On the Dynamics of Sexually Transmitted Diseases Under Awareness and Treatment

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In this paper, we develop and extend the work of Jia and Qin on sexually transmitted disease models with a novel class of non-linear incidence. Awareness plays a central role both in the susceptible and the infectious classes. The Existence, uniqueness, boundedness, and positivity of solutions are systematically established. Concavity arguments and the occurrence of a vertical asymptote are essential in the proof of the existence of a unique endemic equilibrium. Conditions for the stability of all steady states are investigated. In particular, numerical simulations are performed in order to capture the asymptotic behavior of solutions.

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1. INTRODUCTION

Disease incidence plays a crucial role in mathematical epidemiology and it is essential in the computation of the basic reproduction number. Non-linear incidences are known to induce complex or chaotic behavior as oppose to standard incidences frequently used in classical infectious disease models [1–6]. A class of non-linear incidences particularly useful in the modeling of sexually transmitted diseases was introduced in [7] by the authors in the modeling of HIV/AIDS epidemic. The model considered however was not properly conceptualized as the density of individuals with full-blown AIDS not receiving ARV treatment did not bear any influence on the infection rate of the disease. In addition a number of inaccuracies are displayed in this paper like the unknown variable T missing in the third equation of system (2.1) and also a mistake occurred in the computation of the sign of a_3 in the proof of the stability of the endemic equilibrium. The purpose of this paper is to develop and extend the work on [7] by deriving a realistic model for sexually transmitted diseases with a proper non-linear incidence rate with a valid biological significance and perform a full analysis of the resulting model. In Section 2, the model is derived and presented. Well-posedness analysis, positivity and boundedness are considered in Section 3 followed by stability analysis of the critical points of the system in Section 4, numerical simulations in Section 5 and the conclusion.

2. THE MODEL

In this paper, a model with five compartments is formulated with non-linear incidence $Sg(t, I)$ incorporated into it. The incidence is presumed to be a time dependent non-linear response to the

size of the infectious population.

The compartments are denoted by $S(t), I(t), T(t), A(t)$ and $R(t)$ which represent the number of susceptible individuals, the number of infected individuals with the potential of transmitting the disease as they are not under treatment and do not take any form of protection while engaging in sexual activities, the number of individuals under treatment, the number of infectious individuals engaging in safe sex, and heathy individuals that engage in safe sex, respectively, at time t. The model represented in **Figure 1** is governed by the system of nonlinear ordinary differential equations

$$\begin{cases} \dfrac{dS}{dt} = \Gamma - Sg(t, I) - (\omega_1 + d)S, \\[2mm] \dfrac{dI}{dt} = Sg(t, I) + \nu_1 T - (d + \tau_1 + \tau_2)I, \\[2mm] \dfrac{dT}{dt} = \tau_2 I - \nu_2 T - (d + \nu_1)T, \\[2mm] \dfrac{dA}{dt} = \tau_1 I - dA + \nu_2 T, \\[2mm] \dfrac{dR}{dt} = \omega_1 S - dR, \end{cases} \quad (1)$$

endowed with initial conditions

$S(0) \equiv S_0 > 0,\ I(0) \equiv I_0 > 0,\ T(0) \equiv T_0 \geq 0,\ A(0) \equiv A_0 \geq 0,\ R(0) \equiv R_0 \geq 0$.

The parameters in the evolution system (1) are described in **Table 1**:

The total population $N(t)$ is given by $S(t) + I(t) + T(t) + A(t) + R(t)$. By adding all the equations of the system (1), we obtain the rate of change of $N(t)$, which is given by

$$\frac{dN}{dt} = \Gamma - dN \quad (2)$$

and $N(t)$ varies over time and is nearing a stable fixed point $\dfrac{\Gamma}{d}$ as $t \to \infty$. Therefore, the biologically feasible region for the system (1) is given by

$$\Psi = \left\{ (S, I, T, A, R) \in \mathbb{R}^5_+ \mid 0 < S(t) + I(t) + T(t) + A(t) + R(t) \leq \frac{\Gamma}{d} \right\}.$$

It is easy to see that the set Ψ is positively invariant. Next we present a systematic analysis of our evolution equation.

3. MATHEMATICAL ANALYSIS

We start by investigating the well-posedness of the model (1). Given the fact that the variables represent biologically densities, it is important to show that all the variables remain positive at all time.

