frontiers | Frontiers in Applied Mathematics and Statistics

TYPE Original Research
PUBLISHED 16 March 2023
DOI 10.3389/fams.2023.1122330

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SPECIALTY SECTION
This article was submitted to Mathematical Biology, a section of the journal Frontiers in Applied Mathematics and Statistics

RECEIVED 12 December 2022
ACCEPTED 14 February 2023
PUBLISHED 16 March 2023

CITATION
Rayungsari M, Suryanto A, Kusumawinahyu WM and Darti I (2023) Dynamics analysis of a predator–prey fractional-order model incorporating predator cannibalism and refuge. *Front. Appl. Math. Stat.* 9:1122330. doi: 10.3389/fams.2023.1122330

Dynamics analysis of a predator–prey fractional-order model incorporating predator cannibalism and refuge

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In this article, we consider a predator–prey interaction incorporating cannibalism, refuge, and memory effect. To involve the memory effect, we apply Caputo fractional-order derivative operator. We verify the non-negativity, existence, uniqueness, and boundedness of the model solution. We then analyze the local and global stability of the equilibrium points. We also investigate the existence of Hopf bifurcation. The model has four equilibrium points, i.e., the origin point, prey extinction point, predator extinction point, and coexistence point. The origin point is always unstable, while the other equilibrium points are conditionally locally asymptotically stable. The stability of the coexistence point depends on the order of the Caputo derivative, α. The prey extinction point, predator extinction point, and coexistence point are conditionally globally and asymptotically stable. There exists Hopf bifurcation of coexistence point with parameter α. The analytic results of stability properties and Hopf bifurcations are confirmed by numerical simulations.

KEYWORDS
predator-prey system, cannibalism, refuge, Caputo fractional-order derivative, local and global stability analyzes, Hopf bifurcation (critical) value

1. Introduction

Predator–prey interaction, as the basis of the food chain, is among the most essential ecological issues. In numerous published research, mathematical models have been developed to explain the dynamics of Predator–prey interaction, such as by incorporating social behavior [1, 2], age structure [3, 4], ratio-dependent functional response [5, 6], harvesting [7, 8], and so on. The Predator–prey model is still being developed by considering many factors that occur in nature. Cannibalism, the consuming of the same species in whole or in part, is one of its most intriguing aspects since many animals in nature exhibit cannibalistic behaviors, such as carnivore mammals [9–11], fish [12, 13], and spiders [14–16]. Cannibalism may provide adaptive advantages such as exploiting conspecifics as a food source or eliminating possible competitors [17].

Some researchers have investigated the mathematical model involving cannibalism [18–21]. Kang et al. [18] studied a single-species cannibalism model with stage structure. The model studied is a dynamic system of one population such an age structure that divides the population into two classes, i.e., eggs and an adult class consisting of larvae, pupae, queen insects, worker insects, and other types. Zhang et al. [19] analyzed predator–prey models with cannibalism and stage structure in predators so that the model studied was a three-dimensional dynamical model. In Zhang's model, the predator population is divided into two subpopulations, i.e., juvenile and adult predators. The birth rate of juvenile predators is assumed to be proportional to the number of adult predators and follows the Malthus growth model. Predation of prey and juvenile predators by adult predators follows the type-I Holling functional response. Meanwhile, Deng et al. [20] studied a two-dimensional predator–prey model with predator cannibalism.

Aside from cannibalism, another interesting Predator–prey phenomenon to investigate is prey hiding behavior from predator captures and attacks. This is known as refuge behavior in the context of ecology. The mathematical model of Predator–prey interaction with prey refuge has also piqued the interest of researchers [21–25]. Rayungsari et al. [21] modified model proposed by Deng et al. [20] by adding the assumption that there is a refuge in the cannibalized predator population, as much as mP. Moreover, it is also assumed that predators need time to catch and handle the prey, so that the rate of prey predation follows the Holling type-II functional response. The Predator–prey model incorporating predator cannibalism and refuge proposed by Rayungsari et al. [21] is as follows:

$$\begin{aligned}\frac{dN}{dt} &= rN\left(1-\frac{N}{K}\right)-\frac{b_1NP}{k_1+N},\\ \frac{dP}{dt} &= \frac{c_1NP}{k_1+N}+c_2P-eP-\frac{b_2(1-m)P^2}{k_2+(1-m)P},\end{aligned} \tag{1}$$

where $N \geq 0$ and $P \geq 0$ represent prey density and predator density, respectively. The parameters of system (Equation 1) are positive constants described in Table 1. Predator cannibalism is represented by the last term of the second equation in system (Equation 1). The model can be interpreted as follows: In the absence of predator, prey grows logistically with the intrinsic growth rate r and the environmental carrying capacity K. With the presence of the predator, the prey population density decreases by $\frac{b_1NP}{k_1+N}$, where b_1 is the maximum predation rate and k_1 is the half-saturation constant. The predation rate follows Holling type-II functional response with the assumption that predators need time to catch and handle the prey. With the prey predation by predator, the predator population density increases by $\frac{c_1NP}{k_1+N}$, where c_1 is the conversion rate of predation of prey into predator births and $c_1 \leq b_1$. Predators die naturally with the death rate e. The term $\frac{b_2(1-m)P^2}{k_2+(1-m)P}$ depicts the decrease in predator population density caused by cannibalism with saturated a cannibalism rate, which follows Holling type-II functional response,

$$\frac{b_2(1-m)P}{k_2+(1-m)P}. \tag{2}$$

The value of Equation (2) monotonically increases with the supremum b_2. $(1-m)P$ is the amount of the available predator to be cannibalized, as m is the coefficient of refuge. The conversion rate of cannibalism into predator birth (c_2) is assumed to be less than the maximum predator cannibalism rate (b_2).

TABLE 1 Description of parameters.

Parameter	Description
r	Intrinsic growth rate of prey
K	Environmental carrying capacity for prey
b_1	Maximum prey predation rate
k_1	Half-saturation constant of predation
c_1	Conversion rate of prey biomass into predator birth
c_2	Conversion rate of cannibalism into predator birth
e	Predator natural death rate
b_2	Maximum predator cannibalism rate
m	Coefficient of refuge
k_2	Half-saturation constant of predator cannibalism

The model proposed by Rayungsari et al. [21] was constructed in a system of nonlinear differential equations with the first-order derivative, where the change of population density at any time depends on the current population density instantaneously. Whereas in reality, the current condition is also affected by the history of all previous conditions, which is called the memory effect [26]. The phenomenon or systems that have memory and genetic characteristics can be described by a fractional differential system [27]. The definition of fractional-order derivative was first introduced by Liouville [28] motivated by L'Hôpital and Leibniz's critical thinking on derivatives of order $\frac{1}{2}$. Liouville's definition was modified by Riemann by applying a direct generalization of the Cauchy formula and named Riemann–Liouville fractional derivative operator [29]. The fractional-order derivative concept by Liouville and Riemann utilizes Euler's study of fractional integration, which led him to construct the Gamma function as generalization of the factorial concept for fractional numbers [30]. In 1967, Michele Caputo modified the Riemann–Liouville operator so that when solving differential equations, no initial conditions are required. The definition of the modified operator is named by Caputo fractional-order derivative operator. Predator–prey models using Caputo-type fractional derivatives have been widely studied recently [24, 31–33]. Hence, in this article, we modify and analyze the Predator–prey model incorporating predator cannibalism and refuge in Rayungsari et al. [21] by applying the Caputo fractional-order derivative operator.

