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Thermal Timoshenko beam system with suspenders and Kelvin–Voigt damping

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In the present study, we consider a thermal-Timoshenko-beam system with suspenders and Kelvin–Voigt damping type, where the heat is given by Cattaneo's law. Using the existing semi-group theory and energy method, we establish the existence and uniqueness of weak global solution, and an exponential stability result. The results are obtained without imposing the equal-wave speed of propagation condition.
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Timoshenko, thermoelasticity, suspenders, Cattaneo's law, Kelvin–Voigt, well-posedness, stability

1. Problem setting and introduction

In the present study, we consider a cable-suspended beam structure such as the suspension bridge, where the roadbed has a negligible sectional dimension in comparison with its length (span of the bridge). Therefore, it is modeled in Timoshenko theory through a one-dimensional extensible beam, while the (main) suspension cable models an elastic string that is coupled to the deck. The equations of motion describing such Timoshenko-suspended-beam system, see [1–9], are given by

$$\begin{cases} \rho u_{tt} - V_x - Q = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ \rho A\varphi_{tt} - S_x + Q = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ \rho I\psi_{tt} - M_x + S = 0, & \text{in } (0,L)\times\mathbb{R}_+, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is the vertical displacement of the vibrating spring of main cable, $\varphi = \varphi(x,t)$ is the transverse displacement, $\psi = \psi(x,t)$ is the rotation angle, and L, ρ, A, and I are, respectively, length, mass density, cross-section area, and moment of inertia. The constitutive laws V, Q, S, and M are defined by

$$V = \alpha u_x, \;\; Q = \lambda(\varphi - u), \;\; S = kGA(\varphi_x + \psi), \;\; M = EI\psi_x, \tag{1.2}$$

where the physical parameters α, λ, E, G, and k are, respectively, the elastic modulus of the string, the stiffness of elastic springs, the Young's modulus of the beam, the shear modulus, and the shear correction coefficient of the beam. Generally, the system (1.1) is not exponentially stable, see for instance [10, 11], and the references therein. Therefore, we need to introduce a dissipative mechanism to achieve an exponential stability. A common and powerful way of stabilizing hyperbolic systems from mechanical structures in literature is through thermal damping, see [12], where a generalized theory on thermoelasticity is established. Assuming the cable is thermally insulated and consider a stress–strain constitutive law of Kelvin–Voigt type, see [11], then (1.2) takes the form

$$\begin{cases} V = \alpha u_x, \;\; Q = \lambda(\varphi - u), \\ S = kGA(\varphi_x + \psi) + \gamma_1(\varphi_x + \psi)_t - \beta\theta, \\ M = EI\psi_x + \gamma_2\psi_{xt}, \end{cases} \tag{1.3}$$

where γ_0 and γ_1 are damping coefficients, $\theta = \theta(x,t)$ is the temperature difference, and $\beta > 0$ is a coupling constant. When the heat conduction θ in (1.3) is governed by Cattaneo's law [13–15], we have the following:

$$\begin{cases} \rho_3\theta_t + q_x + \beta(\varphi_x + \psi)_t = 0, & \text{in } (0,L)\times\mathbb{R}_+, \\ \tau q_t + \sigma q + \theta_x = 0, & \text{in } (0,L)\times\mathbb{R}_+, \end{cases} \tag{1.4}$$

where $q = q(x,t)$ is the heat flux and ρ_3, τ, and $\sigma > 0$ are coupling constants. Considering linear damping force with damping coefficient γ_0 on the vertical displacement of suspenders and by setting $L = 1, \rho_1 = \rho A, \rho_2 = \rho I, k_1 = kGA$, and $k_2 = EI$, then substituting (1.3) into (1.1) and coupling it with (1.4), we arrive at the following system:

$$\begin{cases} \rho u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) + \gamma_0 u_t = 0, \text{ in } (0,1)\times\mathbb{R}_+ \\ \rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x - \gamma_1(\varphi_x + \psi)_{xt} + \beta\theta_x + \lambda(\varphi - u) = 0, & \text{in } (0,1)\times\mathbb{R}_+, \\ \rho_2\psi_{tt} - k_2\psi_{xx} - \gamma_2\psi_{xxt} + k_1(\varphi_x + \psi) + \gamma_1(\varphi_x + \psi)_t - \beta\theta = 0, & \text{in } (0,1)\times\mathbb{R}_+, \\ \rho_3\theta_t + q_x + \beta(\varphi_x + \psi)_t = 0, & \text{in } (0,1)\times\mathbb{R}_+, \\ \tau q_t + \sigma q + \theta_x = 0, & \text{in } (0,1)\times\mathbb{R}_+. \end{cases} \tag{1.5}$$

We supplement system (1.5) with the boundary conditions as follows:

$$\begin{cases} u(0,t) = u(1,t) = \varphi_x(0,t) = \varphi(1,t) = 0, & t \in \mathbb{R}_+, \\ \psi(0,t) = \psi(1,t) = \theta(0,t) = q(1,t) = 0, & t \in \mathbb{R}_+, \end{cases} \tag{1.6}$$

and the initial data are

$$\begin{cases} u(x,0) = u_0(x), \; \varphi(x,0) = \varphi_0(x), \; \psi(x,0) = \psi_0(x), \; \theta(x,0) = \\ \theta_0(x), \quad x \in (0,1), \\ u_t(x,0) = u_1(x), \; \varphi_t(x,0) = \varphi_1(x), \; \psi_t(x,0) = \psi_1(x), \\ q(x,0) = q_0(x), \; x \in (0,1). \end{cases} \tag{1.7}$$