Lemma 1. *For any non-negative initial conditions* $(S_0, I_0, T_0, A_0, R_0)$, *system (1) has a local solution which is unique.*

Proof: Let $x = (S, I, T, A, R)$, system (1) can be rewritten as $x'(t) = f(t, x(t))$, where $f : \mathbb{R}^6 \to \mathbb{R}^5$ is a C^1 vector field. By the classical differential equation theory, we can confirm that system (1) has a unique local solution defined in a maximum interval $[0, t_m)$. $\quad\square$

Lemma 2. *For any non-negative initial conditions* $(S_0, I_0, T_0, A_0, R_0)$, *the solution of (1) is non-negative and bounded for all* $t \in [0, t_m)$.

Proof: We start by showing positivity of the local solution for any non-negative initial conditions. It is easy to see that $S(t) \geq 0$ for all $t \in [0, t_m)$. Indeed, assume the contrary and let $t_1 > 0$ be the first time such that $S(t_1) = 0$ and $S'(t_1) < 0$. From the first equation of the system (1), we have $S'(t_1) = \Gamma > 0$, which presents a contradiction. Therefore $S(t) \geq 0$ for all $t \in [0, t_m)$. Using the same argument, positivity $I(t),\ T(t),\ A(t)$ and $R(t)$ in the interval $[0, t_m)$ are established. Furthermore from (2), we have that

$$N(t) = \frac{\Gamma}{d} + N(0)e^{-dt} \leq \frac{\Gamma}{d} + N(0). \quad (3)$$

Therefore the solution $N(t)$ is bounded in the interval $[0, t_m)$. $\quad\square$

Theorem 1 *For any non-negative initial conditions* $(S_0, I_0, T_0, A_0, R_0)$, *system (1) has a unique global solution. Moreover, this solution is non-negative and bounded for all* $t \geq 0$.

Proof: The solution does not blow up in a finite time as it is bounded, it is therefore defined at all time $t \geq 0$. Other properties of the solution follow from lemma (1) and lemma (2). $\quad\square$

Setting $m = d + \tau_1 + \tau_2$ and $n = d + \nu_1 + \nu_2$, system (1) transforms into a reduced system

$$\begin{cases} \dfrac{dS}{dt} = \Gamma - Sg(t, I) - (\omega_1 + d)S, \\[2mm] \dfrac{dI}{dt} = Sg(t, I) + \nu_1 T - mI, \\[2mm] \dfrac{dT}{dt} = \tau_2 I - nT, \\[2mm] \dfrac{dA}{dt} = \tau_1 I - dA + \nu_2 T, \\[2mm] \dfrac{dR}{dt} = \omega_1 S - dR. \end{cases} \quad (4)$$

4. MODELS WITH TIME INDEPENDENT NON-LINEAR RESPONSE

In this section, we assume that the non-linear response function is not time dependent, ie $g(t, I) \equiv g(I)$. Following [7], it is further
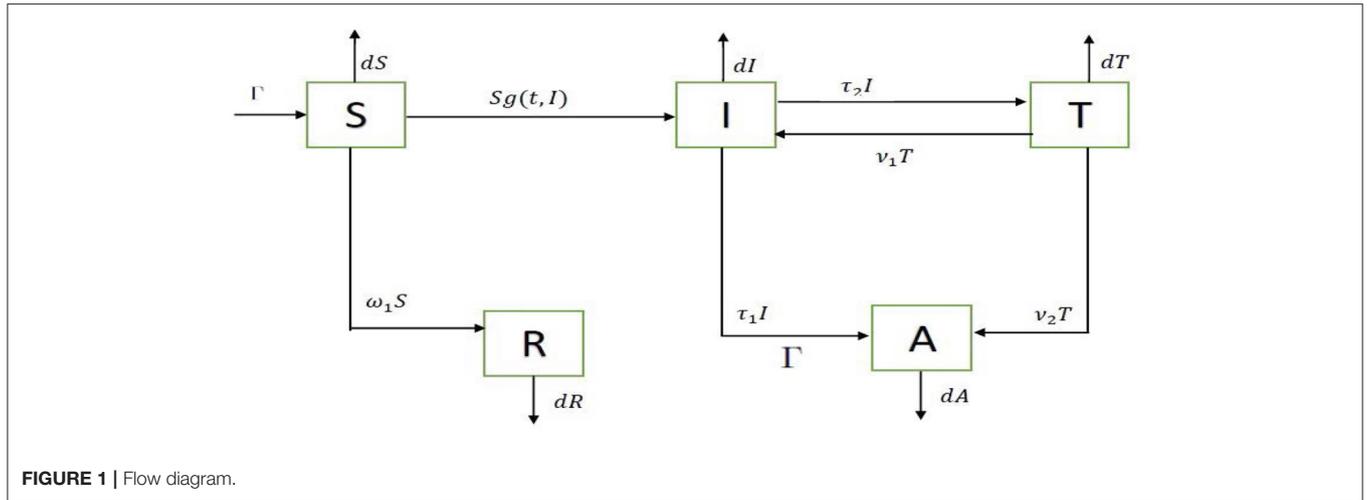
FIGURE 1 | Flow diagram.