This article is organized as follows. In Section 2, model development and basic properties are described. The basic properties consist of verification of the non-negativity, existence, uniqueness, and boundedness of solutions of the model. In Section 3, the results of dynamical analysis are presented. The results consist of the existence and stability of equilibrium points. Both local and global stability are investigated, while the analyzed bifurcation is the Hopf bifurcation. In Section 4, the numerical simulations and intrepretations are carried out to confirm the analytical results. Finally, in Section 5, we draw some concluding remarks.

2. Model development and basic properties

By applying the Caputo fractional-order derivative operator to the left-hand side of system (Equation 1), the model becomes

$$
\begin{aligned}
D_*^\alpha N &= rN\left(1-\frac{N}{K}\right)-\frac{b_1NP}{k_1+N}\\
D_*^\alpha P &= \frac{c_1NP}{k_1+N}+c_2P-eP-\frac{b_2(1-m)P^2}{k_2+(1-m)P}
\end{aligned}, \tag{3}
$$

with $\alpha \in \mathbb{R}$, $0 < \alpha \leq 1$, and D_*^α is the α-order of Caputo fractional derivative operator defined by $D_*^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} x(s)\, ds$.

Since the variables in the system (Equation 3) represent the population densities, the solution of the system must be non-negative. The solution of system (Equation 3) is guaranteed to be non-negative by the following theorem.

THEOREM 1. All solutions of Equation (3) are non-negative for any initial values $(N(0), P(0)) \in \mathbb{R}_+^2$.

Proof. Since $D_*^\alpha = N\left(r\left(1-\frac{N}{K}\right)-\frac{b_1P}{k_1+N}\right)$, then $D_*^\alpha N(0) = 0$ if $N(0) = 0$. $D_*^\alpha N = 0$ means there is no change of prey population density. Consequently, $N(t) = 0, \forall t > 0$. Then, we prove that if $N(0) > 0$ then $N(t) \geq 0$ for every $t > 0$. Suppose that the statement is wrong, so there is $t^* > 0$ such as

$$
\begin{aligned}
&N(t) > 0,\ 0 \leq t < t^*,\\
&N(t) = 0,\ t = t^*,\\
&N(t) < 0,\ t \geq t^*,
\end{aligned} \tag{4}
$$

From Equations (3), (4), we get that $D_*^\alpha N = 0, t = t^*$. Thus, there is no change in the population density of N when $t = t^*$. From the prior statement, $N(t) = 0, t = t^*$, so that $N(t) = 0, t > t^*$. This contradicts the statement that $N(t) < 0$ for $t > t^*$. Therefore, $N(t) \geq 0$ for all $t > 0$ is correct. In the same way, it can be proved that $P(t) \geq 0$ for every $t > 0$.

Next, we show the existence and uniqueness of solution of the system (Equation 3) using Theorem 3.7 in Li et al. [34]. Consider a region $[0, \infty) \times \Omega$, where $\Omega = \{X = (N, P) \in \mathbb{R}_+^2 : c_2 < e\}$. Then, we denote a mapping $F(X) = (F_1(X), F_2(X))$, where

$$
\begin{aligned}
F_1(X) &= rN\left(1-\frac{N}{K}\right)-\frac{b_1NP}{k_1+N},\\
F_2(X) &= \frac{c_1NP}{k_1+N}+c_2P-eP-\frac{b_2(1-m)P^2}{k_2+(1-m)P}.
\end{aligned} \tag{5}
$$

For all $X = (N, P), \bar{X} = (\bar{N}, \bar{P}) \in \Omega$,

$$
\begin{aligned}
||F(X)-F(\bar{X})|| &\leq \left|F_1(X)-F_1(\bar{X})\right| + \left|F_2(X)-F_2(\bar{X})\right|\\
&= \left|\left[rN\left(1-\frac{N}{K}\right)-\frac{b_1NP}{k_1+N}\right]-\left[r\bar{N}\left(1-\frac{\bar{N}}{K}\right)-\frac{b_1\bar{N}\bar{P}}{k_1+\bar{N}}\right]\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\left[\frac{c_1NP}{k_1+N}+c_2P-eP-\frac{b_2(1-m)P^2}{k_2+(1-m)P}\right]\right.\\
&\left.-\left[\frac{c_1\bar{N}\bar{P}}{k+\bar{N}}+c_2\bar{P}-e\bar{P}-\frac{b_2(1-m)\bar{P}^2}{k_2+(1-m)\bar{P}}\right]\right|\\
&\leq \left|rN-\bar{N}\right| + \left|\frac{N^2-\bar{N}^2}{K}\right| + \left|\frac{b_1NP(k_1+\bar{N})-b_1\bar{N}\bar{P}(k_1+N)}{(k_1+N)(k_1+\bar{N})}\right|\\
&+\left|\frac{c_1NP(k_1+\bar{N})-c_1\bar{N}\bar{P}(k_1+N)}{(k_1+N)(k_1+\bar{N})}\right| + \left|(c_2-e)(P-\bar{P})\right|\\
&+\left|\frac{b_2(1-m)(P^2(k_2+(1-m)\bar{P})-(\bar{P}^2(k_2+(1-m)P)}{(k_2+(1-m)P)(k_2+(1-m)\bar{P})}\right|\\
&\leq \left[r+\frac{r(N+\bar{N})}{K}+\frac{(b_1+c_1)k_1P}{(k_1+N)(k_1+\bar{N})}\right]\left|N-\bar{N}\right|\\
&+\left[\frac{(b_1+c_1)\bar{N}}{k_1+\bar{N}}+e-c_2\right.\\
&\left.+\frac{b_2(1-m)\left[k_2(P+\bar{P})+P\bar{P}(1-m)\right]}{(k_2+(1-m)P)(k_2+(1-m)\bar{P})}\right]\left|P-\bar{P}\right|.
\end{aligned}
$$

Since in the following discussion, it can be proved that the system solution (Equation 3) is bounded in Ω, there is a positive constant $M = \max\{N, P\}, \forall t \geq 0$. Hence, we have

$$
\begin{aligned}
||F(X)-F(\bar{X})|| &\leq \left[r+\frac{2M}{K}+\frac{(b_1+c_1)k_1M}{k_1^2}\right]\left|N-\bar{N}\right|\\
&+\left[\frac{(b_1+c_1)M}{k_1}+e-c_2+\frac{b_2(1-m)\left[2k_2M+(1-m)M^2\right]}{k_2^2}\right]\\
&\left|P-\bar{P}\right|\\
&= L_1\left|N-\bar{N}\right| + L_2\left|P-\bar{P}\right|,
\end{aligned}
$$

with

$$
\begin{aligned}
L_1 &= r+\frac{2M}{K}+\frac{(b_1+c_1)k_1M}{k_1^2},\\
L_2 &= \frac{(b_1+c_1)M}{k_1}+e-c_2+\frac{b_2(1-m)\left[2k_2M+(1-m)M^2\right]}{k_2^2}.
\end{aligned}
$$

By choosing a positive constant $L = \max\{L_1, L_2\}$, we get

$$
||F(X)-F(\bar{X})|| \leq L||X-\bar{X}||. \tag{6}
$$

Based on Equation (6), the function $F(X)$ satisfies the Lipschitz condition so that there exist a unique solution $X(t)$ of the system (Equation 3) with any initial value of $X(0) = (N(0), P(0))$. Thus, we derive the following theorem.