The main focus of this article was to investigate system (1.5)−(1.7). We establish the well-posedness and the asymptotic behavior of solution by using the semi-group and the multiplier methods. For related results to system (1.5)−(1.7), we mention the result of Bochicchio et al. [16], where the authors considered system (1.5) with heat conduction governed by Fourier 's law ($\tau = 0$), $\gamma_1 = \gamma_2 = 0$, and linear frictional damping on $(1.5)_1$ and $(1.5)_2$. They proved an exponential stability result and numerical analysis of the system. Very recently, Mukiawa et al. [17] studied (1.5) with general, delay, and weak internal damping on the first equation and established a general stability result. We also mention the study of Enyi [18], the author proved exponential stability results for thermoelastic Timoshenko beam systems with full and partial Kelvin–Voigt damping, where the heat conduction is governed by the Cattaneo law of heat transfer. There are many closely related Timoshenko systems in literature, which have discussed a lack of exponential stability, see [11, 19, 20], and the references therein. In comparison to the present system, there is no ambiguity since the system is fully damped. Another interesting direction that can be considered is a type of thermoelastic system governed by Saint-Venant's principle on bounded bodies, see [21], where the decay estimates of two-temperature model are obtained. For more related results, the reader should consult the following articles [22–26] and the references therein. The rest of this study is organized as follows: In Section 2, we prove an existence and uniqueness result. In Section 3, we state and prove the main stability result.

2. Well-posedness

In this section, we transform system (1.5)−(1.7) into semi-group setting and establish the existence and uniqueness result. Let $\langle .,. \rangle$ and $\|.\|$ denote, respectively, the inner product and the norm in $L^2(0,1)$.

1. We shall convert Problem (1.5)–(1.7) into the Cauchy form

$$W_t + AW = 0, \;\; W(0) = W_0. \tag{2.1}$$

2. Define appropriate spaces and use the semi-group method, see Pazy [27], to establish the well-posedness.

To this end, we set $W = (u, v, \varphi, w, \psi, z, \theta, q)^T$, where $v = u_t$, $w = \varphi_t$, and $z = \psi_t$. Thus, problem (1.5)–(1.7) becomes

$$(P)\begin{cases} W_t + \mathcal{A}W = 0, \\ W(0) = W_0, \end{cases} \tag{2.2}$$

where $W_0 = (u_0, u_1, \varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0)^T$ and

$$\mathcal{A}W = \begin{pmatrix} -v \\ -\frac{\alpha}{\rho}u_{xx} - \frac{\lambda}{\rho}(\varphi - u) + \frac{\gamma_0}{\rho}v \\ -w \\ -\frac{k_1}{\rho_1}(\varphi_x + \psi)_x - \frac{\gamma_1}{\rho_1}(w_x + z)_x + \frac{\beta}{\rho_1}\theta_x + \frac{\lambda}{\rho_1}(\varphi - u) \\ -z \\ -\frac{k_2}{\rho_2}\psi_{xx} - \frac{\gamma_2}{\rho_2}z_{xx} + \frac{k_1}{\rho_2}(\varphi_x + \psi) + \frac{\gamma_1}{\rho_2}(w_x + z) - \frac{\beta}{\rho_2}\theta \\ \frac{1}{\rho_3}q_x + \frac{\beta}{\rho_3}(w_x + z) \\ \frac{\sigma}{\tau}q + \frac{1}{\tau}\theta_x \end{pmatrix}.$$

Next, we define the Sobolev spaces as follows:

$H_a^1(0,1):=\{\phi\in H_0^1(0,1)\ :\phi(0)=0\}$, and $H_a^2(0,1):=\{\phi\in H^2(0,1)\ :\ \phi_x\in H_a^1(0,1)\}$,
$H_*^1(0,1):=\{\phi\in H_0^1(0,1)\ :\phi(1)=0\}$, and $H_*^2(0,1):=\{\phi\in H^2(0,1)\ :\ \phi_x\in H_*^1(0,1)\}$.

The phase space of our problem is the following Hilbert space,

$$\mathscr{H}:=H_0^1(0,1)\times L^2(0,1)\times H_*^1(0,1)\times L^2(0,1)\times H_0^1(0,1)\times L^2(0,1)\times L^2(0,1)\times L^2(0,1).$$