TABLE 1 | Biological meaning of parameters.

Parameter	Biological meaning
Γ	Recruitment rate
d	Natural death rate
τ_1	The rate at which infected individuals adhere to safe sex practices
τ_2	The rate at which infected individuals receive antiviral drugs
ω_1	The rate at which susceptible individuals adhere to safe sex practices
ν_1	The rate of defaulting from treatment
ν_2	The rate at which treated individuals adhere to safe sex practices .

assumed that

$$(H_1) \; : \; g(0) = 0, \; g'(0) > 0, \; g''(I) \le 0 \; \text{ for } I \ge 0,$$

$$(H_2) \; : \; \lim_{I \to 0^+} \frac{g(I)}{I} = k, \; 0 < k < \infty.$$

4.1. The Basic Reproduction Number

In this section, we use the next generation method [8] to obtain the basic reproduction number. Let z be the transpose of (I, A, T, S, R). We rewrite the system (4) in the matrix form

$$\frac{dz}{dt} = \mathcal{F}(z) - \mathcal{V}(z),$$

where

$$\mathcal{F}(z) = \begin{pmatrix} Sg(I) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathcal{V}(z) = \begin{pmatrix} mI - \nu_1 T \\ dA - \tau_1 I - \nu_2 T \\ nT - \tau_2 I \\ Sg(I) + (\omega_1 + d)S - \Gamma \\ dR - \omega_1 S \end{pmatrix}.$$

The disease free equilibrium of system (4) takes the form

$$E^0 = (I^0, A^0, T^0, S^0, R^0) = \left(0, 0, 0, \frac{\Gamma}{\omega_1 + d}, \frac{\omega_1 \Gamma}{d(\omega_1 + d)} \right).$$

Following [8], we compute the basic reproduction number using the formula below

$$\mathcal{R}_0 = \rho(FV^{-1}),$$

where

$$F = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial E} & \frac{\partial \mathcal{F}_1}{\partial I} & \frac{\partial \mathcal{F}_1}{\partial T} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{(E^0, I^0, T^0, S^0, R^0)} = \begin{bmatrix} \frac{\alpha \Gamma}{\omega_1 + d} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} \frac{\partial \mathcal{V}_1}{\partial E} & \frac{\partial \mathcal{V}_1}{\partial I} & \frac{\partial \mathcal{V}_1}{\partial T} \\ \frac{\partial \mathcal{V}_2}{\partial E} & \frac{\partial \mathcal{V}_2}{\partial I} & \frac{\partial \mathcal{V}_2}{\partial T} \\ \frac{\partial \mathcal{V}_3}{\partial E} & \frac{\partial \mathcal{V}_3}{\partial I} & \frac{\partial \mathcal{V}_3}{\partial T} \end{bmatrix}_{(E^0, I^0, T^0, S^0, R^0)} = \begin{bmatrix} m & 0 & -\nu_1 \\ -\tau_1 & d & -\nu_2 \\ -\tau_2 & 0 & n \end{bmatrix}$$

and ρ is the spectral radius of the matrix FV^{-1}. Given the fact that

$$V^{-1} = \frac{1}{d(mn - v_1\tau_2)} \begin{bmatrix} nd & 0 & -v_1 d \\ \tau_1 n - v_2\tau_2 & mn - v_1\tau_2 & v_2 m - v_1\tau_1 \\ \tau_2 d & 0 & md \end{bmatrix},$$

it follows that

$$\mathcal{R}_0 = \frac{\alpha n\Gamma}{(\omega_1 + d)(mn - v_1\tau_2)}. \tag{5}$$

4.1.1. Stability of the Disease-Free Equilibrium

The stability of the disease-free equilibrium will be investigated in this subsection.