THEOREM 2. For the system (Equation 3) with any non-negative initial condition $(N(0), P(0)) \in \Omega$, there exist a unique solution $X(t) \in \Omega$.

Next, due to the limited carrying capacity of the prey and predator resources, the size of both populations in the system (Equation 3) must be limited. Consider a function defined by

$$
V(t) = N(t) + \frac{b_1}{c_1}P(t).
$$

The Caputo derivative α-order of V satisfies,

$$\begin{aligned}
D_*^\alpha V &\le D_*^\alpha N + \frac{b_1}{c_1} D_*^\alpha P \\
&= \left[rN\left(1 - \frac{N}{K}\right) - \frac{b_1 NP}{k_1 + N} \right] \\
&+ \tfrac{b_1}{c_1}\left[\tfrac{c_1 NP}{k_1+N} + c_2 P - eP - \tfrac{b_2(1-m)P^2}{k_2+(1-m)P} \right] \\
&= rN - \frac{r}{K}N^2 + \frac{b_1}{c_1}\left(c_2 - e - \frac{b_2(1-m)P}{k_2 + (1-m)P} \right) P \\
&\le rN - \tfrac{r}{K}N^2 + \tfrac{b_1}{c_1}\left(c_2 - e\right)P.
\end{aligned}$$

For any positive constant φ,

$$\begin{aligned}
D_*^\alpha V + \varphi V &\le rN - \frac{r}{K}N^2 + \frac{b_1}{c_1}(c_2 - e)P + \varphi\left(N + \frac{b_1}{c_1}P\right) \\
&= (r+\varphi)N - \frac{r}{K}N^2 + \frac{b_1}{c_1}(c_2 - e + \varphi)P.
\end{aligned}$$

If $c_2 < e$ and by choosing $0 < \varphi < e - c_2$, we get

$$\begin{aligned}
D_*^\alpha V + \varphi V &< (r+\varphi)N - \frac{r}{K}N^2 \\
&= -\frac{r}{K}\left[\left(N - \frac{(r+\varphi)K}{2r} \right)^2 - \left(\frac{(r+\varphi)K}{2r} \right)^2 \right] \\
&\le \frac{r}{K}\left(\frac{(r+\varphi)K}{2r} \right)^2.
\end{aligned} \tag{7}$$

Based on Equation (7), Generalized Mean Value Theorem in Odibat and Shawagfeh [35], and Lemma 6.1 (Fractional Comparison Principle) in Li et al. [34], we get that,

$$\begin{aligned}
V(t) &\le \left(V(0) - \frac{r}{\varphi K}\left(\frac{(r+\varphi)K}{2r} \right)^2 \right) E_\alpha[-\varphi(t)^\alpha] \\
&+ \frac{r}{\varphi K}\left(\frac{(r+\varphi)K}{2r} \right)^2.
\end{aligned} \tag{8}$$

$E_\alpha[-\varphi(t)^\alpha] \to 0ast \to +\infty$, so that,

$$V(t) \to \frac{r}{\varphi K}\left(\frac{(r+\varphi)K}{2r} \right)^2, t \to +\infty.$$

Hence, we establish the following theorem.

THEOREM 3. All solutions of Equation (2) with initial values $(N(0), P(0)) \in \{(x,y) \in R_+^2 : c_2 < e\}$ are uniformly bounded

3. Dynamical analysis

3.1. Existence of equilibrium points

In the similar way as in Rayungsari et al. [21], the system (Equation 3) has four equilibrium points, namely $E_0 = (0,0)$, $E_1 = (0, P_1)$, $E_2 = (K, 0)$, and $E_3 = (N_3, P_3)$, where $P_1 = \dfrac{k_2(e - c_2)}{(c_2 - e - b_2)(1-m)}$. If $b_2 + e \neq c_1 + c_2$, then N_3 and P_3 in E_3 is obtained from the solution of a cubic equation using the Cardano's formula [36, 37], i.e.,

$$\begin{aligned}
N_3 &= \frac{\sqrt[3]{q_2 \pm \sqrt{q_2^2 + \frac{4}{27}q_1^3}}}{\sqrt[3]{2}} - \frac{q_1\sqrt[3]{2}}{3\sqrt[3]{q_2 \pm \sqrt{q_2^2 + \frac{4}{27}q_1^3}}} - \frac{B}{3A}, \\
P_3 &= \frac{r}{b_1}\left(1 - \frac{N_3}{K}\right)(k_1 + N_3),
\end{aligned} \tag{9}$$

with

$$\begin{aligned}
q_1 &= \frac{3AC - B^2}{3A^2}, \\
q_2 &= \frac{9ABC - 2B^3 - 27A^2D}{27A^3}, \\
A &= \frac{r}{b_1 K}(1-m)(b_2 - c_1 - c_2 + e), \\
B &= \frac{r}{b_1}(1-m)\left[(c_1 + c_2 - e - b_2) - \frac{k_1}{K}(c_1 + 2(c_2 - e - b_2)) \right], \\
C &= (c_1 + c_2 - e)k_2 \\
&+ \tfrac{rk_1}{b_1}(1-m)\left[c_1 + (2 - k_1)(c_2 - e) - 2b_2 + \tfrac{b_2 k_1}{K} \right], \\
D &= k_1\left[k_2(c_2 - e) + \frac{rk_1}{b_1}(1-m)(c_2 - e - b_2) \right].
\end{aligned}$$

Whereas, if $b_2 + e = c_1 + c_2$, we have the value of N_3 and P_3 as follows:

$$N_3 = \frac{-R \pm \sqrt{R^2 - 4QS}}{2Q}, P_3 = \frac{r}{b_1}\left(1 - \frac{N_3}{K}\right)(k_1 + N_3),$$

with

$$\begin{aligned}
Q &= \frac{c_1 r k_1}{b_1 K}(1-m), \\
R &= b_2 k_2 + \frac{rk_1}{b_1}(1-m)\left(k_1(c_1 - b_2) - c_1 + \frac{b_2 k_1}{K} \right), \\
S &= k_1\left[k_2(b_2 - c_1) - \frac{r c_1 k_1}{b_1}(1-m) \right].
\end{aligned}$$

Two of the equilibrium points need existence conditions. E_1 exists in $\mathbb{R}_+^2$ if $0 < c_2 - e < b_2$. The coexistence point E_3 exists in $\mathbb{R}_+^2$ if $q_2^2 + \frac{4}{27}q_1^3 \ge 0$ and $0 < N_3 < K$ for $b_2 + e \neq c_1 + c_2$. Meanwhile, for $b_2 + e = c_1 + c_2$, E_3 exists in $\mathbb{R}_+^2$ if $R^2 - 4QS \ge 0$ and $0 < N_3 < K$.

3.2. Local stability

Local stability of the equilibrium points of Equation (3) are determined by the arguments of the eigenvalues of Jacobian matrix and applying Matignon Local Stability Theorem in Petras [38]. Suppose that E^* is an equilibrium point of system (Equation 3). Based on Matignon Local Stability Theorem, E^* is local asymptotically stable if all of the eigenvalues λ_j of the Jacobian matrix,

$$J(E^*) = \begin{bmatrix} r\left(1 - \dfrac{2N}{K}\right) - \dfrac{b_1 k_1 P}{(k_1 + N)^2} & -\dfrac{b_1 N}{k_1 + N} \\ \dfrac{c_1 k_1 P}{(k_1 + N)^2} & \dfrac{c_1 N}{k_1 + N} + c_2 - e \\ & -\dfrac{b_2(1-m)P\left[2k_2 + (1-m)P\right]}{(k_2 + (1-m)P)^2} \end{bmatrix} \tag{10}$$

that satisfies $|\arg(\lambda_j)| > \dfrac{\alpha\pi}{2}$.