We endow $\mathscr{H}$ with the following inner product:

$$\begin{aligned}\langle W,\widehat{W}\rangle_{\mathscr{H}}:=&\rho\int_0^1 v\widehat{v}dx+\alpha\int_0^1 u_x\widehat{u}_xdx+\lambda\int_0^1(\varphi-u)(\widehat{\varphi}-\widehat{u})dx\\&+\rho_1\int_0^1 w\widehat{w}dx+k_1\int_0^1(\varphi_x+\psi)(\widehat{\varphi}_x+\widehat{\psi})dx+\rho_2\int_0^1 z\widehat{z}dx\\&+k_2\int_0^1\psi_x\widehat{\psi_x}dx+\rho_3\int_0^1\theta\widehat{\theta}dx+\tau\int_0^1 q\widehat{q}dx,\end{aligned}$$

for any $W=(u,v,\varphi,w,\psi,z,\theta,q)^T$, $\widehat{W}=(\widehat{u},\widehat{v},\widehat{\varphi},\widehat{w},\widehat{\psi},\widehat{z},\widehat{\theta},\widehat{q})^T\in\mathscr{H}$, and norm

$$\|W\|_{\mathscr{H}}^2:=\rho\|v\|^2+\alpha\|u_x\|^2+\lambda\|\varphi-u\|^2+\rho_1\|w\|^2+k_1\|\varphi_x+\psi\|^2+\rho_2\|z\|^2+k_2\|\psi_x\|^2+\rho_3\|\theta\|^2+\tau\|q\|^2,$$

for any $W=(u,v,\varphi,w,\psi,z,\theta,q)^T\in\mathscr{H}$.

The domain of $\mathcal{A}$ is defined as,

$$\mathcal{D}(\mathcal{A}):=\left\{(u,v,\varphi,w,\psi,z,\theta,q)\in\mathscr{H}\ \middle|\ \begin{array}{l}u\in H^2(0,1)\cap H_0^1(0,1),\ \ v\in H_0^1(0,1),\\ \varphi\in H_a^2(0,1)\cap H_*^1(0,1),\ \ w\in H_*^1(0,1),\\ \psi,z\in H^2(0,1)\cap H_0^1(0,1),\ \ \theta\in H_a^1(0,1),\\ q\in H_*^1(0,1),\ \ w_x+z\in H_a^1(0,1),\\ \text{and }\ (\varphi_x+\psi)\in H_a^1(0,1)\end{array}\right\}.$$

Lemma 2.1. The operator $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathscr{H}\to\mathscr{H}$ is monotone.

Proof. Let $W=(u,v,\varphi,w,\psi,z,\theta,q)\in\mathscr{H}$, then using integration by parts and the boundary conditions (1.6), we get,

$$\begin{aligned}\langle\mathcal{A}W,W\rangle_{\mathscr{H}}=&\rho\int_0^1\left[-\frac{\alpha}{\rho}u_{xx}-\frac{\lambda}{\rho}(\varphi-u)+\frac{\gamma_0}{\rho}v\right]vdx-\alpha\int_0^1 u_xv_xdx\\&+\lambda\int_0^1(v-w)(\varphi-u)dx\\&+\rho_1\int_0^1\left[-\frac{k_1}{\rho_1}(\varphi_x+\psi)_x-\frac{\gamma_1}{\rho_1}(w_x+z)_x+\frac{\beta}{\rho_1}\theta_x+\frac{\lambda}{\rho}(\varphi-u)\right]wdx\\&-k_1\int_0^1(w_x+z)(\varphi_x+\psi)dx\\&+\rho_2\int_0^1\left[-\frac{k_2}{\rho_2}\psi_{xx}-\frac{\gamma_2}{\rho_2}z_{xx}+\frac{k_1}{\rho_2}(\varphi_x+\psi)+\frac{\gamma_1}{\rho_2}(w_x+z)-\frac{\beta}{\rho_2}\theta\right]zdx\\&-k_2\int_0^1 z_x\psi_xdx+\rho_3\int_0^1\left[\frac{1}{\rho_3}q_x+\frac{\beta}{\rho_3}(w_x+z)\right]\theta dx+\tau\int_0^1\left[\frac{\sigma}{\tau}q+\frac{1}{\tau}\theta_x\right]qdx\\=&\gamma_0\|v\|^2+\gamma_1\|w_x+z\|^2+\gamma_2\|z_x\|^2+\sigma\|q\|^2\geq 0.\end{aligned}$$

Therefore, $\mathcal{A}$ is monotone. □

Lemma 2.2. The operator $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathscr{H}\to\mathscr{H}$ is maximal.

Proof. Let $F=(f^1,f^2,f^3,f^4,f^5,f^6,f^7,f^8)^T\in\mathscr{H}$. We consider the stationary problem

$$W+\mathcal{A}W=F,\tag{2.3}$$

where $W=(u,v,\varphi,w,\psi,z,\theta,q)$. Now, from (2.3), we get,

$$\begin{cases}u-v=f^1, & \text{in } H_0^1(0,1),\\ \rho v-\alpha u_{xx}-\lambda(\varphi-u)+\gamma_0v=\rho f^2, & \text{in } L^2(0,1),\\ \varphi-w=f^3, & \text{in } H_*^1(1,0),\\ \rho_1w-k_1(\varphi_x+\psi)_x-\gamma_1(w_x+z)_x+\beta\theta_x+\lambda(\varphi-u)=\rho_1f^4, & \text{in } L^2(0,1),\\ \psi-z=f^5, & \text{in } H_0^1(1,0),\\ \rho_2z-k_2\psi_{xx}-\gamma_2z_{xx}+k_1(\varphi_x+\psi)+\gamma_1(w_x+z)-\beta\theta=\rho_2f^6, & \text{in } L^2(0,1),\\ \rho_3\theta+q_x+\beta(w_x+z)=\rho_3f^7, & \text{in } L^2(0,1),\\ \tau q+\sigma q+\theta_x=\tau f^8,, & \text{in } L^2(0,1).\end{cases}\tag{2.4}$$