Theorem 2 *The disease free equilibrium E^0 is globally asymptotically stable if $0 < \mathcal{R}_0 < 1$, and unstable if $\mathcal{R}_0 > 1$.*

Proof: The Jacobian matrix (J_{E^0}), evaluated at E^0, is given by

$$J_{E^0} = \begin{pmatrix} -(\omega_1 + d) & \dfrac{-\alpha\Gamma}{\omega_1 + d} & 0 & 0 & 0 \\ 0 & \dfrac{\alpha\Gamma}{\omega_1 + d} - m & v_1 & 0 & 0 \\ 0 & \tau_2 & -n & 0 & 0 \\ 0 & \tau_1 & v_2 & -d & 0 \\ \omega_1 & 0 & 0 & 0 & -d \end{pmatrix}. \tag{6}$$

The characteristic equation that results from the Jacobian matrix (J_{E^0}) is given by $\det(J_{E^0} - \lambda I) = 0$. Thus, we get

$$(d+\lambda)^2 [(\omega_1+d)+\lambda] \left[\left(m - \frac{\alpha\Gamma}{\omega_1 + d} + \lambda \right)(n + \lambda) - v_1\tau_2 \right] = 0. \tag{7}$$

The characteristic equation (7) has three negative real roots, which are

$$\lambda_1 = -d,$$
$$\lambda_2 = \lambda_3 = -(\omega_1 + d),$$

and the other 2 roots, λ_4 and λ_5, are roots of the equation

$$f(\lambda) = \left[\left(m - \frac{\alpha\Gamma}{\omega_1 + d} \right) + \lambda \right](n+\lambda) - v_1\tau_2 \equiv \lambda^2 + a_1\lambda + a_2 = 0, \tag{8}$$

where

$$a_1 = m + n - \frac{\alpha\Gamma}{\omega_1 + d},$$
$$a_2 = mn - v_1\tau_2 - \frac{\alpha n\Gamma}{\omega_1 + d}.$$

We now need to consider the signs of λ_4 and λ_5. Note that

$$mn - v_1\tau_2 > 0.$$

Assuming

$$\mathcal{R}_0 = \frac{\alpha n\Gamma}{(\omega_1 + d)(mn - v_1\tau_2)} < 1,$$

we have that

$$mn - v_1\tau_2 - \frac{\alpha n\Gamma}{\omega_1 + d} = a_2 > 0.$$

Moreover

$$\frac{\alpha n\Gamma}{\omega_1 + d} < mn - v_1\tau_2 < mn.$$

It implies that

$$m - \frac{\alpha\Gamma}{\omega_1 + d} > 0.$$

It follows that $a_1 > 0$. As a result the roots λ_4 and λ_5 are strictly negative. We can conclude that all roots of (7) have negative real parts, therefore, the disease free equilibrium is locally asymptotically stable [8–10]. Furthermore assuming that

$$\mathcal{R}_0 = \frac{\alpha n\Gamma}{(\omega_1 + d)(mn - v_1\tau_2)} > 1,$$

we have that $a_2 < 0$, it follows that the characteristic equation $f(\lambda) = 0$ has a least a strictly positive root. Therefore, the disease free equilibrium E^0 is unstable.

4.2. Existence of an Endemic Equilibrium

In this subsection, we investigate the existence of an endemic equilibrium for the system (4).

Proposition 1. *The system of differential equations (4) admits a unique endemic equilibrium if and only if $\mathcal{R}_0 > 1$.*

Proof: Let $E^* = (S^*, I^*, T^*, A^*, R^*)$ be an equilibrium point. Then the components of E^* satisfy the following set of equations

$$\begin{cases} \Gamma - S^*g(I^*) - (\omega_1 + d)S^* = 0, \\ S^*g(I^*) + v_1 T^* - mI^* = 0, \\ \tau_2 I^* - nT^* = 0, \\ \tau_1 I^* - dA^* + v_2 T^* = 0, \\ \omega_1 S^* - dR^* = 0. \end{cases} \tag{9}$$

From the last three equations of the system (9), we have that

$$T^* = \frac{\tau_2 I^*}{n},$$

$$A^* = \frac{\tau_1 + \dfrac{v_2\tau_2}{n}}{d} I^*,$$

$$R^* = \frac{\omega_1 S^*}{d}.$$

Substituting T^*, A^* and R^* into the first two equations, we obtain

$$\Gamma - S^*g(I^*) - (\omega_1 + d)S^* = 0,$$

$$S^*g(I^*) - \left(\frac{\nu_1\tau_2}{n} - m\right)I^* = 0.$$

It follows that

$$S^* = \frac{\left(m - \frac{\nu_1\tau_2}{n}\right)I^*}{g(I^*)} \qquad (10)$$

and

$$\Gamma - \left(m - \frac{\nu_1\tau_2}{n}\right)I^* - (\omega_1 + d)\frac{\left(m - \frac{\nu_1\tau_2}{n}\right)I^*}{g(I^*)} = 0. \qquad (11)$$

