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On testing exponentiality under Type-I censoring

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Two new goodness-of-fit testing procedures are introduced to test exponentiality when data are subject to Type-I censoring. We proposed four test statistics for this purpose. Under extensive Monte Carlo simulations, we showed that the proposed tests maintain the nominal significance level and show good power for both monotonic and non-monotonic hazard function alternatives even for small samples as $n = 10$. A real dataset is studied for illustrative purposes.

KEYWORDS
beta distribution, binomial distribution, exponential distribution, goodness-of-fit testing, maximum likelihood estimator, order statistics, quantiles, Type-I censoring

1. Introduction

In reliability and life testing problems, Type-I censoring has gained a significant amount of popularity due to the duration of the experiment being fixed prior to it being started and the fact that it is under the control of the experimenter.

It is of interest to study the lifetime of n items by performing a life testing experiment. By controlling the total time, the experiment can be terminated at the time of T, which can be determined before the life testing experiment begins. This means that d observations take the form of $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{d:n}$, and $n - d$ data values are censored, as discussed by Balakrishnan and Cohen [1] and Cohen [2].

The exponential distribution considered in this article is well-known and frequently uses lifetime models. The exponential model is a special case among many important statistical models such as Weibull and gamma distributions. The simplicity and the existence of closed form solutions for many problems make the exponential model appealing, which informs the current study (see also Balakrishnan and Basu [3]). We assume the following form of pdf for the exponential distribution with scale parameter θ

$$f(x) = \frac{1}{\theta} \exp(-\frac{x}{\theta}), \quad x > 0.$$

Suppose n items are placed in a life testing experiment, which will be terminated at a pre-determined time $T > 0$. Let $X_{1:n}, X_{2:n}, \ldots, X_{d:n}$ be the corresponding Type-I censored sample from a distribution function F. Consider the following goodness-of-fit hypothesis

$$H_0 : F(x) = 1 - \exp(-x/\theta), \quad \text{versus} \quad H_1 : F(x) \neq 1 - \exp(-x/\theta), \tag{1}$$

For some positive scale parameter θ. Based on this, the current study was interested in testing for exponentiality.

The maximum likelihood estimator (MLE) of θ, based on censored data $X_{1:n}, X_{2:n}, \ldots, X_{d:n}$ is given by

$$\hat{\theta} = \frac{1}{d}\left\{ \sum_{i=1}^{d} X_{i:n} + (n-d)T \right\}, \tag{2}$$

TABLE 1 Monte Carlo estimate of the coefficient of skewness ($\sqrt{\beta_1}$) and coefficient of kurtosis (β_2) for the test statistics T_1, T_2, T_3, and T_B under the null distribution of exponentiality.

		$F(T) = 1 - \exp(-T/\theta)$						
Coefficient	Test statistic	0.4	0.5	0.6	0.7	0.8	0.9	0.99
$\sqrt{\beta_1}$	T_1	−0.3129	−0.1180	0.0271	0.1912	0.3796	0.5875	0.6320
	T_2	1.3232	1.3133	1.4808	1.6821	1.8963	2.1000	2.0743
	T_3	0.4641	0.4881	0.5641	0.7117	0.8580	1.0095	1.0359
	T_P	1.4783	1.6418	1.6760	1.7500	1.6208	1.4810	1.4247
β_2	T_1	2.9265	2.8364	2.7859	2.8209	2.9906	3.3985	3.4865
	T_2	6.4788	5.8247	7.1227	7.7983	8.9460	10.4060	9.8452
	T_3	3.4250	3.2778	3.3825	3.6152	3.9251	4.5293	4.5615
	T_P	7.3660	8.6962	8.5972	9.1348	7.7253	6.4801	5.9467

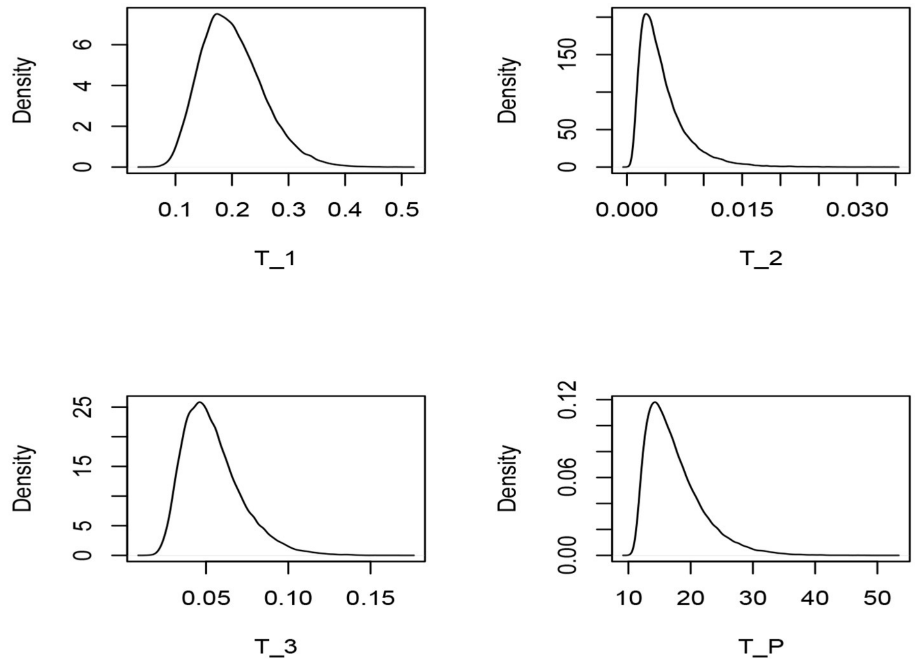

FIGURE 1
Monte Carlo simulated pdf curves for the test statistics T_1, T_2, T_3, and T_P under the assumption of exponential model.

