \rho_0 \mu^0 k^{-1} \sim \rho_0 g_0^0 k^{-1} \end{aligned} \tag{9}$$

For the density fluctuations, the exponent -1 of the k spectrum is the same in the anisotropic and heterogeneous Rayleigh-Taylor mixing and in the isotropic and homogeneous Kolmogorov turbulence. In either case the dynamics is specific and is balanced per unit mass (rather than per unit volume), as displayed in the independence of the invariant forms of these processes—μ or ε—on the fluid density ρ_0 [10, 53].

3.5. Link to other modeling approaches

We further illustrate in step-by-step derivations that our results on Rayleigh-Taylor mixing are consistent with other models of realistic turbulent processes and with some empirical models [53, 67–75].

In modeling realistic turbulent processes, the spectral density $E(k)$ is often related to the energy dissipation rate ε, the wavevector k, and the process time scale τ as $\varepsilon \sim \tau k^4 E^2$ [73].

In canonical turbulence the time-scale is $\tau \sim (k^3 E)^{-1/2}$, leading to $\varepsilon \sim (k^3 E)^{-1/2} k^4 E^2 \sim k^{5/2} E^{3/2}$ and, due to the invariance of the energy dissipation rate ε, to the spectral density $E(k) \sim \varepsilon^{2/3} k^{-5/3}$. In RT mixing, the time-scale is $\tau \sim (g_0 k)^{-1/2}$, leading to $\varepsilon \sim (g_0 k)^{-1/2} k^4 E^2 \sim$

$g_0^{-1/2}\, k^{7/2}\, E^2$ and the spectral density $E \sim \varepsilon^{1/2}\, g_0^{1/4}\, k^{-7/4}$. By further accounting for the scale-dependence of the energy dissipation rate $\varepsilon \sim g_0^{3/2}\, k^{-1/2} \sim \mu^{3/2}\, k^{-1/2}$, we obtain the spectral density in Rayleigh-Taylor mixing with constant acceleration as $E \sim g_0 k^{-2} \sim \mu k^{-2}$, similarly to the foregoing [73, 74]:

$$\varepsilon \sim \tau\, k^4 E^2, \quad \tau \sim (g_0 k)^{-1/2}, \quad \varepsilon \sim g_0^{3/2}\, k^{-1/2} \sim \mu^{3/2}\, k^{-1/2} \Rightarrow E \sim g_0 k^{-2} \sim \mu\, k^{-2} \tag{10}$$

By considering the Rayleigh-Taylor time scale $\tau \sim (g_0 k)^{-1/2}$ and by formally treating the energy dissipation rate ε, the empirical model [74] identifies the spectral density as $E \sim k^{-7/4}$. We derive this result from the spectral density $E \sim \mu\, k^{-2}$ in Rayleigh-Taylor mixing, by accounting for the scale dependence of the energy dissipation rate $\varepsilon \sim \mu^{3/2}\, k^{-1/2}$, the invariant form of the rate of momentum loss $\mu \sim g_0$ and the time-scale $\tau \sim (g_0 k)^{-1/2}$ as:

$$\begin{aligned} &E \sim \mu k^{-2} \sim g_0 k^{-2} \sim \left(\mu^{3/2}\, k^{-1/2}\right)^{1/2} g_0^{1/4}\, k^{-7/4} \\ &\sim \left(g_0^{3/2}\, k^{-1/2}\right)^{1/2} g_0^{1/4}\, k^{-7/4} \quad \Rightarrow \\ &E \sim \varepsilon^{1/2}\, g_0^{1/4}\, k^{-7/4} \quad \Rightarrow \\ &E \sim \varepsilon^{1/2}\, (g_0 k)^{1/4}\, k^{-2} \sim \varepsilon^{1/2}\, \tau^{-1/2}\, k^{-2} \sim \mu k^{-2} \end{aligned} \tag{11}$$