From $(2.4)_1$, $(2.4)_3$, and $(2.4)_5$, we have $v=u-f^1$, $w=\varphi-f^3$, and $z=\psi-f^5$, respectively. Therefore, (2.4) becomes,

$$\begin{cases}(\rho+\gamma_0)u-\alpha u_{xx}-\lambda(\varphi-u)=\underbrace{\rho f^1+\gamma_0f^1+\rho f^2}_{g_1}, & \text{in } L^2(0,1),\\ \rho_1\varphi-(k_1+\gamma_1)(\varphi_x+\psi)_x+\beta\theta_x+\lambda(\varphi-u)=\underbrace{\rho_1f^3+\rho_1f^4-\gamma_1f^3_{xx}-\gamma_1f^5_x}_{g_2}, & \text{in } H^{-1}(0,1),\\ \rho_2\psi-(k_2+\gamma_2)\psi_{xx}+(k_1+\gamma_1)(\varphi_x+\psi)-\beta\theta=\underbrace{\gamma_1f^3_x+\rho_2f^5+\gamma_1f^5-\gamma_2f^5_{xx}+\rho_2f^6}_{g_3}, & \text{in } H^{-1}(0,1),\\ \rho_3\theta+q_x+\beta(\varphi_x+\psi)=\underbrace{\beta f^3_x+\beta f^5+\rho_2f^7}_{g_4}, & \text{in } L^2(0,1),\\ (\tau+\sigma)q+\theta_x=\underbrace{\tau f^8}_{g_5}, & \text{in } L^2(0,1).\end{cases}\tag{2.5}$$

We define the following bilinear form $\mathcal{B}$ on $\mathbb{H}\times\mathbb{H}$ and linear form $\mathscr{L}$ on $\mathbb{H}$, where $\mathbb{H}:=H_0^1(0,1)\times H_*^1(0,1)\times H_0^1(0,1)\times L^2(0,1)\times L^2(0,1)$, as follows:

$$\begin{aligned}&\mathcal{B}((u,\varphi,\psi,\theta,q),(u^*,\varphi^*,\psi^*,\theta^*,q^*))\\&:=(\rho+\gamma_0)\int_0^1 uu^*dx+\alpha\int_0^1 u_xu_x^*dx+\lambda\int_0^1(\varphi-u)(\varphi^*-u^*)dx\\&+\rho_1\int_0^1\varphi\varphi^*dx+(k_1+\gamma_1)\int_0^1(\varphi_x+\psi)(\varphi_x^*+\psi^*)dx\\&+\rho_2\int_0^1\psi\psi^*dx+(k_2+\gamma_2)\int_0^1\psi_x\psi_x^*dx+\rho_3\int_0^1\theta\theta^*dx\\&+(\tau+\sigma)\int_0^1 qq^*dx,\end{aligned}$$

and

$$\mathscr{L}((u^*,\varphi^*,\psi^*,\theta^*,q^*)):=\int_0^1 u^*g_1dx+\int_0^1\varphi^*g_2dx+\int_0^1\psi^*g_3dx+\int_0^1 g_4\theta^*dx+\int_0^1 g_5q^*dx,$$

for every $(u, \varphi, \psi, \theta, q)$, $(u^*, \varphi^*, \psi^*, \theta^*, q^*) \in \mathbb{H}$.

When $\mathbb{H}$ is endowed with the following norm,

$$\begin{aligned}\|(u, \varphi, \psi, \theta, q)\|^2_{\mathbb{H}} =& \rho\|u\|^2 + \alpha\|u_x\|^2 + \lambda\|\varphi - u\|^2 + \\ & \rho_1\|\varphi\|^2 + k_1\|\varphi_x + \psi\|^2 \\ & + \rho_2\|\psi\|^2 + k_2\|\psi_x\|^2 + \rho_3\|\theta\|^2 + \tau\|q\|^2,\end{aligned}$$

it is easy to see that $\mathcal{B}$ is a continuous and coercive bilinear form on $\mathbb{H} \times \mathbb{H}$, and $\mathscr{L}$ is a linear continuous form on $\mathbb{H}$. Therefore, by the Lax–Milgram theorem, there exists a unique $(u, \varphi, \psi, \theta, q) \in \mathbb{H}$ such that

$$\mathcal{B}((u, \varphi, \psi, \theta, q), (u^*, \varphi^*, \psi^*, \theta^*, q^*)) =$$

$$\mathscr{L}((u^*, \varphi^*, \psi^*, \theta^*, q^*)), \ \forall\, (u^*, \varphi^*, \psi^*, \theta^*, q^*) \in \mathbb{H}.$$

It follows from $(2.4)_1$, $(2.4)_3$, and $(2.4)_5$ that $v \in H^1_0(0,1)$, $w \in H^1_*(0,1)$, and $z \in H^1_0(0,1)$, respectively. Then, using regularity theory, it follows from $(2.5)_1$, $(2.5)_2$, and $(2.5)_3$, that $u, \varphi, \psi \in H^2(0,1)$. Moreover, from $(2.5)_4$ and $(2.5)_5$, we deduce that $\theta \in H^1_a(0,1)$ and $q \in H^1_*(0,1)$. Therefore, $W = (u, v, \varphi, w, \psi, z, \theta, q) \in \mathcal{D}(\mathcal{A})$ and satisfies (2.3), that is, $\mathcal{A}$ is maximal.