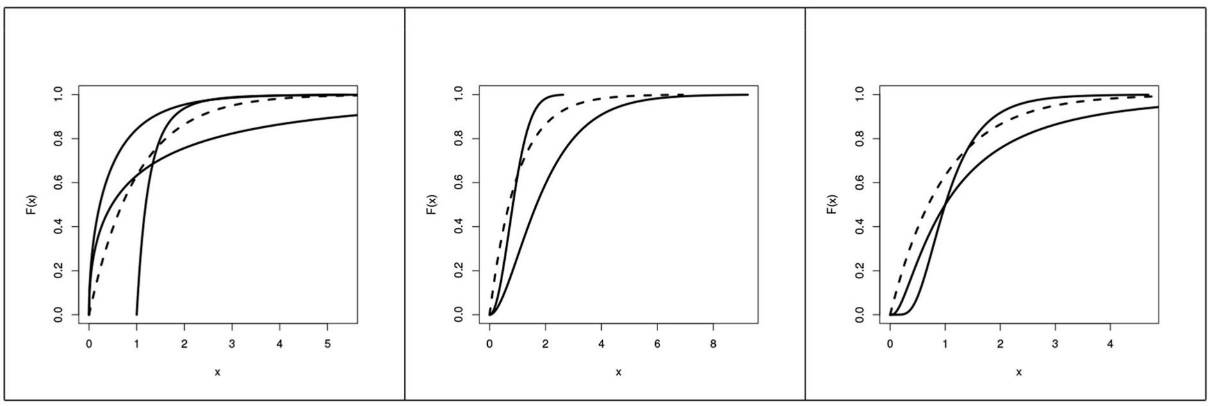

FIGURE 2
CDF of the alternative groups $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ (left to right) and the CDF of a standard exponential distribution (dashed).

TABLE 2 Empirical significance level for the test statistics T_1, T_2, T_3, and T_P when $n = 10, 20, 30$ with $\alpha = 0.10$ and 10^5 iterations.

		$F(T) = 1 - \exp(-T/\theta)$						
Alt. model	Test statistic	0.4	0.5	0.6	0.7	0.8	0.9	0.99
$n = 10$	T_1	0.0991	0.0993	0.0998	0.1004	0.1036	0.0975	0.0987
	T_2	0.0999	0.1003	0.0997	0.1013	0.0974	0.0995	0.1006
	T_3	0.0996	0.0999	0.0989	0.1016	0.0963	0.0991	0.1005
	T_P	0.0989	0.0974	0.1018	0.0995	0.0971	0.09993	0.1026
$n = 20$	T_1	0.0993	0.1013	0.0998	0.1004	0.0994	0.1035	0.0983
	T_2	0.0989	0.1014	0.0969	0.0994	0.1008	0.1028	0.1001
	T_3	0.0994	0.1003	0.0975	0.1007	0.1027	0.1015	0.1007
	T_P	0.0995	0.0981	0.0977	0.1012	0.0999	0.1025	0.1021
$n = 30$	T_1	0.1002	0.0981	0.1011	0.1042	0.1011	0.1010	0.0972
	T_2	0.1001	0.0993	0.1004	0.1026	0.1034	0.1003	0.0988
	T_3	0.0996	0.1002	0.0998	0.1010	0.1035	0.0998	0.0994
	T_P	0.0985	0.0998	0.1010	0.1013	0.1023	0.0976	0.1001

Provided that $d \geq 1$. However, hereafter we assume that $d \geq 1$ and that at least one example of censored data are observed.

Pearson [4] was the first to study the problem of goodness-of-fit, which is a statistical procedure for testing the suitability of a specific model to describe a given set of complete or censored data. For a detailed discussion of this problem see D'Agostino and Stephens [5], Huber-Carol et al. [6], and Nikulin and Chimitova [7] among others.

Stephens [8] proposed a version of the Cramer-von Mises and Anderson-Darling goodness-of-fit test statistics for Type-I censored data. Pakyari and Balakrishnan [9] studied a goodness-of-fit testing procedure for the exponential distribution when the available data are Type-I censored. They studied the goodness-of-fit testing problem for the exponential model by treating the Type-I censored data as a complete sample and then performing classical goodness-of-fit tests for complete data.

Their method considered the Type-I censored sample $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{d:n}$ as order statistics from a complete sample of size d, from a right-truncated exponential distribution at time T.

Pakyari and Resalati Nia [10] extended their work to test other lifetime models such as Weibull and log-normal distributions. We use this idea to present test statistics based on order statistics in Section 2. For the various goodness-of-fit procedures available for censored data see also Balakrishnan et al. [11], Balakrishnan et al. [12], Doring and Cramer [13], Lim and Park [14], Lin et al. [15], Noughabi [16], Pakyari and Balakrishnan [17], Pakyari [18], Pakyari and Baklizi [19], Park and Pakyari [20], and Qi et al. [21].

This article presents new testing procedures for testing the goodness-of-fit of the exponential model when data are Type-I censored. We study several testing procedures in this regard such as tests based on order statistics, tests based on quantiles, and tests based on binomial distribution. However, our proposed method is based on order statistics followed by tests based on quantiles. We investigate the empirical power of the proposed tests through an extensive Monte Carlo simulation study.

This study aims to provide some easy yet powerful goodness-of-fit testing procedures for exponentiality, which is known to be a special case among many well-applied lifetime models.

The paper is structured as follows. Section 2 introduces some test statistics which are constructed based on order statistics. In Section 3 we propose a test statistic based on a linear combination of quantiles vector. Tests based on binomial distribution are discussed in Section 4. In Section 5, we investigate the validity of the proposed tests by calculating the empirical significance levels and comparing them with the nominated levels. We then perform a Monte Carlo simulation study to access the empirical power of the proposed tests so that we can compare them with the power of some known tests described in the literature on this subject. Finally, we explain the proposed tests using a real data example.

2. Tests based on order statistics

Note that conditional on $D = d$,

$$(X_{1:n}, \ldots, X_{d:n}) \overset{\mathrm{d}}{=} (V_{1:d}, \ldots, V_{d:d}),$$

Where the order statistics $V_{1:d}, \ldots, V_{d:d}$ are a random sample of size d from exponential distribution but right truncated at T; see Arnold et al. [22] and David and Nagaraja [23].

On finding the MLE of θ, it will be useful to transform the Type-I censored sample $X_{1:n}, X_{2:n}, \ldots, X_{d:n}$ to the complete uniformly distributed sample $U_{1:d}, U_{2:d}, \ldots, U_{d:d}$ using the following transformation:

$$U_{i:d} = \frac{1 - \exp(-X_{i:n}/\hat{\theta})}{1 - \exp(-T/\hat{\theta})}, \qquad \text{for} \quad i = 1, \ldots, d. \quad (3)$$

Therefore, testing that the Type-I censored data $X_{1:n}, X_{2:n}, \ldots, X_{d:n}$ follow exponential distribution is equivalent to testing that the complete data $U_{1:d}, U_{2:d}, \ldots, U_{d:d}$ follow a uniform distribution.

If we then let $v_i = U_{i:d} - \frac{i}{n+1}$ be the deviation of each order statistics $U_{i:d}$ from its expected value, then several goodness-of-fit test statistics can be considered:

TABLE 3 Empirical power for various alternative models, $n = 10$, $\alpha = 0.10$ and 10^5 iterations.