THEOREM 4. The origin point $E_0(0,0)$ is always unstable.

Proof. The Jacobian matrix for $E_0 = (0,0)$ is

$$J(E_0) = \begin{bmatrix} r & 0 \\ 0 & c_2 - e \end{bmatrix}, \tag{11}$$

so the eigenvalues are $\lambda_1 = r$ and $\lambda_2 = c_2 - e$. The argument of the first eigenvalue is $|\arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$. If $c_2 > e$ then $|\arg(\lambda_2)| = 0 < \frac{\alpha\pi}{2}$ (E_0 is an unstable source), while if $c_2 > e$ then $|\arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$ (E_0 is an unstable saddle node).

THEOREM 5. Prey extinction point $E_1\,(0, P_1)$ is local asymptotically stable if $r < \frac{b_1 P_1}{k_1}$ and unstable saddle node if $r > \frac{b_1 P_1}{k_1}$.

Proof. By substituting $E_1 = (0, P_1)$ to Equation (10), we get the Jacobian matrix for E_1,

$$J(E_1) = \begin{bmatrix} r - \dfrac{b_1 P_1}{k_1} & 0 \\ \dfrac{c_1 P_1}{k_1} & \dfrac{(c_2 - e)(c_2 - e - b_2)}{b_2} \end{bmatrix}. \tag{12}$$

The eigenvalues are $\lambda_1 = r - \frac{b_1 P_1}{k_1}$ and $\lambda_2 = \frac{(c_2 - e)(c_2 - e - b_2)}{b_2}$. Based on the existence condition of E_1, then λ_2 is the negative real number and $|\arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$. Hence, the local stability of E_1 depends on λ_1. If $r < \frac{b_1 P_1}{k_1}$, $\lambda_1 < 0$, and $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$ so that E_1 is local asymptotically stable. Otherwise, if $r > \frac{b_1 P_1}{k_1}$ then $\lambda_1 > 0$, $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$, and E_1 is an unstable saddle node.

THEOREM 6. The predator extinction point $E_2(K, 0)$ is local asymptotically stable if $e > \frac{c_1 K}{k_1 + K} + c_2$ and unstable saddle node if $e < \frac{c_1 K}{k_1 + K} + c_2$.

Proof. With the same way, we get the Jacobian matrix for E_2 as follows:

$$J(E_2) = \begin{bmatrix} -r & -\dfrac{b_1 K}{k_1 + K} \\ 0 & \dfrac{c_1 K}{k_1 + K} + c_2 - e \end{bmatrix}. \tag{13}$$

The eigenvalues are $\lambda_1 = -r$ and $\lambda_2 = \frac{c_1 K}{k_1 + K} + c_2 - e$. It is clear that $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$. E_2 is local asymptotically stable if $|\arg(\lambda_2)| > \frac{\alpha\pi}{2}$, i.e., for $e > \frac{c_1 K}{k_1 + K} + c_2$. If $e < \frac{c_1 K}{k_1 + K} + c_2$, $|\arg(\lambda_2)| = 0 < \frac{\alpha\pi}{2}$, and E_2 is an unstable saddle node.

For existence point $E_3(N_3, P_3)$, the Jacobian matrix is as follows:

$$J(E_3) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \tag{14}$$

where

$$\begin{aligned} J_{11} &= \frac{rN_3}{k_1 + N_3}\left(1 - \frac{k_1 + 2N_3}{K}\right), \\ J_{12} &= -\frac{b_1 N_3}{k_1 + N_3}, \\ J_{21} &= \frac{c_1 k_1 r}{b_1(k_1 + N_3)}\left(1 - \frac{N_3}{K}\right), \\ J_{22} &= -\frac{b_1 b_2 k_2 r(1 - m)\left(1 - \frac{N_3}{K}\right)(k_1 + N_3)}{\left(b_1 k_2 + r(1 - m)\left(1 - \frac{N_3}{K}\right)(k_1 + N_3)\right)^2}. \end{aligned} \tag{15}$$

Thus, the eigenvalues are obtained from the following quadratic equation.

$$\lambda^2 - trace(J(E_3)) + \det(J(E_3)) = 0, \tag{16}$$

where

$$\begin{aligned} \det(J(E_3)) &= J_{11}J_{22} - J_{12}J_{21} \\ &= -\frac{r^2 b_1 b_2 k_2 (1 - m) N_3 \left(1 - \frac{N_3}{K}\right)}{\left(b_1 k_2 + r(1 - m)\left(1 - \frac{N_3}{K}\right)(k_1 + N_3)\right)^2}\left(1 - \frac{k_1 + 2N_3}{K}\right) \\ &\quad + \frac{c_1 k_1 r N_3}{(k_1 + N_3)^2}\left(1 - \frac{N_3}{K}\right) \end{aligned} \tag{17}$$

and

$$\begin{aligned} trace(J(E_3)) &= J_{11} + J_{22} \\ &= \frac{rN_3}{k_1 + N_3}\left(1 - \frac{k_1 + 2N_3}{K}\right) \\ &\quad - \frac{b_1 b_2 k_2 r(1 - m)\left(1 - \frac{N_3}{K}\right)(k_1 + N_3)}{\left(b_1 k_2 + r(1 - m)\left(1 - \frac{N_3}{K}\right)(k_1 + N_3)\right)^2}. \end{aligned} \tag{18}$$

Suppose that

$$a = \frac{b_1 b_2 k_2 (1 - m)\left(1 - \frac{N_3}{K}\right)(k_1 + N_3)^2}{N_3\left(b_1 k_2 + r(1 - m)\left(1 - \frac{N_3}{K}\right)(k_1 + N_3)\right)^2} > 0, \tag{19}$$

then

$$\begin{aligned} trace(J(E_3)) &= \frac{rN_3}{k_1 + N_3}\left(1 - \frac{k_1 + 2N_3}{K}\right) - \frac{arN_3}{k_1 + N_3} \\ &= \frac{rN_3}{k_1 + N_3}\left(1 - a - \frac{k_1 + 2N_3}{K}\right) \\ &= \frac{rN_3}{k_1 + N_3}\left(\frac{K - aK - k_1 - 2N_3}{K}\right). \end{aligned} \tag{20}$$

Suppose that d is the discriminant of Equation (16), i.e.,

$$d = trace(J(E_3))^2 - 4\det(J(E_3)). \tag{21}$$

The cases are divided into two parts, those are for $d \geq 0$ and for $d < 0$.

1. Case 1 ($d \geq 0$)
   For this case, if $k_1 > K - 2N_3$, we have $\det(J) > 0$ and $trace(J) < 0$. Therefore, the eigenvalues (solutions of Equation 16) are real and negative. Consequently, $|\arg(\lambda_j)| = \pi > \frac{\alpha\pi}{2}$ for $j = 1, 2$ and E_3 is local asymptotically stable.