□

On account of Lemma 2.1 and Lemma 2.2, we apply the semi-group theory for linear operator, see [27], and immediately have the following result.

Theorem 2.1. Let $W_0 \in \mathscr{H}$ be given, then the Cauchy Problem (2.2) has a unique local weak solution,

$$W \in \mathcal{C}([0, T_m), \mathscr{H}), \text{ for some, } T_m > 0.$$

Remark 2.1. One can easily compute [see (3.3)] that the solution

$$W = (u, u_t, \varphi, \varphi_t, \psi, \psi_t, \theta, q)$$

of (1.5)–(1.7) is given by Theorem 2.1 that satisfies

$$\|W(t)\|^2_{\mathscr{H}} \le C\|W_0\|^2_{\mathscr{H}} \ \forall\, t \ge 0.$$

Thus, the solution W is global, that is, if $W_0 \in \mathscr{H}$ then $W \in \mathcal{C}([0, \infty), \mathscr{H})$.

Now, due to the density of $\mathcal{D}(\mathcal{A})$ in $\mathscr{H}$, we can announce the following result.

Theorem 2.2. Given $W_0 \in \mathcal{D}(\mathcal{A})$, then problem (1.5)–(1.7) has a unique global solution in the class

$$W \in \mathcal{C}([0, \infty), \mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1([0, \infty), \mathscr{H}).$$

3. Stability result

This section is devoted to the exponential stability of system (1.5)–(1.7). The energy functional associated with problem (1.5) − (1.7) is defined by

$$\begin{aligned}\mathcal{E}(t) =& \frac{1}{2}\left[\rho\|u_t\|^2 + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \alpha\|u_x\|^2 + \lambda\|(\varphi - u)\|^2\right] \\ & + \frac{1}{2}\left[k_1\|\varphi_x + \psi\|^2 + k_2\|\psi_x\|^2 + \rho_3\|\theta\|^2 + \tau\|q\|^2\right].\end{aligned} \tag{3.1}$$

The main stability result is as follows:

Theorem 3.1. The energy functional $\mathcal{E}(t)$ defined in (3.1) decays exponentially as time approaches infinity. That is, there exist two constants $K, \delta > 0$ such that

$$\mathcal{E}(t) \le K e^{-\delta t}, \ \forall\, t \ge 0. \tag{3.2}$$

3.1. Proof of Theorem 3.1

We provide several Lemmas to facilitate the proof of Theorem (3.1).

Lemma 3.1. Let $(\varphi, \psi, \theta, q)$ be the solution of (1.5). Then, the energy functional (3.1) satisfies

$$\mathcal{E}'(t) = -\gamma_0\|u_t\|^2 - \gamma_1\|\varphi_{xt} + \psi_t\|^2 - \gamma_2\|\psi_{xt}\|^2 - \sigma\|q\|^2 \le 0, \ \forall\, t \ge 0. \tag{3.3}$$

Proof. Multiplying $(1.5)_1$ by u_t, $(1.5)_2$ by φ_t, $(1.5)_3$ by ψ_t, $(1.5)_4$ by θ, $(1.5)_5$ by q, integrating over $(0,1)$, using integration by parts and the boundary conditions (1.6), we have,

$$\frac{1}{2}\frac{d}{dt}\left(\rho\|u_t\|^2 + \alpha\|u_x\|^2 + \lambda\|(\varphi - u)\|^2\right) - \lambda\langle(\varphi - u), \varphi_t\rangle + \gamma_0\|u_t\|^2 = 0, \tag{3.4}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi_t\|^2 + k_1\|\varphi_x + \psi\|^2\right) - k_1\langle(\varphi_x + \psi), \psi_t\rangle + \gamma_1\langle(\varphi_x + \psi)_t, \varphi_{xt}\rangle \\ + \lambda\langle(\varphi - u), \varphi_t\rangle - \beta\langle\theta, \varphi_{xt}\rangle = 0,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi_t\|^2 + k_2\|\psi_x\|^2\right) + \gamma_2\|\psi_{xt}\|^2 + k_1\langle(\varphi_x + \psi), \psi_t\rangle + \\ \gamma_1\langle(\varphi_x + \psi)_t, \psi_t\rangle - \beta\langle\theta, \psi_t\rangle = 0,\end{aligned} \tag{3.6}$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_3\|\theta\|^2\right) + \langle\theta, q_x\rangle + \beta\langle\theta, (\varphi_x + \psi)_t\rangle = 0, \tag{3.7}$$

and

$$\frac{1}{2}\frac{d}{dt}\left(\tau\|q\|^2\right) + \sigma\|q\|^2 - \langle q_x, \theta\rangle = 0. \tag{3.8}$$

Adding (3.4)–(3.8), we obtain,

$$\frac{d}{dt}\mathcal{E}(t) = -\gamma_0\|u_t\|^2 - \gamma_1\|\varphi_{xt} + \psi_t\|^2 - \gamma_2\|\psi_{xt}\|^2 - \sigma\|q\|^2 \le 0, \ \forall\, t \ge 0. \tag{3.9}$$

The computations above are done for regular solution. However, the result remains true for weak solution by density argument. □

Remark 3.1. The lemma above implies that the energy (3.1) is decreasing and bounded above by $E(0)$.