		$F(T) = 1 - \exp(-T/\theta)$						
Alt. model	Test statistic	0.4	0.5	0.6	0.7	0.8	0.9	0.99
Gamma (0.5, 1.0)	T_1	0.0778	0.0777	0.0913	0.1235	0.1969	0.3119	0.4422
	T_2	0.0012	0.0007	0.0009	0.0024	0.0299	0.1901	0.4574
	T_3	0.0011	0.0007	0.0008	0.0023	0.0327	0.1960	0.4764
	T_P	0.4067	0.3124	0.2478	0.1807	0.1395	0.1072	0.0896
	KS	0.2890	0.3023	0.3161	0.3265	0.3365	0.3455	0.3506
	CM	0.2839	0.3082	0.3284	0.3467	0.3635	0.3769	0.3819
	AD	**0.4832**	**0.4985**	**0.5122**	**0.5260**	**0.5397**	**0.5525**	**0.5584**
Gamma (2.0, 1.0)	T_1	0.0397	0.0407	0.0644	0.1058	0.1436	0.1422	0.1123
	T_2	0.3674	0.5499	0.7112	0.8199	0.8802	0.8950	0.6350
	T_3	**0.3840**	**0.5633**	**0.7193**	**0.8246**	**0.8839**	**0.8966**	**0.6469**
	T_P	0.1654	0.0755	0.0501	0.0594	0.1118	0.2311	0.5117
	KS	0.0172	0.0424	0.0864	0.1431	0.2090	0.2766	0.3428
	CM	0.0069	0.0223	0.0562	0.1129	0.1920	0.2658	0.3914
	AD	0.0047	0.0142	0.0373	0.0785	0.1417	0.2080	0.3416
Weibull (0.5, 1.0)	T_1	0.1485	0.1728	0.2136	0.2776	0.3750	0.4877	0.5944
	T_2	0.0111	0.0128	0.0178	0.0326	0.1094	0.4173	0.7360
	T_3	0.0089	0.0103	0.0124	0.0249	0.1087	0.4295	0.7533
	T_P	0.2883	0.2541	0.2361	0.2189	0.2154	0.2383	0.3204
	KS	0.3261	0.3632	0.3973	0.4314	0.4673	0.5112	0.5845
	CM	0.3032	0.3489	0.3919	0.4325	0.4786	0.5328	0.6230
	AD	**0.5493**	**0.5829**	**0.6126**	**0.6427**	**0.6740**	**0.7120**	**0.7748**
Weibull (2.0, 1.0)	T_1	0.0376	0.0369	0.0307	0.0201	0.0509	0.1888	0.4095
	T_2	0.5026	0.5577	0.5126	0.4000	0.3229	0.4234	0.6259
	T_3	**0.5287**	**0.5974**	**0.5737**	0.4755	0.3515	0.3589	0.5843
	T_P	0.1327	0.2337	0.4356	**0.6517**	**0.8180**	**0.8659**	**0.8547**
	KS	0.1131	0.2463	0.4020	0.5421	0.6515	0.6683	0.6739
	CM	0.0764	0.2132	0.4021	0.5831	0.7252	0.7489	0.7691
	AD	0.0500	0.1570	0.3262	0.5066	0.6699	0.7084	0.7374
Log-normal (0, 0.5)	T_1	0.0075	0.0023	0.0020	0.007	0.0583	0.2516	0.5692
	T_2	0.7264	0.9087	0.9413	0.8844	0.8173	0.8399	0.8377
	T_3	**0.7545**	**0.9270**	**0.9682**	**0.9338**	0.8551	0.8193	0.7770
	T_P	0.0833	0.1759	0.4220	0.6734	0.8444	**0.9309**	**0.9738**
	KS	0.0597	0.2801	0.5690	0.7498	0.8349	0.8715	0.8759
	CM	0.0279	0.2098	0.5221	0.7516	**0.8600**	0.9069	0.9119
	AD	0.0184	0.1528	0.4393	0.6924	0.8226	0.8822	0.8999
Log-normal (0, 1.0)	T_1	0.0206	0.0308	0.0435	0.0521	0.0633	0.0685	0.0870
	T_2	0.4181	0.4111	0.4024	0.3848	0.3698	**0.3350**	0.1544
	T_3	**0.4529**	**0.4418**	**0.4322**	**0.4166**	**0.3911**	0.3347	0.1486
	T_P	0.1014	0.1195	0.1350	0.1402	0.1479	0.1560	**0.1982**
	KS	0.0878	0.0922	0.1091	0.1152	0.1211	0.1299	0.1318
	CM	0.0598	0.0951	0.1211	0.1275	0.1297	0.1315	0.1354
	AD	0.0378	0.0645	0.0861	0.0999	0.1008	0.1023	0.1059

(Continued)

TABLE 3 (Continued)

		$F(T) = 1 - \exp(-T/\theta)$						
Alt. model	Test statistic	0.4	0.5	0.6	0.7	0.8	0.9	0.99
Lomax (1, 4.0)	T_1	0.0070	0.0051	0.0067	0.0117	0.0299	0.0743	0.1590
	T_2	0.0001	0.0002	0.0001	0.0004	0.0022	0.0361	0.1706
	T_3	0.0001	0.0001	0.0001	0.0002	0.0021	0.0364	0.1781
	T_P	**0.5390**	**0.3998**	**0.2704**	**0.1647**	0.1073	0.0784	0.0727
	KS	0.0999	0.1073	0.1163	0.1275	0.1395	0.1495	0.1582
	CM	0.0931	0.1035	0.1153	0.1287	0.1423	0.1554	0.1658
	AD	0.1038	0.1131	0.1245	0.1391	**0.1545**	**0.1700**	**0.1881**

Bold values indicate the largest estimated power in each case.

$$T_1 = \max_{1\le i\le d}(v_i) + \max_{1\le i\le d}(-v_i), \quad T_2 = \sum_{i=1}^{d} \frac{v_i^2}{d}, \qquad T_3 = \sum_{i=1}^{d} \frac{|v_i|}{d}. \tag{4}$$

Large values of these statistics will tend to reject the null hypothesis of exponentiality. In Section 5, we use the Monte Carlo simulation to determine the upper tail of the simulated values of the statistics T_1, T_2, and T_3 as critical points for testing exponentiality.