2. Case 2 ($d < 0$)
   In case ($d < 0$), the eigenvalues are complex number with non-zero imaginary part $\lambda = \dfrac{trace(J(E_3)) + \sqrt{d}}{2}$ and $\bar{\lambda} = \dfrac{trace(J(E_3)) - \sqrt{d}}{2}$. Suppose that
   (a) If $k_1 > K - 2N_3 - aK$, then $trace(J) < 0$ so that $Re(\lambda) < 0$ and E_3 is local asymptotically stable since $|\arg(\lambda)| = |\arg(\bar{\lambda})| = \pi > \dfrac{\alpha\pi}{2}$.
   (b) If $k_1 < K - 2N_3 - aK$, then $trace(J) > 0$ so that $Re(\lambda) > 0$ and E_3 is local asymptotically stable if $|\arg(\lambda)| > \dfrac{\alpha\pi}{2}$.

Hence, we establish the following theorem.

THEOREM 7. Suppose that $d = trace(J(E_3))^2 - 4\det(J(E_3))$ with $trace(J(E_3))$ and $\det(J(E_3))$ are the trace and determinant of matrix $J(E_3)$ in Equation (14). $E_3 = (N_3, P_3)$ is locally asymptotically stable if one of the following conditions are satisfied.

1. $d \geq 0$ and $k_1 > K - 2N_3$,
2. $d < 0$ and $k_1 > K - 2N_3 - aK$,
3. $d < 0$, $k_1 < K - 2N_3 - aK$, and $|\arg(\lambda)| = \left|\dfrac{Im(\lambda)}{Re(\lambda)}\right| = \left|\dfrac{\lambda - \bar{\lambda}}{\lambda + \bar{\lambda}}\right| > \dfrac{\alpha\pi}{2}$,

with a is as in Equation (19).

3.3. Global stability

Next, we investigate the global stability of E_1, E_2, and E_3. For this aim, we use the help of Lemma 3.1 in Vargas-De-Leon [39] and Generalized Lasalle Invariance Principle in Huo et al. [40].

For prey extinction point $E_1(0, P_1)$, we consider a Lyapunov function,

$$V_1(N,P) = N + \frac{b_1}{c_1}\left(P - P_1 - P_1 \ln\frac{P}{P_1}\right).$$

The Caputo derivative α-order of V_1 is as follows:

$$\begin{aligned}
D_*^\alpha V_1 \;&\leq\; D_*^\alpha N + \frac{b_1}{c_1}\left(\frac{P - P_1}{P}\right) D_*^\alpha P \\
&= rN\left(1 - \frac{N}{K}\right) - \frac{b_1 NP}{k_1 + N} \\
&+ \frac{b_1}{c_1}\left(\frac{P - P_1}{P}\right)\left(\frac{c_1 NP}{k_1 + N} + c_2 P - eP - \frac{b_2(1-m)P^2}{k_2 + (1-m)P}\right) \\
&= rN\left(1 - \frac{N}{K}\right) - \frac{b_1 NP_1}{k_1 + N} \\
&+ \frac{b_1}{c_1 P_1}\left(P - P_1\right)\left(\frac{k_2(c_2 - e)P_1 + k_2(e - c_2)P}{k_2 + (1-m)P}\right) \\
&= rN\left(1 - \frac{N}{K}\right) - \frac{b_1 NP_1}{k_1 + N} \\
&- \frac{b_1}{c_1 P_1}\left(P - P_1\right)^2\left(\frac{k_2(c_2 - e)}{k_2 + (1-m)P}\right) \\
&\leq rN\left(1 - \frac{N}{K}\right) - \frac{b_1 NP_1}{k_1 + N}.
\end{aligned}$$

If $r < \dfrac{b_1 P_1}{k_1}$, then we have,

$$\begin{aligned}
D_*^\alpha V_1 \;&\leq\; rN\left(1 - \frac{N}{K}\right) - \frac{rk_1 N}{k_1 + N} \\
&= \frac{rN}{K(k_1 + N)}\left(KN - k_1 N - N^2\right).
\end{aligned}$$

$D_*^\alpha V_1 = 0$ only if $N = 0$. For $N > 0$, if $K \leq k_1$, then $D_*^\alpha V_1 \leq 0$ and according to Generalized Lasalle Invariance Principle [40], E_1 is globally asymptotically stable. We write the global stability conditions of E_1 in the following theorem.

THEOREM 8. If $E_1 = (0, P_1)$ exists, then E_1 is globally asymtotically stable if $r < \dfrac{b_1 P_1}{k_1}$ and $K \leq k_1$.

Then, we construct a Lyapunov function as follows:

$$V_2(N,P) = \frac{c_1}{b_1}\left(N - K - K\ln\frac{N}{K}\right) + P,$$

for $E_2(K, 0)$. We have,

$$\begin{aligned}
D_*^\alpha V_2 \;&\leq\; \frac{c_1}{b_1}\left(\frac{N - K}{N}\right) D_*^\alpha N + D_*^\alpha P \\
&= \frac{c_1}{b_1}\left(\frac{N - K}{N}\right)\left(rN\left(1 - \frac{N}{K}\right) - \frac{b_1 NP}{k_1 + N}\right) + \frac{c_1 NP}{k_1 + N} \\
&\quad + c_2 P - eP - \frac{b_2(1-m)P^2}{k_2 + (1-m)P} \\
&= -\frac{c_1 r}{b_1 K}\left(N - K\right)^2 \\
&+ P\left(\frac{c_1 K}{k_1 + N} + c_2 - e - \frac{b_2(1-m)P}{k_2 + (1-m)P}\right) \\
&\leq P\left(\frac{c_1 K}{k_1 + N} + c_2 - e - \frac{b_2(1-m)P}{k_2 + (1-m)P}\right) \\
&\leq P\left(\frac{c_1 K}{k_1 + N} + c_2 - e\right).
\end{aligned}$$

Suppose that $e > \dfrac{c_1 K}{k_1} + c_2$. Thus, we have,

$$\begin{aligned}
D_*^\alpha V_2 \;&\leq\; P\left(\frac{c_1 K}{k_1 + N} + c_2 - \left(\frac{c_1 K}{k_1} + c_2\right)\right) \\
&= P\left(\tfrac{c_1 K}{k_1 + N} - \tfrac{c_1 K}{k_1}\right) \leq 0.
\end{aligned}$$

We get that $D_*^\alpha V_2 \leq 0, \forall (N,P) \in \mathbb{R}_+^2$. Hence, E_2 is globally asymptotically stable with the condition as in the following theorem.

THEOREM 9. E_2 is globally asymtotically stable if $e > \dfrac{c_1 K}{k_1} + c_2$.