Now, we construct a suitable Lyapunov functional L such that

$$a_1\mathcal{E}(t) \le L(t) \le a_2\mathcal{E}(t), \ \ \forall\, t \ge 0, \tag{3.10}$$

for some $a_1, a_2 > 0$, and show that L satisfies for some $\eta > 0$

$$L'(t) \le -\eta L(t), \ \forall\, t \ge 0, \tag{3.11}$$

from which, we obtain

$$L(t) \le L(0)e^{-\varpi t}, \ \forall\, t \ge 0, \tag{3.12}$$

for some $\varpi > 0$. The exponential decay of L in (3.12) will then imply the exponential decay of the energy functional $\mathcal{E}(t)$. To achieve (3.10)–(3.12), we define L as follows:

$$L(t) := NE(t) + N_1G_1(t) + N_2G_2(t) + N_3G_3(t), \quad t \ge 0, \tag{3.13}$$

for some $N, N_1, N_2, N_3 > 0$ to be specified later, and

$$\begin{aligned} G_1(t) &= \rho\langle u_t(t), u(t)\rangle + \rho_1\langle \varphi_t(t), \varphi(t)\rangle + \rho_2\langle \psi_t(t), \psi(t)\rangle + \frac{\gamma_0}{2}\|u(t)\|^2,\\ G_2(t) &= \tau\rho_3\langle \theta(t), Q(t)\rangle, \qquad \text{where } Q(x,t) = \int_0^x q(y,t)dy,\\ G_3(t) &= -\rho_1\rho_3\langle \theta(t), \Phi_t(t)\rangle, \qquad \text{where } \Phi(x,t) = \int_0^x \varphi(y,t)dy. \end{aligned} \tag{3.14}$$

Let us mention that routine computations, applying Young's, Cauchy–Schwarz, and Poincaré's inequalities give (3.10). Next, we provide some Lemmas needed to establish (3.11)–(3.12).

Lemma 3.2. The functional G_1, along the solution of system (1.5)—(1.7) satisfies the estimate

$$\begin{aligned} G_1'(t) \le& -\alpha\|u_x\|^2 - \lambda\|\varphi - u\|^2 - \frac{k_1}{2}\|\varphi_x + \psi\|^2 - \frac{k_2}{2}\|\psi_x\|^2 + \rho\|u_t\|^2\\ &+\rho_1\|\varphi_t\|^2 + c_1\|\psi_{xt}\|^2 + c_2\|\varphi_{xt} + \psi_t\|^2 + c_3\|\theta\|^2, \ \forall\, t \ge 0. \end{aligned} \tag{3.15}$$

Proof. Differentiating G_1, using $(1.5)_1$, $(1.5)_2$, and $(1.5)_3$, then applying integration by parts and the boundary conditions (1.6), we obtain

$$\begin{aligned} G_1'(t) =& \rho\|u_t\|^2 + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 - \alpha\|u_x\|^2 - \lambda\|\varphi - u\|^2 - k_1\\ &\|\varphi_x + \psi\|^2 - k_2\|\psi_x\|^2 - \gamma_1\langle(\varphi_x + \psi), (\varphi_{xt} + \psi_t)\rangle -\\ &\gamma_2\langle \psi_x, \psi_{xt}\rangle + \beta\langle(\varphi_x + \psi), \theta\rangle. \end{aligned} \tag{3.16}$$

Exploiting Young's and Poincaré's inequalities, we obtain,

$$\begin{aligned} G_1'(t) \le& \rho\|u_t\|^2 + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_{xt}\|^2 -\\ &\alpha\|u_x\|^2 - \lambda\|\varphi - u\|^2 - k_1\|\varphi_x + \psi\|^2\\ &- k_2\|\psi_x\|^2 + \frac{k_1}{4}\|\varphi_x + \psi\|^2 + \frac{\gamma_1^2}{k_1}\|\varphi_{xt} + \psi_t\|^2 +\\ &\frac{k_2}{2}\|\psi_x\|^2 + \frac{\gamma_2^2}{2k_2}\|\psi_{xt}\|^2 + \frac{k_1}{4}\|\varphi_x + \psi\|^2 +\\ &\frac{\beta^2}{k_1}\|\theta\|^2 = -\alpha\|u_x\|^2 - \lambda\|\varphi - u\|^2 - \frac{k_1}{2}\|\varphi_x + \psi\|^2 -\\ &\frac{k_2}{2}\|\psi_x\|^2 + \rho\|u_t\|^2 + \rho_1\|\varphi_t\|^2 +\\ &\left(\rho_2 + \frac{\gamma_2^2}{2k_2}\right)\|\psi_{xt}\|^2 + \frac{\gamma_1^2}{k_1}\|\varphi_{xt} +\\ &\psi_t\|^2 + \frac{\beta^2}{k_1}\|\theta\|^2. \end{aligned} \tag{3.17}$$