3. Test based on quantiles

Note that the order statistics $U_{i:d}$ defined by Equation (3) follow beta distribution with parameters $(i, d-i+1)$. Define

$$p_i = F_B(u_{i:d}; i, d-i+1), \tag{5}$$

Where $F_B(x; \alpha, \beta) = \frac{B(x;\alpha,\beta)}{B(\alpha,\beta)}$ is the cumulative distribution function (CDF) and $B(\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt$ is the incomplete beta function. p_i is the CDF of the beta distribution with parameters $(i, d-i+1)$ evaluated at $u_{i:d}$.

The quantiles vector

$$P = (p_1, p_2, \ldots, p_d)',$$

can be used as a measure of goodness-of-fit. Extreme values of p_i, i.e. values close to zero or one are signs of "badness-of-fit"! It is noteworthy that, although p_i's are uniformly distributed over $(0, 1)$, they are not statistically independent.

We propose a test statistic in terms of a linear combination of p_i and $1 - p_i$ as follows:

$$T_P = -\sum_{i=1}^{d}\{w\log(p_{(i)}) + (1-w)\log(1-p_{(i)})\}, \tag{6}$$

Where $w = \frac{i-1}{d}$, for $i = 1, 2, \ldots, d$ and $p_{(i)}$'s are the ordered values of p_i arranged from smallest to largest. Note that the test statistic T_P will be calculated for values of $p_{(i)}$ in the interval $(0, 1)$, i.e. we exclude the cases with $p_{(i)} = 0$ or $p_{(i)} = 1$. Note also that whilst $u_{i:d}$'s are ordered in terms of their values, the p_i's are not necessarily ordered. Moreover, the test statistic T_P, will be large whenever one of p_i's are close to zero or one. Hence, large values of T_P provide evidence that the null hypothesis H_0 of exponentiality should be rejected.

4. Test based on binomial distribution

Let the discrete random variable D denote the number of observed failures before the termination time T. Then, under the validity of the null hypothesis, under the exponentiality of the model, D follows a binomial distribution with parameters (n, p), where $p = F(T) = 1 - \exp(-T/\theta)$.

Hence, testing the null hypothesis of exponentiality (1), is equivalent to performing a binomial test say

$$H_0: p = F(T), \quad \text{versus} \quad H_1: p \neq F(T). \tag{7}$$

Note that if we assume that the null hypothesis is true, i.e. under the validity of the exponential model, we expect to observe $nF(T)$ failures. The usual binomial test may then be used to find the associated p-value.

For large values of sample size n, the binomial distribution is well approximated by the Gaussian model in which a z-test is performed to the test statistic Z, using continuity correction given by

$$T_B = \frac{d - nF(T) \pm 0.5}{\sqrt{nF(T)(1-F(T))}}. \tag{8}$$

However, using the Monte Carlo simulation we found that the test statistic T_B does not maintain the nominated significance level for small sample sizes even for sample sizes $n \le 40$, so we did not include the power of T_B in our simulation study.

In the following section, we perform a Monte Carlo simulation to assess the power of the proposed tests for various alternative models, and for a combination of various sample sizes n and censoring proportion $1 - F(T) = \exp(-T/\theta)$.

5. Simulation study

In this section, the performance of our proposed tests will be evaluated by studying the empirical significance level and the empirical power through extensive Monte Carlo simulations. We used the R pseudo-random generator with 50,000 iterations.

First, we investigate the null distribution of the test statistics presented in the previous section using the Monte Carlo estimate of the coefficient of skewness ($\sqrt{\beta_1}$) and the coefficient of kurtosis (β_2) when the underlying distribution is standard exponential. The results are shown in Table 1. The coefficient of skewness ($\sqrt{\beta_1}$) and the coefficient of kurtosis (β_2) are defined as:

TABLE 4 Empirical power for various alternative models, $n = 20$, $\alpha = 0.10$ and 10^5 iterations.

		$F(T) = 1 - \exp(-T/\theta)$						
Alt. model	Test statistic	0.4	0.5	0.6	0.7	0.8	0.9	0.99
Gamma (0.5, 1.0)	T_1	0.0182	0.0268	0.0494	0.1055	0.2309	0.4381	0.6051
	T_2	0.0006	0.0006	0.0004	0.0003	0.0272	0.2949	0.6679
	T_3	0.0001	0.0002	0.0006	0.0004	0.0275	0.2874	0.6760
	T_P	0.6579	0.5661	0.4720	0.3892	0.3055	0.2363	0.1861
	KS	0.4719	0.4969	0.5208	0.5401	0.5576	0.5709	0.5810
	CM	0.4953	0.5280	0.5566	0.5794	0.5995	0.6164	0.6266
	AD	**0.6914**	**0.7111**	**0.7270**	**0.7421**	**0.7552**	**0.7680**	**0.7769**
Gamma (2.0, 1.0)	T_1	0.0356	0.0965	0.1981	0.3072	0.3605	0.2873	0.1951
	T_2	0.6529	0.8402	**0.9382**	**0.9795**	**0.9915**	0.9935	0.8940
	T_3	**0.6665**	**0.8433**	0.9361	0.9786	0.9904	**0.9936**	**0.9105**
	T_P	0.1004	0.0320	0.0282	0.0630	0.1487	0.3430	0.7356
	KS	0.0497	0.1052	0.1788	0.2587	0.3443	0.4377	0.5446
	CM	0.0241	0.0693	0.1465	0.2450	0.3573	0.4805	0.6217
	AD	0.0159	0.0507	0.1171	0.2093	0.3208	0.4511	0.6081
Weibull (0.5, 1.0)	T_1	0.1243	0.1889	0.3036	0.4549	0.6325	0.7829	0.8686
	T_2	0.0022	0.0037	0.0055	0.0153	0.1068	0.6130	0.9267
	T_3	0.0015	0.0020	0.0026	0.0077	0.1024	0.6148	0.9325
	T_P	0.5391	0.5188	0.5017	0.4984	0.5127	0.5502	0.6573
	KS	0.5517	0.6032	0.6489	0.6919	0.7363	0.7868	0.8607
	CM	0.5672	0.6247	0.6772	0.7243	0.7714	0.8223	0.8926
	AD	**0.7700**	**0.8068**	**0.8383**	**0.8657**	**0.8907**	**0.9169**	**0.9516**
Weibull (2.0, 1.0)	T_1	0.0519	0.0555	0.0338	0.0150	0.0747	0.4667	0.8202
	T_2	0.7815	0.8200	0.7588	0.6440	0.6007	0.8118	0.9483
	T_3	**0.8037**	**0.8603**	**0.8401**	0.7603	0.6606	0.7770	0.9449
	T_P	0.1333	0.3359	0.6336	**0.8765**	**0.9710**	**0.9899**	**0.9900**
	KS	0.2490	0.4472	0.6458	0.8029	0.8966	0.9106	0.9257
	CM	0.2103	0.4486	0.6851	0.8538	0.9423	0.9620	0.9711
	AD	0.1719	0.4019	0.6487	0.8345	0.9328	0.9702	0.9662
Log-normal (0, 0.5)	T_1	0.0044	0.0019	0.0019	0.0022	0.0834	0.5793	0.9512
	T_2	0.9313	0.9958	0.9992	0.9965	0.9942	0.9978	0.9979
	T_3	**0.9448**	**0.9972**	**0.9998**	**0.9991**	**0.9967**	0.9968	0.9947
	T_P	0.0524	0.2508	0.6612	0.9224	0.9884	**0.9992**	**0.9999**
	KS	0.1964	0.6236	0.8898	0.9691	0.9902	0.9954	0.9963
	CM	0.1181	0.5572	0.8886	0.9762	0.9942	0.9978	0.9986
	AD	0.0844	0.5029	0.8771	0.9771	0.9956	0.9985	0.9991
Log-normal (0, 1.0)	T_1	0.0275	0.0468	0.0650	0.0755	0.0826	0.0858	0.1259
	T_2	0.6370	0.6378	0.6074	0.5851	0.5637	0.5291	0.2006
	T_3	**0.6861**	**0.6900**	**0.6663**	**0.6440**	**0.6086**	**0.5443**	0.2060
	T_P	0.1123	0.1518	0.1748	0.1960	0.2092	0.2297	**0.3296**
	KS	0.1059	0.1216	0.1487	0.1524	0.1682	0.1719	0.2016
	CM	0.1351	0.1477	0.1586	0.1782	0.1943	0.2059	0.2206
	AD	0.1251	0.1337	0.1408	0.1757	0.1845	0.1958	0.2147