To investigate the global stability of coexistence point, we consider a Lyapunov function

$$V_3(N,P) = N - N_3 - N_3 \ln\frac{N}{N_3} + \frac{b_1}{c_1}\left(P - P_3 - P_3 \ln\frac{P}{P_3}\right),$$

where N_3 and P_3 as in Equation (9). The α-order derivative of V_3 satisfies

$$\begin{aligned} D_*^{\alpha} V_3 \leq & \left(1-\frac{N_3}{N}\right)\left(rN\left(1-\frac{N}{K}\right)-\frac{b_1 NP}{k_1+N}\right) \\ & + \frac{b_1}{c_1}\left(1-\frac{P_3}{P}\right)\left(\frac{c_1 NP}{k_1+N}+c_2P-eP-\frac{b_2(1-m)P^2}{k_2+(1-m)P}\right) \\ = & (N-N_3)\left[r\left(\frac{N_3-N}{K}\right)-\frac{b_1k_1(P-P_3)}{(k_1+N)(k_1+N_3)}\right] \\ & + \frac{b_1}{c_1}(P-P_3)\left[\frac{c_1k_1(N-N_3)}{(k_1+N)(k_1+N_3)}\right. \\ & \left. - \frac{b_2k_2(1-m)(P-P_3)}{(k_2+(1-m)P)(k_2+(1-m)P_3)}\right] \\ = & -\frac{r}{K}(N-N_3)^2-\frac{b_1(N-N_3)(N_3P-NP_3)}{(k_1+N)(k_1+N_3)} \\ & - \frac{b_1b_2k_2(1-m)(P-P_3)^2}{c_1(k_2+(1-m)P)(k_2+(1-m)P_3)}. \end{aligned}$$

Consider a domain $\Omega^* = \left\{(N,P)\middle|\frac{P}{P_3}>\frac{N}{N_3}>1\right\}$. Then, $D_*^{\alpha}V_3 < 0$ and E_3 is globally asymptotically stable in Ω^*. Hence, we derive the following theorem.

THEOREM 10. E_3 is globally asymptotically stable in the domain $\Omega^* = \left\{(N,P)\middle|\frac{P}{P_3}>\frac{N}{N_3}>1\right\}$.

3.4. Existence of Hopf bifurcation

THEOREM 11. If $d < 0$ and $k_1 < K - 2N_3 - aK$ with a is given in Equation (19), then E_3 undergoes Hopf bifurcation when the order of Caputo derivative, α, pass α^* with

$$\alpha^* = \tan^{-1}\left|\frac{Im(\lambda^*)}{Re(\lambda^*)}\right| \tag{22}$$

and λ^* is an eigenvalue of E_3.

Proof. Suppose that $d < 0$ and $k_1 < K - 2N_3 - aK$. Then, the eigenvalues of $J(E_3)$ are a pair of complex number $\lambda_1 = \lambda^*$ and $\lambda_2 = \bar{\lambda}^*$ with positive real part. Suppose that

$$f(\alpha) = \frac{\alpha\pi}{2} - \min|\arg(\lambda_i)_{i=1,2}|.$$

For $\alpha = \alpha^*$ with

$$\alpha^* = \tan^{-1}\left|\frac{Im(\lambda^*)}{Re(\lambda^*)}\right|,$$

we have $f(\alpha^*) = 0$ and $\left.\frac{df(\alpha)}{d\alpha}\right|_{\alpha=\alpha_*} = \frac{\pi}{2} \neq 0$. According to Theorem 3 in Li and Wu [41], E_3 undergoes Hopf bifurcation at $\alpha = \alpha^*$.

4. Numerical simulations

In this section, numerical simulations of the model (Equation 3) are carried out using Matlab software and the predictor–corrector scheme, which is introduced by Diethelm et al. [42]. The purposes of the numerical simulations are to confirm the dynamics analysis results and the existence of Hopf bifurcation. Since there are no available data related to our proposed model yet, the parameter values are chosen hypothetically in Table 2.

TABLE 2 Parameter values.

Parameter	Simulation 1	Simulation 2	Simulation 3
r	1	1	1
K	1	1	1
b_1	0.5	0.5	0.3
k_1	0.3	0.3	0.3
c_1	0.2	0.1	0.2
c_2	0.2	0.2	0.12
e	0.1	0.3	0.02
b_2	0.3	0.3	0.35
m	0.3	0.3	0.3
k_2	1	1	1

For parameter values in Simulation 1, E_1 exists, i.e., $E_1 = (0, 0.7143)$ and the local stability condition in Theorem 5 is satisfied. We conduct numerical simulations with several values of α. The results in Figure 1 show that the solutions tend to the prey extinction point for all α values chosen. This is consistent with the analytical results since the Jacobi matrix eigenvalues are negative real numbers, which involve E_1 always asymptotically stable with the selected parameter values for any order derivative of the $\alpha \in (0, 1]$. However, we can see a difference in the solution's behavior for each α. The lower the α value, the slower the solution reaches E_1.

For the second simulation, we use the same parameter values but c_1 and e (see Table 2). As a result, the existence condition for E_1 is not satisfied, so the point does not exist. It means that the prey can survive from extinction. For the predator extinction point $E_2(1, 0)$, the stability condition in Theorem 6 is satisfied and E_2 is asymptotically stable for any fractional order of $\alpha \in (0, 1]$. It fits the numerial simulation results in Figure 2. Represented by some values of α, we can see that the solutions with initial value close to E_2 go to E_2. With a greater α value, the solution will reach the predator extinction point faster.

Next, we demonstrate the effect of the derivative order on the behavior of the solution, with $0.8 \leq \alpha \leq 1$. The parameter values in the last column of Table 2 were chosen. With those parameter values, the coexistence point exists, i.e., $E_3(0.1423, 1.2645)$, which has the eigenvalues $\lambda^* = 0.0232 + 0.1589i$ and $\bar{\lambda}^* = 0.0232 - 0.1589i$. The parameter values satisfy $k_1 < K - 2N_3 - aK$ and the discriminant of the quadratic equation of the eigenvalues is negative, i.e., $d = -0.1010$. Based on the Theorem 7, the stability of E_3 is determined by the argument of the order derivative α. The threshold is $\alpha^* = 0.9077$, which satisfies $\alpha^* < \frac{2}{\pi}\left|\frac{\lambda^* - \bar{\lambda}^*}{\lambda^* + \bar{\lambda}^*}\right|$.

From the bifurcation diagram in Figure 3, we can see that for $\alpha < \alpha^*$, the solutions tend to E_3. Meanwhile, for $\alpha > \alpha^*$, the

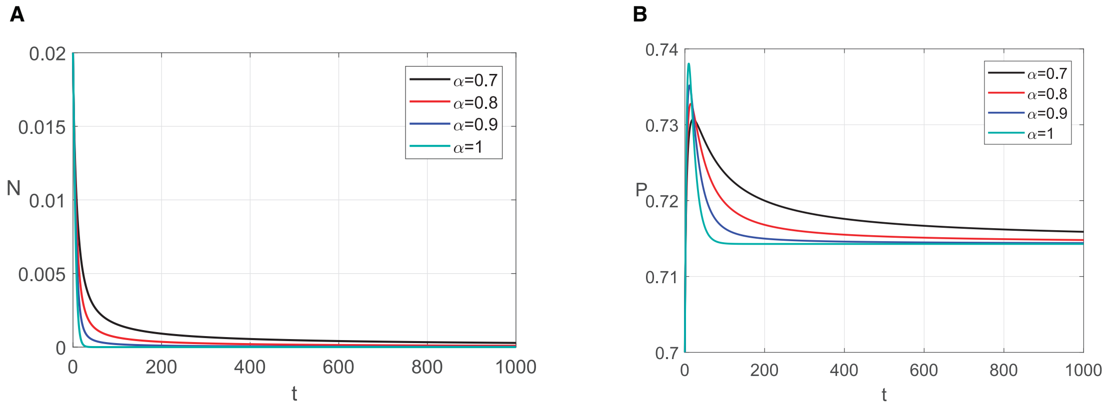

FIGURE 1
Graphic solutions of Simulation 1. **(A)** Solutions of *N* with respect to time *t*. **(B)** Solutions of *P* with respect to time *t*.