By setting $c_1 = \rho_2 + \frac{\gamma_2^2}{2k_2}, c_2 = \frac{\gamma_1^2}{k_1}$, and $c_3 = \frac{\beta^2}{k_1}$, we obtain (3.15). □

Lemma 3.3. The functional G_2, along the solution of system (1.5)—(1.7), satisfies the estimate

$$G_2'(t) \le -\frac{\rho_3}{2}\|\theta\|^2 + c_4\|\varphi_{xt} + \psi_t\|^2 + c_5\|q\|^2, \ \forall\, t \ge 0. \tag{3.18}$$

Proof. Differentiation of G_2, using $(1.5)_3$ and $(1.5)_4$, and applying integration by parts leads to

$$G_2'(t) = -\rho_3\|\theta\|^2 + \tau\|q\|^2 - \tau\beta\langle(\varphi_{xt} + \psi_t), Q(t)\rangle - \sigma\rho_3\langle \theta, Q(t)\rangle,$$

where

$$Q(x,t) = \int_0^x q(y,t)dy.$$

Using Cauchy–Schwarz, we note that

$$\|Q\|^2 = \int_0^1\left(\int_0^x q(y,t)dy\right)^2 dx \le \|q\|^2.$$

It follows by Young's and Cauchy–Schwarz inequalities that

$$\begin{aligned} G_2'(t) \le& -\rho_3\|\theta\|^2 + \tau\|q\|^2 + \frac{\tau\beta}{2}\|\varphi_{xt} + \psi_t\|^2 + \frac{\tau\beta}{2}\|Q\|^2 +\\ &\frac{\rho_3}{2}\|\theta\|^2 + \frac{\sigma^2\rho_3}{2}\|Q\|^2 \le -\frac{\rho_3}{2}\|\theta\|^2 +\\ &\frac{\tau\beta}{2}\|\varphi_{xt} + \psi_t\|^2 + \left(\tau + \frac{\tau\beta}{2} + \frac{\sigma^2\rho_3}{2}\right)\|q\|^2. \end{aligned} \tag{3.19}$$

Hence, we obtain (3.18), with $c_4 = \frac{\tau\beta}{2}$ and $c_5 = \left(\tau + \frac{\tau\beta}{2} + \frac{\sigma^2\rho_3}{2}\right)$. □

Lemma 3.4. The functional G_3, along the solution of (1.5), satisfies, the estimate

$$\begin{aligned} G_3'(t) \le& -\frac{\beta\rho_1}{2}\|\varphi_t\|^2 + \epsilon_1\|\varphi_x + \psi\|^2 + \epsilon_2\|\varphi - u\|^2 + c_6\|\psi_{xt}\|^2 + c_7\|\varphi_{xt} + \psi_t\|^2\\ &+c_8\left(1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right)\|\theta\|^2 + c_9\|q\|^2, \forall\, t \ge 0. \end{aligned} \tag{3.20}$$

Proof. Differentiation of G_3, using $(1.5)_2$ and $(1.5)_4$, integration by parts and boundary conditions, we get,

$$\begin{aligned} G_3'(t) =& -\beta\rho_1\|\varphi_t\|^2 - \rho_1\langle q, \varphi_t\rangle + \beta\rho_1\langle \psi_t, \Phi_t(t)\rangle - \rho_3k_1\langle \theta, (\varphi_x + \psi)\rangle\\ &-\rho_3\gamma_1\langle \theta, (\varphi_{xt} + \psi_t)\rangle + \lambda\rho_3\langle \theta, \Omega(t)\rangle + \rho_3\beta\|\theta\|^2, \end{aligned}$$

where,

$$\Phi_t(x,t) = \int_0^x \varphi_t(y,t)dy \ \text{ and } \ \Omega(x,t) = \int_0^x (\varphi(y,t) - u(y,t))dy.$$

Exploiting Cauchy–Schwarz inequality, we see that

$$\|\Phi_t\|^2 \le \|\varphi_t\|^2 \ \text{ and } \ \|\Omega(t)\|^2 \le \|(\varphi - u)\|^2.$$

Therefore, using Young's, Cauchy–Schwarz, and Poincaré's inequalities, we get

$$\begin{aligned}
G_3'(t) \leq & -\beta\rho_1\|\varphi_t\|^2 + \frac{\beta\rho_1}{4}\|\varphi_t\|^2 + \frac{\rho_1}{\beta}\|q\|^2 + \\
& \frac{\beta\rho_1}{4}\|\Phi_t\|^2 + \beta\rho_1\|\psi_t\|^2 + \epsilon_1\|\varphi_x+\psi\|^2 + \frac{(\rho_3 k_1)^2}{4\epsilon_1}\|\theta\|^2 + \\
& \frac{\rho_3\gamma_1}{2}\|\theta\|^2 + \frac{\rho_3\gamma_1}{2}\|\varphi_{xt}+\psi_t\|^2 \\
& + \epsilon_2\|\Omega(t)\|^2 + \frac{(\lambda\rho_3)^2}{4\epsilon_2}\|\theta\|^2 + \beta\rho_3\|\theta\|^2 \\
\leq & -\frac{\beta\rho_1}{2}\|\varphi_t\|^2 + \epsilon_1\|\varphi_x+\psi\|^2 + \epsilon_2\|\varphi-u\|^2 + \beta\rho_1\|\psi_{xt}\|^2 + \\
& \frac{\rho_3\gamma_1}{2}\|\varphi_{xt}+\psi_t\|^2 + \left(\beta\rho_3 + \frac{(\rho_3k_1)^2}{4\epsilon_1} + \frac{(\lambda\rho_3)^2}{4\epsilon_2}\right)\|\theta\|^2 + \\
& \frac{\rho_1}{\beta}\|q\|^2.
\end{aligned}$$