(Continued)

TABLE 4 (Continued)

		$F(T) = 1 - \exp(-T/\theta)$						
Alt. model	Test statistic	0.4	0.5	0.6	0.7	0.8	0.9	0.99
Lomax (1, 4.0)	T_1	0.0004	0.0001	0.0005	0.0026	0.0156	0.0749	0.1865
	T_2	0.0001	0.0001	0.0001	0.0001	0.0006	0.0425	0.2331
	T_3	0.0002	0.0001	0.0001	0.0001	0.0006	0.0397	0.2393
	T_P	**0.7237**	**0.5520**	**0.3769**	**0.2429**	0.1547	0.1097	0.0889
	KS	0.1122	0.1237	0.1374	0.1553	0.1754	0.1960	0.2217
	CM	0.1090	0.1242	0.1424	0.1660	0.1901	0.2161	0.2421
	AD	0.1178	0.1318	0.1498	0.1730	**0.1993**	**0.2280**	**0.2481**

Bold values indicate the largest estimated power in each case.

$$\sqrt{\beta_1} = \frac{E[T - E(T)]^3}{[Var(T)]^{3/2}}, \tag{9}$$

and

$$\beta_2 = \frac{E[T - E(T)]^4}{[Var(T)]^2}. \tag{10}$$

From Table 1, it is clear that the null distribution of all the test statistics are far from normality, as $\sqrt{\beta_1}$ and β_2 are not close to 0 and 3 respectively, which are the coefficients of skewness and kurtosis of normal distribution. This is also evident from Figure 1, which depicts the simulated pdf curves for the test statistics under the validity of the null hypothesis. Indeed, it has been observed that all the test statistics are skewed to the right. Hence, we use empirical critical values to perform goodness-of-fit tests.

We compare the empirical power of the proposed tests to those of the EDF-based test statistics proposed by Pettitt and Stephens [24] and Stephens [8].

Stephens [8] studied the modification of the Kolmogorov-Smirnov statistic for the Type-I censored data from an exponential model in the form of:

$${}_1D_{T:n} = \max_{1\le i\le d}\left\{\frac{i}{n} - u_{(i)}, u_{(i)} - \frac{i-1}{n}, u_{(d+1)} - \frac{d}{n}\right\}, \tag{11}$$

Where $u_{(i)} = 1 - \exp(-x_{i:n}/\hat{\theta})$ and $u_{(d+1)} = 1 - \exp(-T/\hat{\theta})$ with $\hat{\theta}$ being the MLE of the scale parameter θ given by Equation (2).

Pettitt and Stephens [24] also studied the Cramér-von Mises statistic ${}_1W^2_{T:n}$ and the Anderson-Darling statistic ${}_1A^2_{T:n}$ under Type-I censoring in the form of:

$${}_1W^2_{T:n} = \sum_{i=1}^{d+1}\left(u_{(i)} - \frac{2i-1}{2n}\right)^2 + \frac{d+1}{12n^2} + \frac{n}{3}\left(u_{(d+1)} - \frac{d+1}{n}\right)^3, \tag{12}$$

and

$$\begin{aligned}{}_1A^2_{T:n} = & -\frac{1}{n}\sum_{i=1}^{d+1}(2i-1)\{\log u_{(i)} - \log(1-u_{(i)})\} \\ & - 2\sum_{i=1}^{d+1}\log(1-u_{(i)}) \\ & - \frac{1}{n}\{(d-n+1)^2\log(1-u_{(d+1)}) - (d+1)^2\log u_{(d+1)} \\ & + n^2u_{(d+1)}\}.\end{aligned} \tag{13}$$

We considered seven alternative models in three groups $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ based on their behavior of hazard functions as follows:

$\mathcal{G}_1$: **Group I alternative (Monotonic decreasing hazard rates):**

1. Gamma distribution with shape parameter $\alpha = 0.5$ and scale parameter $\beta = 1.0$, denoted by Gamma (0.5, 1.0).
2. Weibull distribution with shape parameter $a = 0.5$ and scale parameter $b = 1.0$, denoted by Weibull (0.5, 1.0).
3. Lomax distribution with shape parameter $d = 4.0$ and scale parameter $c = 1.0$, denoted by Lomax (1.0, 4.0).

$\mathcal{G}_2$: **Group II alternative (Monotonic increasing hazard rates):**

1. Gamma distribution with shape parameter $\alpha = 2.0$ and scale parameter $\beta = 1.0$, denoted by Gamma (2.0, 1.0).
2. Weibull distribution with shape parameter $a = 2.0$ and scale parameter $b = 1.0$, denoted by Weibull (2.0, 1.0).