Next we set

$$h(I) := \frac{(\omega_1 + d)\left(m - \frac{\nu_1\tau_2}{n}\right)I}{\Gamma - \left(m - \frac{\nu_1\tau_2}{n}\right)I}. \qquad (12)$$

It is enough to show that there exists a point $I^* \in \mathbb{R}^+$ such that $h(I^*) = g(I^*)$. In other words, we will show that the curves of the functions h and g intersect at a point I^*.

Note that

$$I = \frac{n\Gamma}{mn - \nu_1\tau_2}$$

is a vertical asymptote of the function $h(I)$. For all

$$0 < I < \frac{n\Gamma}{mn - \nu_1\tau_2},$$

we have that

$$h'(I) =$$

$$\frac{(\omega_1 + d)\left(m - \frac{\nu_1\tau_2}{n}\right)\left[\Gamma - \left(m - \frac{\nu_1\tau_2}{n}\right)I\right] + (\omega_1 + d)\left(m - \frac{\nu_1\tau_2}{n}\right)^2 I}{\left[\Gamma - \left(m - \frac{\nu_1\tau_2}{n}\right)I\right]^2}$$

$$= \frac{\Gamma(\omega_1 + d)\left(m - \frac{\nu_1\tau_2}{n}\right)}{\left[\Gamma - \left(m - \frac{\nu_1\tau_2}{n}\right)I\right]^2} > 0$$

and

$$h''(I) = \frac{2\left[\Gamma - \left(m - \frac{\nu_1\tau_2}{n}\right)I\right]\left(m - \frac{\nu_1\tau_2}{n}\right)^2\Gamma(\omega_1 + d)}{\left[\Gamma - \left(m - \frac{\nu_1\tau_2}{n}\right)I\right]^4}$$

$$= \frac{2\Gamma(\omega_1 + d)\left(m - \frac{\nu_1\tau_2}{n}\right)^2}{\left[\Gamma - \left(m - \frac{\nu_1\tau_2}{n}\right)I\right]^3} > 0.$$

It follows that the function h is increasing and concave upward in the interval

$$\left[0, \frac{n\Gamma}{mn - \nu_1\tau_2}\right)$$

with a vertical asymptote at the right end of the interval. Note that the function g is increasing and concave downward in the closed interval

$$\left[0, \frac{n\Gamma}{mn - \nu_1\tau_2}\right].$$

As a result if

$$g'(0) > h'(0) = \frac{(\omega_1 + d)\left(m - \frac{\nu_1\tau_2}{n}\right)}{\Gamma},$$

which is equivalent to the condition $\mathcal{R}_0 > 1$, then Equation (12) has a unique root I^* in the interval

$$\left(0, \frac{n\Gamma}{mn - \nu_1\tau_2}\right).$$

Furthermore if

$$I > \frac{n\Gamma}{mn - \nu_1\tau_2},$$

then $h(I) < 0$. There is no intersection point with $g(I)$ since g is a positive function. Therefore there exists a unique endemic equilibrium point $E^* = (S^*, I^*, T^*, A^*, R^*)$ provided that $\mathcal{R}_0 > 1$. In addition if

$$g'(0) \geq h'(0) = \frac{(\omega_1 + d)\left(m - \frac{\nu_1\tau_2}{n}\right)}{\Gamma},$$

equivalent to the condition $\mathcal{R}_0 \leq 1$, there is no endemic equilibrium for the system (4).

4.2.1. Stability of the Endemic Equilibrium

Lemma 3. *Let $g(I)$ be a positive smooth function defined on the interval $[0, \infty)$. Suppose that assumptions H_1 and H_2 hold, then the following inequality is satisfied*

$$1 - \frac{Ig'(I)}{g(I)} \geq 0 \text{ for any } I > 0. \qquad (13)$$

Proof:
We have that

$$\frac{d[g(I) - Ig'(I)]}{dI} = -Ig''(I) \geq 0$$

as $g''(I) \leq 0$. This implies that the function $g(I) - Ig'(I)$ is increasing on the interval $[0, \infty)$. Given the fact that $g(0) - 0g'(0) = 0$, it follows that $g(I) - Ig'(I) \geq 0$. $\qquad \square$

Theorem 3 *If $\mathcal{R}_0 > 1$, then the endemic equilibrium E^* is locally asymptotically stable.*