The phenomenological model [75] postulates that in Rayleigh-Taylor mixing the spectral density is the same as in canonical turbulence $E \sim k^{-5/3}$. We reproduce this prospect from the spectral density defined by the invariant form of Rayleigh-Taylor mixing $E \sim \mu\, k^{-2}$, with $\mu \sim g_0$, and with relations of the rates of momentum loss and energy dissipation as $\mu \sim \varepsilon^{2/3}\, k^{1/3}$:

$$E \sim \mu k^{-2} \sim \left(\varepsilon^{2/3}\, k^{1/3}\right) k^{-2} \sim \varepsilon^{2/3}\, k^{-5/3} \tag{9}$$

The model further states that in Rayleigh-Taylor mixing the viscous scale vanishes with time [75]. For testing this statement, we consider the local Reynolds number set by the invariant form of Rayleigh-Taylor mixing, $\mathrm{Re}_l \sim \mu_l^{1/2}\, l^{3/2}/\nu$, with $\mu_l \sim \mu \sim g_0$, relate the rates of momentum loss and energy dissipation, $\mu_l \sim \varepsilon_l^{2/3}\, l^{-1/3}$, and derive:

$$\mathrm{Re}_l \sim \frac{\mu_l^{1/2}\, l^{3/2}}{\nu} \Rightarrow l \sim \frac{\mathrm{Re}_l^{2/3}\, \nu^{2/3}}{\mu_l^{1/3}} \sim \frac{\mathrm{Re}_l^{2/3}\, \nu^{2/3}}{\left(\varepsilon_l^{2/3}\, l^{-1/3}\right)^{1/3}} \Rightarrow l \sim \frac{\mathrm{Re}_l^{3/4}\, \nu^{3/4}}{\varepsilon_l^{1/4}} \Rightarrow l_\nu \sim \frac{\nu^{3/4}}{\varepsilon_l^{1/4}} \tag{12}$$

The model's result can be further reproduced with a formal replacement $\varepsilon_l \sim \varepsilon$ and substitution $\varepsilon \sim g_0^2 t$.

The results of empirical models of Rayleigh-Taylor mixing can be obtained within group theory approach by formally treating the energy dissipation rate and by masking its scale-dependence [53, 74, 75].

3.6. Velocity structure function

To conclude this section, we consider the velocity structure function, S_n, of the order n, $n \in N$, with the dimension $(\mathrm{m\, s}^{-1})^n$ [17–19]. It scales with the length scale l as $S_n \sim (\varepsilon_l\, l)^{n/3}$ in Kolmogorov turbulence, and as $S_n \sim (\mu_l\, l)^{n/2}$ in Rayleigh-Taylor mixing [10–12, 17–19, 23, 48]. Since in Rayleigh-Taylor mixing the energy dissipation rate is scale-dependent, $\varepsilon_l \sim \mu_l^{3/2}\, l^{1/2}$, we obtain

$$S_n \sim (\varepsilon_l\, l)^{n/3} \sim \left(\mu_l^{3/2}\, l^{1/2}\, l\right)^{n/3} \sim (\mu_l\, l)^{n/2} \Rightarrow S_n \sim (\mu_l\, l)^{n/2} \tag{13}$$

The structure functions in Rayleigh-Taylor mixing and in Kolmogorov turbulence are consistent with one another and are set by their respective invariant forms. In Rayleigh-Taylor mixing the structure function has a steeper dependence on the order number when compared to Kolmogorov turbulence [10–12, 17–19, 23, 48].

In isotropic homogeneous turbulence in realistic environments the structure function is known to depart from the Kolmogorov scenario: It exhibits intermittency and multi-fractality mathematically, is influenced by the flow structures physically, and has remarkable statistics [17, 66, 76–79]. We believe that the approaches developed for canonical turbulence [66, 76, 78] can be generalized to the case of Rayleigh-Taylor mixing with variable accelerations [8–11], to be done in the future.