$\mathcal{G}_3$: **Group III alternative (Non-monotonic hazard rates):**

1. Log-normal distribution with location parameter $\mu = 0$ and scale parameter $\sigma = 0.5$, denoted by Log-normal (0, 0.5).
2. Log-normal distribution with location parameter $\mu = 0$ and scale parameter $\sigma = 1.0$, denoted by Log-normal (0, 1.0).

The following forms of probability density functions were used here.

The gamma distribution with density function

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left\{-\frac{x}{\beta}\right\},\quad x>0,$$

Where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter.

The Weibull distribution with density function

$$f(x;a,b) = \frac{a}{b}\left(\frac{x}{b}\right)^{a-1}e^{-(\frac{x}{b})^a},\quad x>0,$$

Where $a > 0$ and $b > 0$ are the shape and scale parameters, respectively.

TABLE 5 Empirical power for various alternative models, $n = 30$, $\alpha = 0.10$ and 10^5 iterations.

		$F(T) = 1 - \exp(-T/\theta)$						
Alt. model	Test statistic	0.4	0.5	0.6	0.7	0.8	0.9	0.99
Gamma (0.5, 1.0)	T_1	0.0048	0.0109	0.0327	0.0962	0.2567	0.5326	0.7399
	T_2	0.0001	0.0001	0.0002	0.0002	0.0238	0.3757	0.8141
	T_3	0.0001	0.0002	0.0002	0.0006	0.0235	0.3597	0.8173
	T_P	0.7943	0.7199	0.6393	0.5516	0.4584	0.3630	0.3011
	KS	0.6189	0.6510	0.6755	0.6984	0.7179	0.7335	0.7442
	CM	0.6552	0.6889	0.7169	0.7405	0.7602	0.7773	0.7875
	AD	**0.8231**	**0.8421**	**0.8564**	**0.8688**	**0.8790**	**0.8889**	**0.8956**
Gamma (2.0, 1.0)	T_1	0.0679	0.1829	0.3564	0.5111	0.5527	0.4292	0.2861
	T_2	0.8100	0.9397	**0.9877**	**0.9978**	**0.9995**	**0.9997**	0.9749
	T_3	**0.8174**	**0.9403**	0.9865	0.9975	0.9994	0.9997	**0.9809**
	T_P	0.0465	0.0157	0.0272	0.0768	0.2016	0.4617	0.8659
	KS	0.0705	0.1449	0.2520	0.3591	0.4632	0.5754	0.7054
	CM	0.0392	0.1060	0.2289	0.3642	0.4970	0.6326	0.7788
	AD	0.0304	0.0916	0.2076	0.3432	0.4860	0.6337	0.7861
Weibull (0.5, 1.0)	T_1	0.1063	0.2064	0.3789	0.6082	0.8000	0.9226	0.9644
	T_2	0.0006	0.0011	0.0027	0.0093	0.1105	0.7425	0.9816
	T_3	0.0003	0.0005	0.0009	0.0030	0.1057	0.7379	0.9832
	T_P	0.7134	0.6959	0.6936	0.7015	0.7182	0.7574	0.8521
	KS	0.7110	0.7616	0.8054	0.8434	0.8782	0.9141	0.9576
	CM	0.7397	0.7937	0.8378	0.8748	0.9074	0.9378	0.9724
	AD	**0.8853**	**0.9133**	**0.9343**	**0.9518**	**0.9653**	**0.9778**	**0.9906**
Weibull (2.0, 1.0)	T_1	0.0748	0.0734	0.0351	0.0155	0.0991	0.6881	0.9623
	T_2	0.9047	0.9309	0.8946	0.8136	0.7968	0.9501	0.9957
	T_3	**0.9191**	**0.9567**	**0.9482**	0.9044	0.8559	0.9368	0.9958
	T_P	0.1608	0.4555	0.7979	**0.9618**	**0.9962**	**0.9996**	**0.9996**
	KS	0.3515	0.6020	0.8006	0.9230	0.9747	0.9884	0.9982
	CM	0.3275	0.6297	0.8462	0.9548	0.9908	0.9977	0.9985
	AD	0.3016	0.6056	0.8362	0.9529	0.9900	0.9978	0.9989
Log-normal (0, 0.5)	T_1	0.0027	0.0020	0.0021	0.0012	0.1044	0.7779	0.9974
	T_2	0.9818	0.9997	**1.0000**	0.9999	0.9999	**1.0000**	**1.0000**
	T_3	**0.9871**	**0.9998**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
	T_P	0.0453	0.3513	0.8408	0.9869	0.9996	**1.0000**	**1.0000**
	KS	0.3141	0.8019	0.9784	0.9979	0.9998	0.9999	**1.0000**
	CM	0.2203	0.7550	0.9811	0.9986	0.9999	**1.0000**	**1.0000**
	AD	0.1789	0.7453	0.9830	0.9992	0.9999	**1.0000**	**1.0000**
Log-normal (0, 1.0)	T_1	0.0376	0.0642	0.0848	0.1039	0.1037	0.0934	0.1576
	T_2	0.7867	0.7779	0.7559	0.7290	0.7092	0.6692	0.2460
	T_3	**0.8318**	**0.8333**	**0.8160**	**0.7918**	**0.7637**	**0.6916**	0.2606
	T_P	0.1375	0.1923	0.2390	0.2657	0.2814	0.3159	**0.4363**
	KS	0.1175	0.1204	0.1403	0.1626	0.1711	0.1795	0.2113
	CM	0.1393	0.1424	0.1501	0.1646	0.1840	0.1985	0.2214
	AD	0.1492	0.1428	0.1595	0.1778	0.1921	0.2044	0.2379

(Continued)

TABLE 5 (Continued)

		$F(T) = 1 - \exp(-T/\theta)$						
Alt. model	Test statistic	0.4	0.5	0.6	0.7	0.8	0.9	0.99
Lomax (1, 4.0)	T_1	0.0001	0.0001	0.0002	0.0007	0.0090	0.0728	0.2148
	T_2	0.0001	0.0001	0.0001	0.0001	0.0032	0.0401	0.2905
	T_3	0.0001	0.0001	0.0001	0.0001	0.0002	0.0361	0.2948
	T_P	**0.7973**	**0.6338**	**0.4552**	**0.2943**	0.1863	0.1256	0.1059
	KS	0.1212	0.1391	0.1633	0.1887	0.2167	0.2443	0.2718
	CM	0.1216	0.1442	0.1721	0.2046	0.2385	0.2711	0.2970
	AD	0.1293	0.1494	0.1764	0.2097	**0.2456**	**0.2821**	**0.3185**

Bold values indicate the largest estimated power in each case.