Proof : The Jacobian matrix of the endemic equilibrium is given by

$$
J_{E^*} = \begin{pmatrix}
-g(I^*) - (\omega_1 + d) & S^*g'(I^*) & 0 & 0 & 0 \\
g(I^*) & S^*g'(I^*) - m & \nu_1 & 0 & 0 \\
0 & \tau_2 & -n & 0 & 0 \\
0 & \tau_1 & \nu_2 & -d & 0 \\
\omega_1 & 0 & 0 & 0 & -d
\end{pmatrix}. \quad (14)
$$

The characteristic equation that results from the Jacobian matrix (J_{E^*}) is given by $\det(J_{E^*} - \lambda I) = 0$. Thus, we get

$$(d + \lambda)^2(\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3) = 0, \quad (15)$$

where

$b_1 = g(I^*) + \omega_1 + d + m + n - S^*g'(I^*),$
$b_2 = (m + n)[g(I^*) + \omega_1 + d] - S^*g'(I^*)(\omega_1 + d + n) + mn - \nu_1\tau_2,$
$b_3 = (mn - \nu_1\tau_2)[g(I^*) + \omega_1 + d] - n(\omega_1 + d)S^*g'(I^*).$

The characteristic Equation (15) has a negative real double root

$$\lambda_1 = \lambda_2 = -d,$$

and three other roots, λ_3, λ_4 and λ_5, which are the roots of the equation

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0. \quad (16)$$

From Lemma (3), we have that

$$1 - \frac{I^*}{g(I^*)}g'(I^*) \geq 0. \quad (17)$$

It follows that

$$
\begin{aligned}
m - S^*g'(I^*) &= m - \frac{\left(m - \frac{\nu_1\tau_2}{n}\right)I^*}{g(I^*)}g'(I^*) \\
&> m\left[1 - \frac{I^*}{g(I^*)}g'(I^*)\right] \geq 0. \quad (18)
\end{aligned}
$$

Hence,

$b_1 = g(I^*) + \omega_1 + d + m + n - S^*g'(I^*) > 0 \quad \text{using 18,}$
$b_2 = (m+n)[g(I^*) + \omega_1 + d] - S^*g'(I^*)(\omega_1 + d + n) + mn - \nu_1\tau_2$
$\quad = (m+n)g(I^*) + (\omega_1 + d)\left[m + n - S^*g'(I^*)\right] - nS^*g'(I^*) + mn - \nu_1\tau_2$
$\quad = (m + n)g(I^*) + (\omega_1 + d)\left[m + n - S^*g'(I^*)\right] + n\left[m - \frac{\left(m - \frac{\nu_1\tau_2}{n}\right)I^*}{g(I^*)}g'(I^*)\right] - \nu_1\tau_2$
$\quad = (m + n)g(I^*) + (\omega_1 + d)\left[m + n - S^*g'(I^*)\right] + (mn - \nu_1\tau_2)\left[1 - \frac{I^*}{g(I^*)}g'(I^*)\right] > 0 \quad \text{using 17,}$

and

$b_3 = (mn - \nu_1\tau_2)[g(I^*) + \omega_1 + d] - n(\omega_1 + d)S^*g'(I^*)$
$\quad = (mn - \nu_1\tau_2)\left[g(I^*) + \omega_1 + d\right]$
$\qquad - (\omega_1 + d)\left[\frac{(mn - \nu_1\tau_2)I^*}{g(I^*)}g'(I^*)\right]$
$\quad = (mn - \nu_1\tau_2)\left[g(I^*) + \omega_1 + d - (\omega_1 + d)\frac{I^*}{g(I^*)}g'(I^*)\right]$
$\quad = (mn - \nu_1\tau_2)\left\{g(I^*) + (\omega_1 + d)\left[1 - \frac{I^*}{g(I^*)}g'(I^*)\right]\right\} > 0$

using 17.