4. Invariant forms and control dimensional parameters

Symmetries and their associated invariant forms are common in physical processes. They relate the process insights to the control dimensional parameter and enable the problem solution. We give some examples to accentuate the role of the control dimensional parameters and the associated invariant forms in understanding complex processes [3–7, 10–12, 15–17].

For gravitational process, the invariance of the gravitational constant G with the dimension $\mathrm{kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2}$ is compatible with the Kepler's third law, $L^3 \sim t^2$ [17]. In standard diffusion the invariance of diffusion coefficient D with the dimension $\mathrm{m}^2\mathrm{s}^{-1}$ leads to the diffusion scaling law $L \sim t^{1/2}$ and the Gaussian distribution [17].

In canonical turbulence, the invariance of the energy dissipation rate ε is associated with the power E_0 of the external source supplying energy to the system at a constant rate, both having the dimension $\mathrm{m}^2\mathrm{s}^{-3}$ [17–19]. This leads to the scaling laws for the length $L \sim t^{3/2}$ and the velocity $v \sim t^{1/2}$ and displays the stochastic nature of canonical turbulence having normal distribution of velocity fluctuations.

In Rayleigh-Taylor mixing, the invariance of the rate of loss of specific momentum μ is associated with the acceleration strength g_0, both having the dimension $\mathrm{m\, s}^{-2}$. This leads to the scaling laws for the length $L \sim t^2$ and the velocity $L \sim t$ and exhibits the deterministic nature of Rayleigh-Taylor dynamics having ballistic velocity fluctuations [10–12].

We thus find that the scaling laws, spectral shapes, properties of correlations and fluctuations in canonical turbulence and in Rayleigh-Taylor mixing are set by their invariant forms and the associated control dimensional parameters, $\varepsilon \sim \epsilon_0$ and $\mu \sim g_0$, respectively [10–12, 17–19, 47, 48, 53]. These theoretical insights call for experimental investigations. As noted by the 1923 Nobel Laureate Robert A. Millikan, "The fact that Science walks forward on two feet, namely theory and experiment... Sometimes it is one foot which is put forward first, sometimes the other, but continuous progress is only made by the use of both—by theorizing and then testing, or by finding new relations in the process of experimenting and then bringing the theoretical foot up and pushing it on beyond, and so on in unending alternations."[1]

5. Discussion

We considered two paradigmatic complexities of non-equilibrium dynamics—canonical turbulence and Rayleigh-Taylor mixing. These processes are a long-standing theoretical challenge, requiring one to solve the system of conservation laws—non-linear partial differential equations, augmented also in the Rayleigh-Taylor case with the singular boundary value problem at the unstable interface and the ill-posed initial value problem. We handle the mathematical challenges on the basis of the physical concept of symmetries and reveal the effect of invariant forms on attributes and characteristics of these processes. We assess that in Rayleigh-Taylor mixing the correlations are stronger, the velocity spectra are steeper and the deterministic conditions are more authoritative, than in Kolmogorov turbulence. For the first time, to our knowledge, we interlink the scaling laws and spectral shapes of these processes, and identify the decisive role of the control dimensional parameters in their non-equilibrium dynamics.

The concept of symmetries advanced our understanding of dynamical systems and enabled development of universal theoretical models of pattern formation in idealistic systems. Our work finds that symmetries and invariant forms encapsulate information on characteristics of non-equilibrium dynamics and are associated with dimensional parameters controlling physical processes. They can provide an important insight on properties of realistic complex systems. This expands the range of applicability of dynamical systems beyond traditional frameworks, and allow us to systematically investigate a broad range of phenomena, having considerable scientific and technological importance, and including supernovae, climate change, plasma fusion, and purification of water [1–63] (see text footnote 1).

1 The Nobel Prize. Available online at: https://www.nobelprize.org/.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

SIA contributed to conceptualization, formal analysis, investigation, methodology, resources, and writing—original draft.

Funding

The author thanks the National Science Foundation (USA) (Award No. 1404449) and the Australian Research Council (AUS) (Award No. LE220100132).

Conflict of interest

The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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