The log-normal distribution with density function

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma\, x}\exp\left\{-\frac{(\log x-\mu)^2}{2\sigma^2}\right\},\quad x>0,$$

Where $-\infty < \mu < \infty$ is the mean and $\sigma > 0$ is the standard deviation of the transformed normal distribution.

Finally, the Lomax distribution (also known as Pareto Type II), with probability density function

$$f(x;c,d) = \frac{d}{c\,(1+x/c)^{1+d}},\quad x>0,$$

With the scale parameter $c > 0$ and the shape parameter $d > 0$.

The plot of CDFs of the alternative distributions in groups $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are depicted in Figure 2.

For a comprehensive discussion of these distributions, one may refer to Johnson et al. [25, 26] and Kleiber and Kotz [27].

Verifying the empirical significance level is of great importance for the validity of any goodness-of-fit test statistic. To assess the validity of our tests we investigate the empirical significance level by generating 100,000 Type-I censored random data from the exponential distribution with a rate equal to one (standard exponential). We considered a combination of various sample sizes n and proportions (probability) of failures $F(T) = 1 - \exp(-T)$. The empirical significance levels at nominated level $\alpha = 0.10$ are tabulated in Table 2. The values in this table confirm the validity of our proposed tests in terms of preserving the nominated significance level.

The power of the proposed tests together with the powers associated with the classical EDF-based tests are recorded in Tables 3–5 for sample sizes $n = 10$, $n = 20$, and $n = 30$, respectively for the three alternative groups $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$. Figures 4–6 depict the corresponding heatmaps to provide better visualization of the results. The greyscale is given in Figure 7.

The test statistics T_3 and T_P outperformed the classical EDF-based statistics for groups $\mathcal{G}_2$ and $\mathcal{G}_3$, respectively for the monotonic increase and non-monotonic hazard function alternatives for all sample sizes considered here. The test statistic T_2 also had the best power in some cases in groups $\mathcal{G}_2$ and $\mathcal{G}_3$. However, in the group $\mathcal{G}_1$ alternative for monotonic decreasing hazard functions, the EDF-based test statistic AD performed better than the other tests. In Table 5, for log-normal (0, 0.5) alternative and $n = 30$, the empirical powers are equal to 1.00 for most tests when the censoring proportion $F(T)$ is at least 60%. This shows the consistency of the test statistics considered here. Moreover, as one would expect the empirical power values of all the tests considered here increase when the sample size n increases and/or when the censoring proportion $F(T)$ increases.

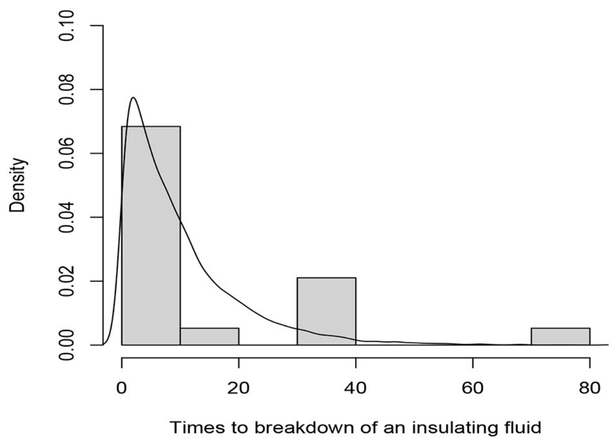

FIGURE 3
Histogram of the complete data and fitted exponential pdf curve of the data in Table 6.

In summary, for the monotonically increasing and non-monotonic hazard rate alternatives, we recommend using the test statistics T_3 and T_P. For the Lomax model alternative, we recommend T_P for a small amount of censoring proportion and the AD statistic for large values of $F(T)$.

6. Numerical example

In this section, we study a numerical example to illustrate our proposed procedure and test statistics. The data concerning the times to breakdown of an insulating fluid tested at 34 kilovolts for $n = 19$ insulating fluids (see Nelson [28], Table 1.1, page 105).

Suppose we decided to terminate the experiment at time $T = 15$ so any data larger than 15 is censored. The complete and the Type-I censored data are summarized in Table 6.

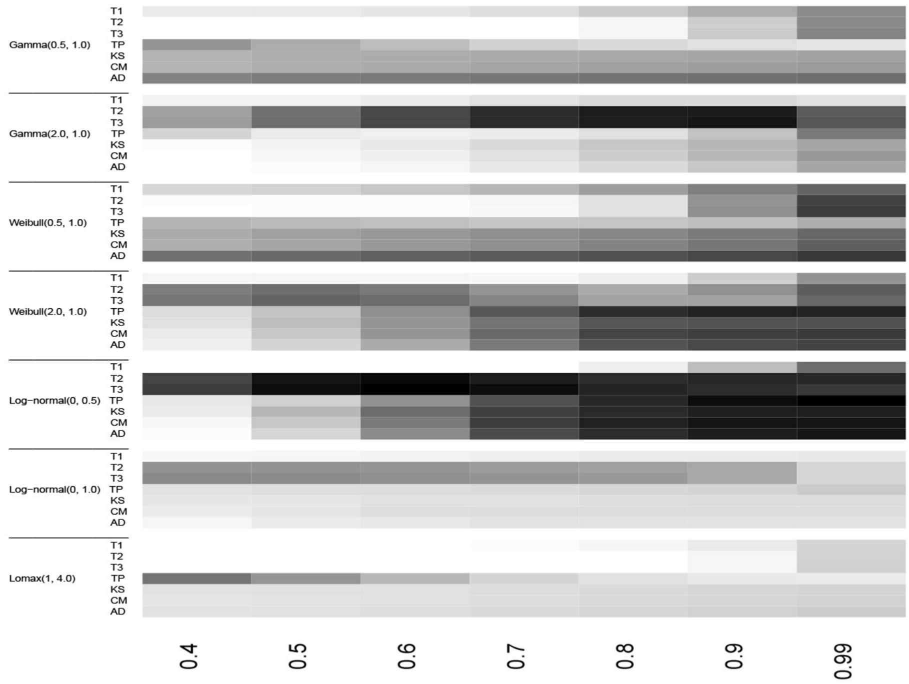

FIGURE 4
Heatmap of the empirical powers listed in Table 3 for $n = 10$, $\alpha = 0.10$ and 10^5 iterations.