Thus, taking $c_6 = \beta\rho_1, c_7 = \frac{\rho_3\gamma_1}{2}$, $c_8 = \max\left\{\beta\rho_3, \frac{(\rho_3 k_1)^2}{4}, \frac{(\lambda\rho_3)^2}{4}\right\}$, and $c_9 = \frac{\rho_1}{\beta}$, we obtain (3.20). □

Now, we give the proof of Theorem 3.1.

Proof. Using Lemma 3.1 and Lemmas 3.2–3.4, it follows from (3.13) that

$$\begin{aligned}
L'(t) \leq & -[\gamma_0 N - \rho N_1]\|u_t\|^2 - \rho_1\left[\frac{\beta}{2}N_3 - N_1\right]\|\varphi_t\|^2 - \\
& [\gamma_2 N - c_1 N_1 - c_6 N_3]\|\psi_{xt}\|^2 \\
& - \alpha N_1\|u_x\|^2 - [\lambda N_1 - \epsilon_2 N_3]\|\varphi-u\|^2 - \left[\frac{k_1}{2}N_1 - \epsilon_1 N_3\right] \\
& \|\varphi_x+\psi\|^2 - \frac{k_2}{2}N_1\|\psi_x\|^2 - [\gamma_1 N - c_2N_1 - c_4N_2 - c_7N_3] \\
& \|\varphi_{xt}+\psi_t\|^2 - \left[\frac{\rho_3}{2}N_2 - c_3N_1 - c_8N_3\left(1+\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\right)\right] \\
& \|\theta\|^2 - [N\sigma - c_5N_2 - c_9N_3]\|q\|^2.
\end{aligned} \tag{3.21}$$

By setting

$$N_1 = 1,\ \epsilon_1 = \frac{k_1}{4N_3},\ \epsilon_2 = \frac{\lambda}{2N_3},$$

we obtain

$$\begin{aligned}
L'(t) \leq & -[\gamma_0 N - \rho]\|u_t\|^2 - \rho_1\left[\frac{\beta}{2}N_3 - 1\right]\|\varphi_t\|^2 - \\
& [\gamma_2 N - c_6N_3 - c_1]\|\psi_{xt}\|^2 - \alpha\|u_x\|^2 - \frac{\lambda}{2}\|\varphi-u\|^2 - \\
& \frac{k_1}{4}\|\varphi_x+\psi\|^2 - \frac{k_2}{2}\|\psi_x\|^2 \\
& - [\gamma_1 N - c_4N_2 - c_7N_3 - c_2]\|\varphi_{xt}+\psi_t\|^2 \\
& - \left[\frac{\rho_3}{2}N_2 - c_8N_3\left(1+\frac{4N_3}{k_1}+\frac{2N_3}{\lambda}\right) - c_3\right] \\
& \|\theta\|^2 - [N\sigma - c_5N_2 - c_9N_3]\|q\|^2.
\end{aligned} \tag{3.22}$$

Now, we choose N_3 large so that

$$\frac{\beta}{2}N_3 - 1 > 0.$$

Next, we select N_2 large enough such that

$$\frac{\rho_3}{2}N_2 - c_8N_3\left(1+\frac{4N_3}{k_1}+\frac{2N_3}{\lambda}\right) - c_3 > 0.$$

Lastly, we choose N very large enough so that (3.10) remain valid and

$$\gamma_0 N - \rho > 0,\ \gamma_2 N - c_6N_3 - c_1 > 0,\ \gamma_1 N - c_4N_2 - c_7N_3 - c_2 > 0,\ N\sigma - c_5N_2 - c_9N_3 > 0.$$

Thus, we have

$$\begin{aligned}
L'(t) \leq & -\eta \\
& \left(\|u_t\|^2 + \|\varphi_t\|^2 + \|\psi_{xt}\|^2 + \|u_x\|^2 + \|\varphi-u\|^2 + \|\varphi_x+\psi\|^2\right) \\
& - \eta\left(\|\varphi_{xt}+\psi_t\|^2 + \|\psi_x\|^2 + \|\theta\|^2 + \|q\|^2\right),
\end{aligned} \tag{3.23}$$

for some $\eta > 0$. Using (3.1) and Poincaré's inequality, we get

$$L'(t) \leq -\eta_1 L(t),\ \forall\, t \geq 0, \tag{3.24}$$

for some positive constant η_1. Integrating (3.24) over $(0, t)$ yields for some $\varpi > 0$

$$L(t) \leq L(0)e^{-\varpi t},\ \forall\, t \geq 0. \tag{3.25}$$

Hence, the exponential estimate of the energy functional $\mathcal{E}(t)$ in (3.2) follows from (3.25) and the equivalent relation (3.10). □

Data availability statement

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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