Moreover we have that

$b_1 b_2 - b_3$
$= b_1\left\{(m + n)g(I^*) + (\omega_1 + d)\left[m + n - S^*g'(I^*)\right]\right\}$
$\quad - (mn - \nu_1\tau_2)g(I^*)$
$\quad + b_1\left\{(mn - \nu_1\tau_2)\left(1 - \frac{I^*}{g(I^*)}g'(I^*)\right)\right\}$
$\quad - \left\{(mn - \nu_1\tau_2)\left[(\omega_1 + d)\left(1 - \frac{I^*}{g(I^*)}g'(I^*)\right)\right]\right\}.$

It follows that

$b_1 b_2 - b_3$
$> n\left\{(m+n)g(I^*) + (\omega_1+d)\left[m+n-S^*g'(I^*)\right]\right\} - mn\,g(I^*) +$
$\left\{(mn-\nu_1\tau_2)\left[(\omega_1+d)\left(1-\frac{I^*}{g(I^*)}g'(I^*)\right)\right][g(I^*)+m-S^*g'(I^*)]\right\}$
$> 0.$

As a result, by the Routh−Hurwitz stability criterion [11], all the roots of the characteristic polynomial (15) have strictly negative real parts. Therefore the endemic equilibrium is locally asymptotically stable.
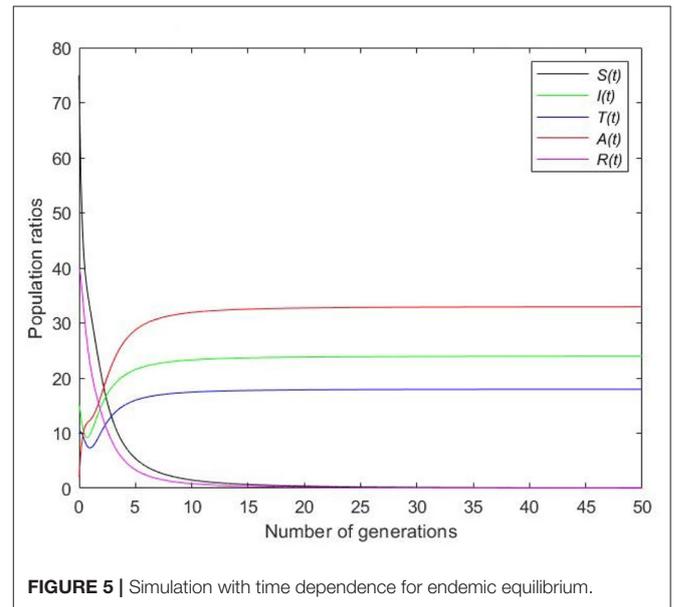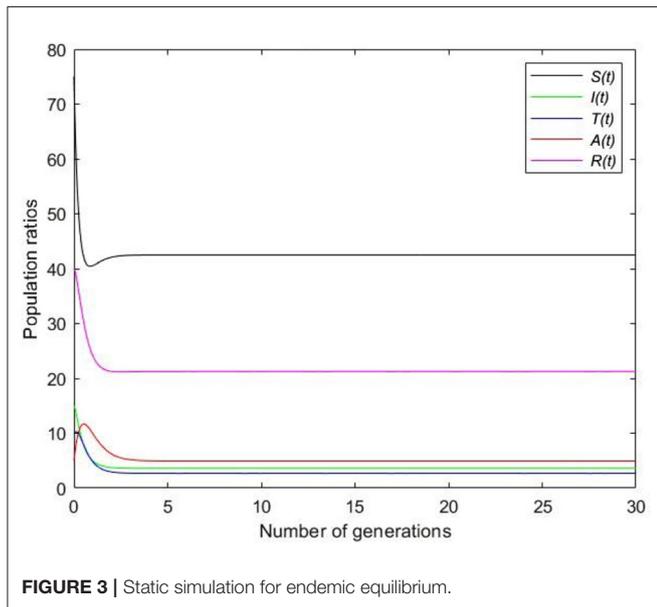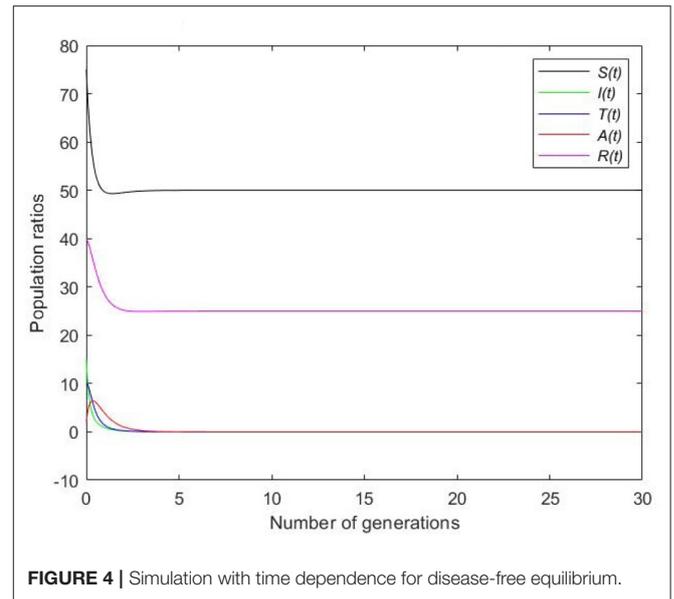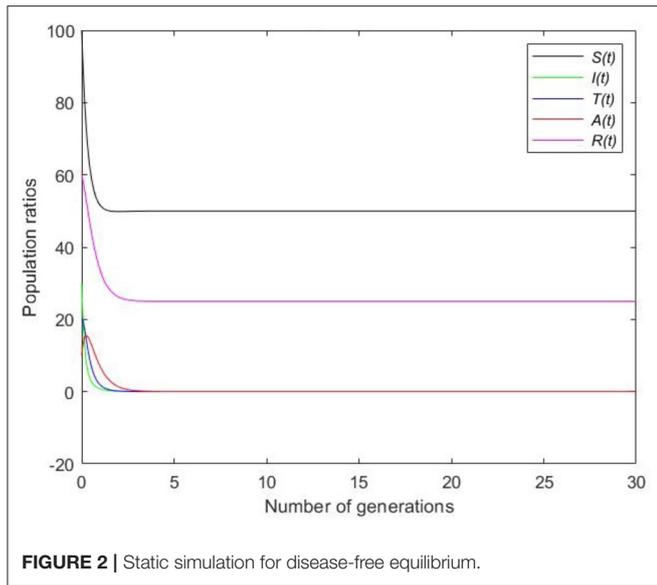
5. NUMERICAL SIMULATIONS