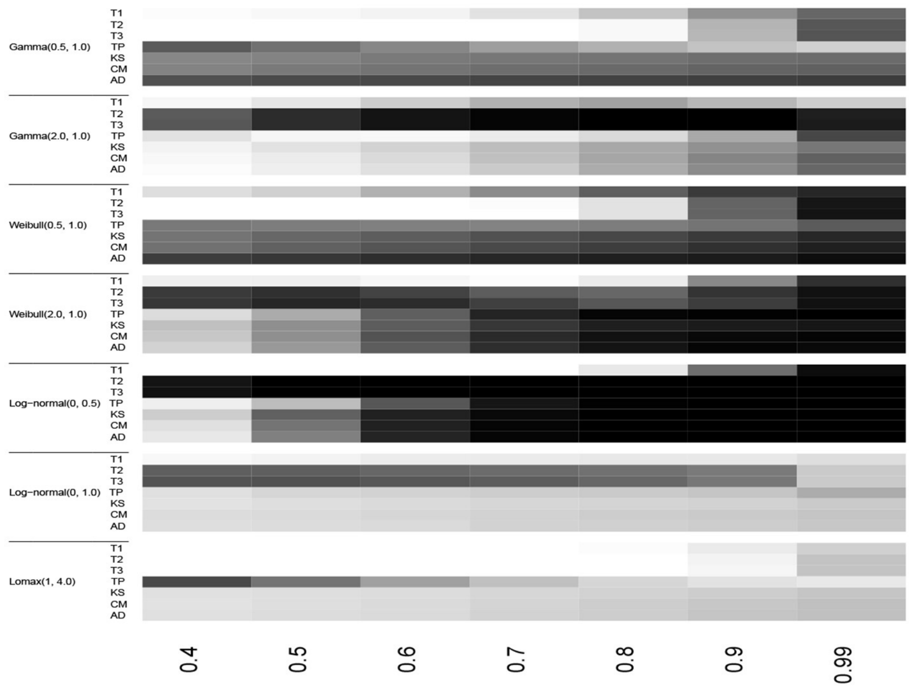

FIGURE 5
Heatmap of the empirical powers listed in Table 4 for $n = 20$, $\alpha = 0.10$ and 10^5 iterations.

The value of d is found to be $d = 14$ and using Equation (2), the MLE of θ is $\hat{\theta} = 10$. Hence, $F(T) = F(15) = 1-\exp(-15/10) = 0.78$.

The values of the test statistics and the associated p-values are given in Table 7. The p-values are sufficiently large for all test statistics and thereby the null hypothesis of exponentiality is not significant and the exponential model fits the data. The histogram of the complete data and the fitted exponential pdf curve with scale parameter $\theta = 10$ are depicted in Figure 3.

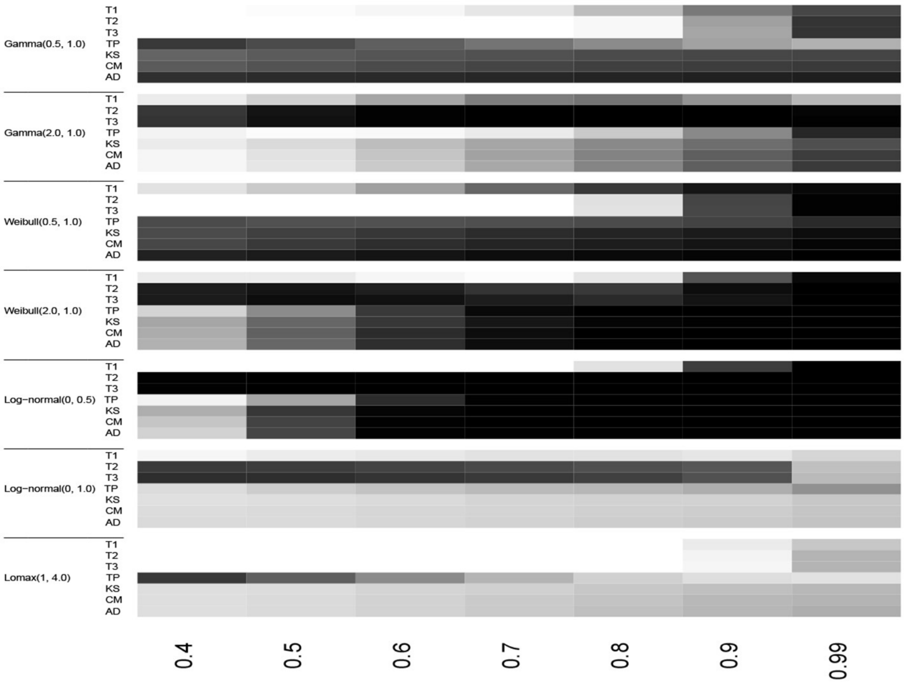

FIGURE 6
Heatmap of the empirical powers listed in Table 5 for $n = 30$, $\alpha = 0.10$ and 10^5 iterations.

TABLE 6 Insulating fluid tested at 34 kilovolts data.

Complete data	0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	6.50
	7.35	8.01	8.27	12.06	31.75	32.52	33.91	36.71	72.89	
Type-I censored data	0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	6.50
	7.35	8.01	8.27	12.06						

TABLE 7 Test values of test statistics and their corresponding p-values for the data in Table 6 when testing for the exponential model.

Criterion	T_1	T_2	T_3	T_P
Test statistic	0.2438	0.0080	0.0748	9.0941
p-value	0.6224	0.8257	0.8110	0.6066

FIGURE 7
The greyscale used in Figures 4–6.

7. Concluding remarks

In this paper, we proposed some new goodness-of-fit tests for exponentiality when the available data are Type-I censored. We employed two methods for this purpose: the first was based on the distance between the observed order statistics and its theoretical mean under the assumption of exponentiality.

The second method was based on the values of quantiles of uniform order statistics, which are known to follow the beta distribution, as is the fact that under the assumption of the null hypothesis, most of the quantiles p_i's should be close to 0.5. We proposed test statistics based on the weighted mean of the logarithm of p_i.

Among the four test statistics presented in this article, the test statistic T_3, based on order statistics, exhibits the most powerful test followed by the test statistic T_4, which is based on quantiles.

The large sample properties of the proposed estimators will be examined in a separate future study through Monte Carlo simulation.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

Sections 1–4 were prepared by RP. Sections 5, 6 were prepared by OA-H (60%) and RP (40%). All authors contributed to the article and approved the submitted version.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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