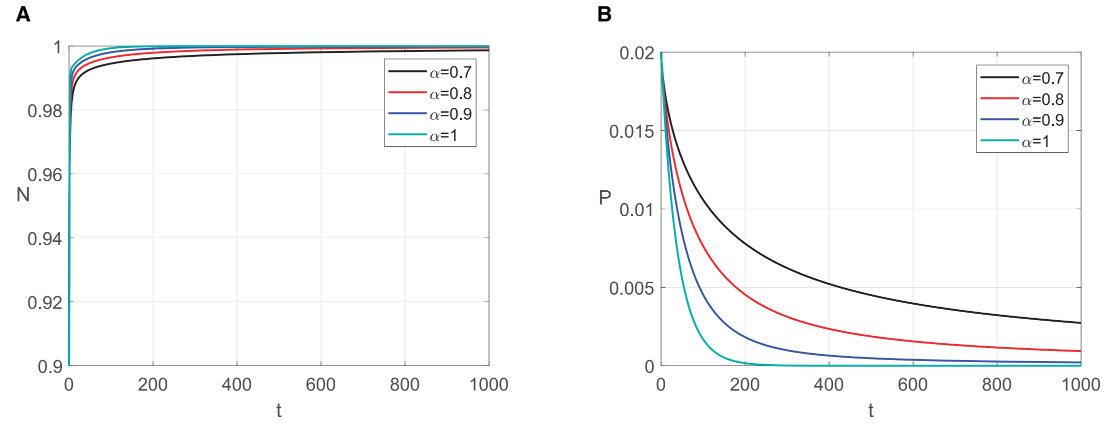

FIGURE 2
Graphic solutions of Simulation 2. **(A)** Solutions of *N* with respect to time *t*. **(B)** Solutions of *P* with respect to time *t*.

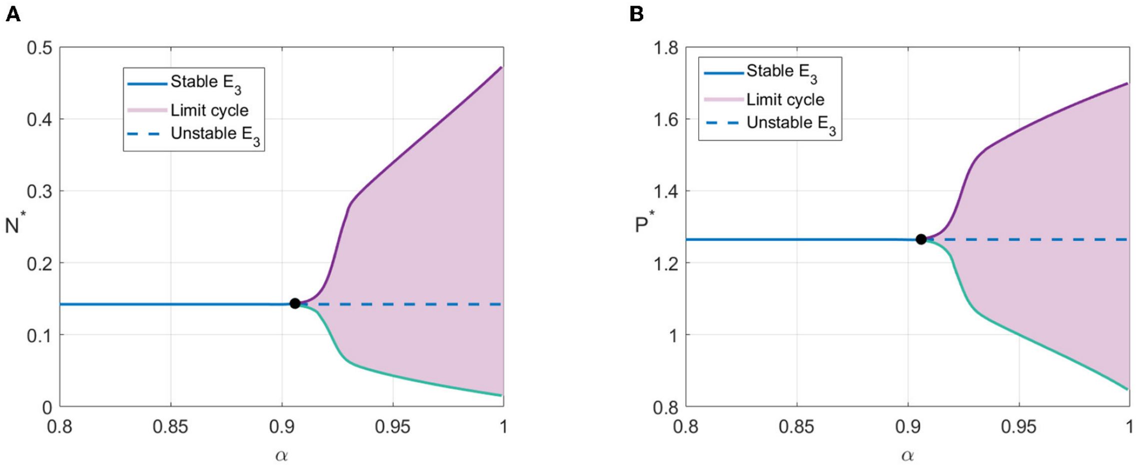

FIGURE 3
Bifurcation diagram with α as bifurcation parameter. **(A)** Value of N^* with respect to derivative order α. **(B)** Value of P^* with respect to derivative order α.

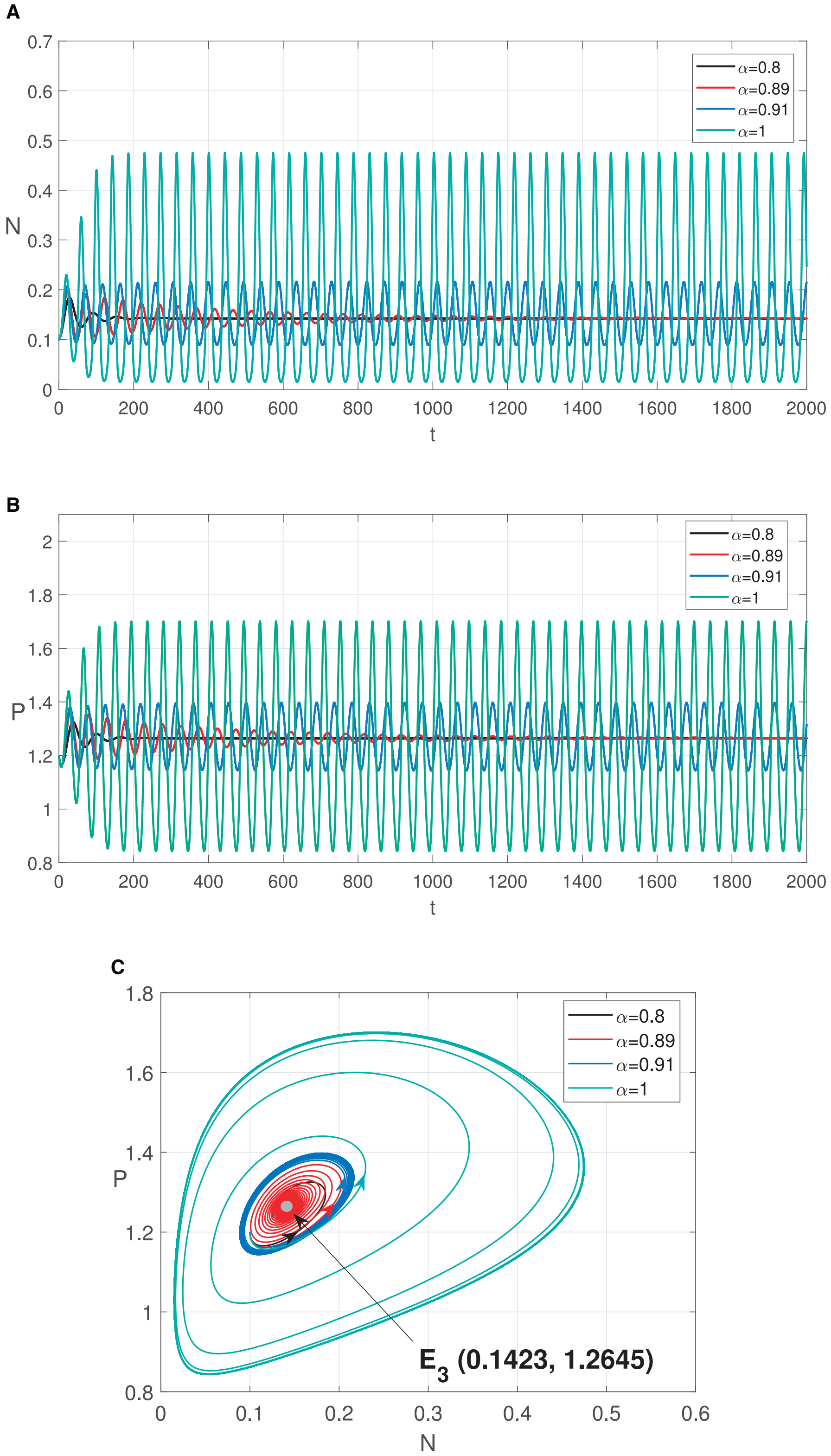

FIGURE 4
Graphic solutions of Simulation 3. **(A)** Solutions of *N* with respect to time *t*. **(B)** Solutions of *P* with respect to time *t*. **(C)** Phase portraits.