In this section, we provide numerical simulations for the evolution system of ordinary differential equations (1) to support the theoretical findings. Without loss of generality we set

$$g(I) = \frac{\alpha(t)I}{1 + \beta(t)I}.$$

Note that conditions of assumptions (H_1) and (H_2) are satisfied. Furthermore we let

$$\Gamma = 150, \omega_1 = 1, d = 2, \nu_1 = 1,$$
$$\nu_2 = 1, \tau_1 = 2, \tau_2 = 3.$$

FIGURE 2 | Static simulation for disease-free equilibrium.

FIGURE 4 | Simulation with time dependence for disease-free equilibrium.

FIGURE 3 | Static simulation for endemic equilibrium.

FIGURE 5 | Simulation with time dependence for endemic equilibrium.

Next we explore two scenarios involving static simulations and time-dependent simulations respectively.

5.1. Static Simulations

Picking $\alpha = \frac{1}{12}$ and $\beta = 1$ and substituting in the expression of the basic reproduction number, we get that $\mathcal{R}_0 = \frac{2}{3} < 1$. According to Theorem 2 the disease-free equilibrium, $E^0 = (50, 0, 0, 0, 25)$, is globally asymptotically stable. In **Figure 2**, it clearly shows that the disease eventually dies out.

Picking $\alpha = \frac{1}{5}$ and $\beta = 0.1$ and substituting in the expression of the basic reproduction number, we get that $\mathcal{R}_0 = 1.6 > 1$. According to Theorem 3 the endemic equilibrium, $E^* = (42.5, 3.6, 2.7, 4.9, 21.2)$, is locally asymptotically stable. In **Figure 3**, all the graphs converge to the endemic equilibrium.

5.2. Time-Dependent Simulations

Picking $\alpha(t) = \frac{1}{5+2t}$, it can be observed in **Figure 4** that graphs converge to the disease free equilibrium as time increases. It therefore suggests the global asymptotical stability of the disease free equilibrium and the extension of the disease in time.

Picking $\alpha(t) = t^2 + \frac{5}{4}$, **Figure 5** clearly shows that the susceptible population vanishes in a short span of time and the disease essentially affects all people in the population.

6. CONCLUDING REMARKS

In this paper, we formulated and investigated a mathematical model describing the dynamics on sexually transmitted disease

models with a novel class of non-linear incidence. We showed that the derived non-autonomous system of differential equations governing the evolution of the process was well-posed and the solution happened to be positive and bounded. The role of awareness in the susceptible and infectious classes was explored and investigated. A vertical asymptote and concavity arguments were critical in the proof of existence of an endemic equilibrium for the system and its asymptotical stability. In particular, numerical simulations were performed in order to predict the asymptotic behavior of solutions and support the theoretical findings.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

All authors contributed to all critical aspects of the analysis and simulations. All authors contributed to the article and approved the submitted version.

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