solutions tend to limit cycle around E_3. As confirmation of the bifurcation diagram, two α values satisfying $\alpha < \alpha^*$, i.e., $\alpha = 0.8$ and $\alpha = 0.89$, and two α values satisfying $\alpha.\alpha^*$, i.e., $\alpha = 0.91$ and $\alpha = 1$, are selected to simulate the solutions of N and P with respect to time. For $\alpha = 0.8$ and $\alpha = 0.89$, the solutions tend to E_3. The solution with $\alpha = 0.89$ oscillates longer than $\alpha = 0.8$ before

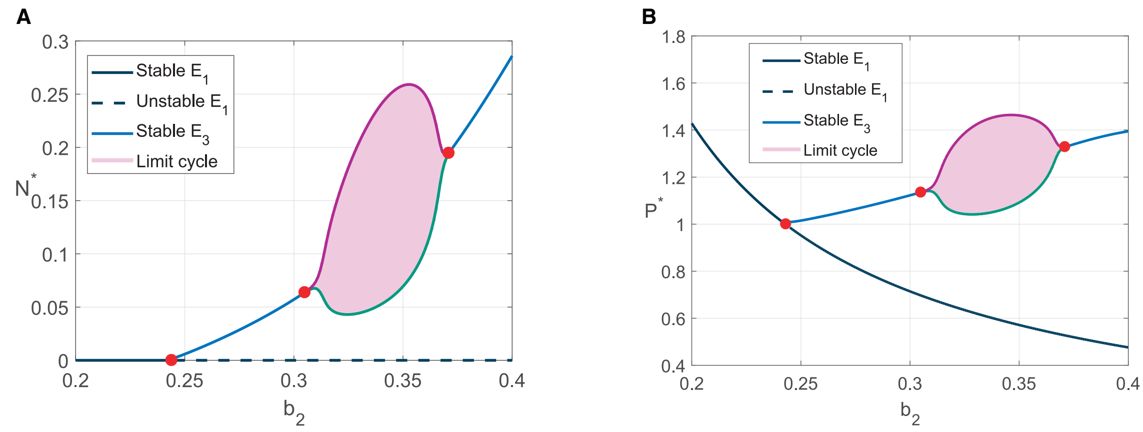

FIGURE 5
Bifurcation diagram with b_2 as bifurcation parameter. **(A)** The value of N^* for $0.2 \leq b_2 \leq 0.4$. **(B)** The value of P^* for $0.2 \leq b_2 \leq 0.4$.

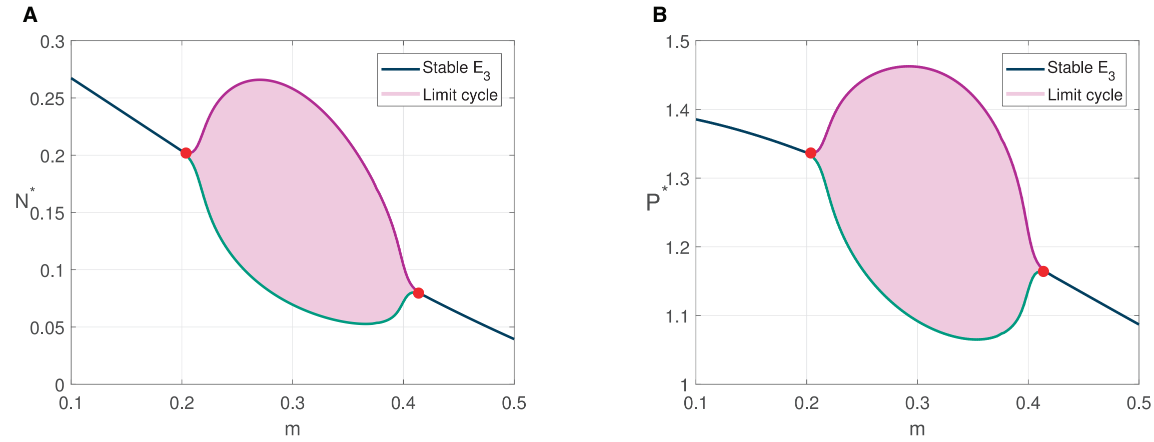

FIGURE 6
Bifurcation diagram with m as bifurcation parameter. **(A)** The value of N^* for $0.1 \leq m \leq 0.5$. **(B)** The value of P^* for $0.1 \leq m \leq 0.5$.

finally convergent to E_3. Meanwhile, for $\alpha = 0.91$ and $\alpha = 1$, each solution convergent to a limit cycle. The amplitude of the limit cycle solution with $\alpha = 1$ is greater than $\alpha = 0.91$.

Numerical simulations in Figures 3, 4 show the existence of Hopf bifurcation in system (3) with α as bifurcation parameter. In addition, the system also undergoes one-parameter Hopf bifurcation with other bifurcation parameters such as cannibalism rate (b_2) and refuge coefficient (m). The bifurcation diagrams are shown in Figures 5, 6, respectively.

For bifurcation diagram with parameter b_2, we have three bifurcation points, i.e., $b_2^* = 0.2429$, $b_2^{**} = 0.306$, and $b_2^{***} = 0.372$. For $b_2 < b_2^*$, the solutions convergent to prey extinction point E_1. It is in accordance with the analytical result since the stability condition of E_1 is satisfied. When the predator cannibalism rate is increased pass b_2^*, E_1 is unstable, and the solutions convergent to the coexistence point, which means the predator survive from extinction. The solutions tend to limit cycle when $b_2^{**} < b_2 < b_2^{***}$. For bifurcation diagram with parameter m, we have two bifurcation points, i.e., m^*, m^{**}. For $m < m^*$, the solutions convergent to coexistence point. The solutions tend to limit cycle in the refuge coefficient range $m^* < m < m^{**}$.

5. Conclusion

A first-order system of Predator–prey interaction incorporating predator cannibalism and refuge is modified by applying Caputo fractional-order derivative operator. We verify the non-negativity, existence, uniqueness, and boundedness of the model solution. The local and global stability of equilibrium points are then examined. In addition, the existence of Hopf bifurcation is investigated. There are four equilibrium points in the model: the origin point, the prey extinction point, the predator extinction point, and the coexistence point. Since the eigenvalues are real numbers, the first three equilibrium points have the same local stability as the first-order system. However, the local stability of the coexistence point differs from that of the first-order system. The coexistence point with positive real-part eigenvalues is locally asymptotically stable in the modified system as long as the absolute of the eigenvalue arguments are bigger than $\frac{\alpha\pi}{2}$. Even though it is based on different theories, the global stability properties of all equilibrium points are identical to those in the first-order system. Under certain conditions, the Hopf bifurcation exists for the coexistence point. Numerical simulations confirmed the analytical results of stability properties and the existence of Hopf bifurcation.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

AS and WMK: conceptualization. MR, AS, and ID: methodology. MR: software, data curation, writing—original draft preparation, and visualization. AS, WMK, and ID: validation, writing—reviewing and editing, and supervision. MR and AS: formal analysis. MR, AS, WMK, and ID: investigation. AS: resources and project administration. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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